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Rainbow Turán methods for trees

Vic Bednar Neal Bushaw

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
U.S.A.

Abstract

The rainbow Turán number, a natural extension of the well studied traditional Turán number, was introduced in 2007 by Keevash, Mubayi, Sudakov and Verstraëte. The rainbow Turán number of a graph H, $\mathrm{ex}^\star(n, H)$, is the largest number of edges for an n vertex graph G which can be properly edge-colored with no rainbow H subgraph. We explore the reduction method for finding upper bounds on rainbow Turán numbers, and use this to inform results for the rainbow Turán numbers of double stars, caterpillars, and perfect binary trees. In addition, we define k-unique colorings and the related k-unique Turán numbers. We provide preliminary results on this new variant on the classic problem.

1 Introduction

The forbidden subgraph problem is perhaps the prototypical question in extremal graph theory. How do we maximize the number of edges in an n-vertex graph, while forbidding some fixed subgraph?

The earliest famous instance of this sort of problem dates to 1907; now known as Mantel's Theorem, this gives the largest possible number of edges in an n-vertex graph that does not contain any triangles [13][1]. In the 1940's Turán generalized Mantel's result to forbidding any particular complete graph, rather than just the triangle. For a fixed graph F, the *Turán number*, $\mathrm{ex}(n, F)$[2], is the largest number of edges of any graph G on n vertices which is F-free. In this context, we refer to F as the *forbidden subgraph* and H as the *host graph*. Throughout, we will use $|H|$ and $\|H\|$ to denote the number of vertices and edges in H, respectively.

[1]It is worth noting that although Mantel is typically given credit for this reference, the history is somewhat more complicated. Mantel submitted the problem to *Nieuwe Opgaven*, where it was distributed loose-leaf to the members of the Royal Dutch Mathematical Society. Eventually, this was reprinted in a digest version, along with a solution due to Wythoff (although it is further noted in this version that solutions were also submitted by H. Gouwentak, W. Mantel, J. Teixeira de Mattos, and Dr. F. Schuh, but only Wythoff's proof appears).

[2]From this point forward we will refer to this as the traditional Turán number.

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The Erdős-Stone[3] theorem states that $\mathrm{ex}(n, F) = \frac{1}{\chi(F)-1}\binom{n}{2} + o(n^2)$, where $\chi(F)$ is the chromatic number of the forbidden subgraph F. As the theorem gives degenerate results for graphs whose chromatic number is two, much of the research into traditional Turán numbers is centered around bipartite graphs. While many results address specific bipartite graphs or specific families of bipartite graphs, there is still little known for bipartite graphs as a whole (for an exhaustive survey, see [7]).

In this paper we consider two closely-related variants of the traditional Turán problem. The first, whose systemic study was formalized in 2007 by Keevash, Mubayi, Sudakov, and Verstraëte, adds in a proper edge coloring of the host graph and requires that the forbidden subgraph be "rainbow" — every edge colored a distinct color. The *rainbow Turán number* of a graph F, denoted $\mathrm{ex}^\star(n, F)$, is the largest possible number of edges among those n-vertex graphs which can be properly edge-colored in a way that contains no rainbow F subgraph. There are two important aspects to this seemingly simple definition. First, there must exist a graph with n vertices and $\mathrm{ex}^\star(n, F)$ edges which admits at least one proper edge coloring that does not contain a rainbow F subgraph. Additionally, every proper edge coloring of every graph with n vertices and at least $\mathrm{ex}^\star(n, F)+1$ edges must contain a rainbow F subgraph. Of particular use to us in this context is the *chromatic index of G*, written $\chi'(G)$; this is the fewest number of colors possible among all proper edge-colorings of G. The most famous bound on this well studied graph invariant is due to Vizing [15] who showed that $\chi'(G)$ is at most one larger than the maximum degree of any vertex in G, $\Delta(G)$. Since any coloring trivially uses at least $\Delta(G)$ colors, this shows that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

The second variation (introduced in this paper) considers the case where a specific number of edges in F must be assigned distinct colors. Given graphs G, F, a natural number k, and a proper edge coloring $\phi : E(G) \to \mathbb{N}$, we say that G contains an *exactly k-unique copy of F* if there are exactly k edges whose color appears exactly once on F. Similarly, G contains a *k-unique copy of F* when there are *at least* k edges whose color appears exactly once on F. We define a new parameterized family of extremal functions, which run from the traditional Turán number to the rainbow Turán number. Given a forbidden graph F and a natural number k, the *k-unique Turán number* $\mathrm{ex}_k(n, F)$ is the largest number of edges in an n vertex graph which has some proper edge coloring that contains no k-unique F.[3]

In this paper we consider the rainbow Turán problem and the k-unique Turán problem for families of small trees. In particular, we focus on the family of double stars — for any $r, s \in \mathbb{N}$, the double star $DS_{r,s}$ is the tree formed by taking a single edge yx and appending r leaves to y and s leaves to x. It is worth noting that all trees of diameter 3 are double stars and vice versa.

[3]We note that throughout, we will be interested in non-exact colorings — that is, k-unique as opposed to exactly k-unique. A larger number of uniquely colored edges will be good for us. We also emphasize that throughout, we mean *edge-colorings when we refer to colorings.*

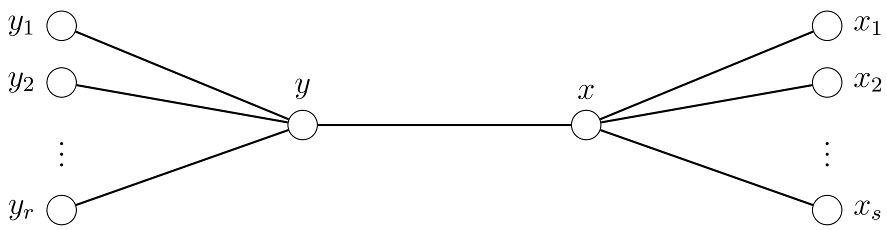

Figure 1: $DS_{r,s}$

2 Previous Results

In [12], the authors introduced the rainbow Turán problem, and further conjectured that $\text{ex}^\star(n, C_{2k}) = O(n^{1+1/k})$. In 2013, Das, Lee, and Sudakov showed $\text{ex}^\star(n, C_{2k}) = O(n^{1+\frac{(1+\epsilon_k)\ln k}{k}})$ with $\epsilon_k \to 0$ as $k \to \infty$ in [2]. O. Janzer proved the conjecture in its entirety (in and in fact a much stronger result) to cover all theta graphs in [9]. A 2016 paper by Johnston, Palmer and Sarkar [10] disproved a conjecture on paths from [12] showing that $\text{ex}^\star(n, P_l) \leq \lceil \frac{3l-2}{2} \rceil n$, and provided exact results for some particular l. Shortly after, the authors of [5] improved this bound to $\text{ex}^\star(n, P_l) < \left(\frac{9(l-1)}{7} + 2 \right) n$. A lower bound for the rainbow Turán number of paths was given in [11] as $\text{ex}^\star(n, P_k) \geq \frac{k}{2} n + O(1)$. The same paper provided bounds on the rainbow Turán number of caterpillars and brooms, as well as an exact result for a specific family or brooms.

3 Summary Of Results

From [12] we have that the rainbow Turán number satisfies the inequality

$$\text{ex}(n, F) \leq \text{ex}^\star(n, F) \leq \text{ex}(n, H) + o(n^2).$$

In conjunction with the Erdős-Stone theorem, this result gives the asympotic *rainbow Turán density* $\frac{\text{ex}^\star(n,F)}{\binom{n}{2}}$. Because of this, we restrict our consideration of rainbow Turán numbers to bipartite graphs, where the Erdős-Stone theorem tells us only that $\text{ex}^\star(n, H) = o(n^2)$. In particular, we present results for the family of double stars as defined in Section 1. Additionally, we present results on k-unique Turán numbers. These are of particular interest due to the fact that we have the following chain of inequalities for any graph F:

$$\text{ex}(n, F) = \text{ex}_0(n, F) \leq \text{ex}_1(n, F) \leq \cdots \leq \text{ex}_{\|H\|}(n, F) = \text{ex}^\star(n, F)$$

Part of the motivation for the k-unique Turán number is that they could allow us to improve existing lower bounds for the rainbow Turán numbers by finding k-unique Turán numbers and increasing the value of k.

In order to consider k-unique Turán numbers, we begin by determining which exactly k-unique values are possible for particular graphs. With that in mind, we define the *k-spectrum* of a graph F, $\mathrm{Spec}(F)$, as the set of k for which F admits an exactly k-unique-coloring. We say that a graph has *full spectrum* if $\mathrm{Spec}(F) = \{0, 1, \ldots, \|F\|-2, \|F\|\}$. We note suspicious omission of $\|F\|-1$ from this list — note that having $(\|F\|-1)$ uniquely colored edges is impossible, since this leaves one edge whose color can neither be unique nor match the color of any other edges.

Theorem 4.1 gives some criteria for some graphs which have a complete k-spectrum. Then, in Corollary 4.2 and Corollary 4.3 we present a complete description of the k-spectrum of cycles and paths respectively. We conclude our focus on k-spectrum with Lemma 4.4 which describes the k-spectrum for all double stars. We have just seen that Theorem 4.5 presents an upper bound for $\mathrm{ex}_k(n, DS_{r,s})$ when $DS_{r,s}$ admits an exactly k-unique coloring; then, Lemma 4.6 extends the result to cover $\mathrm{ex}_k(n, DS_{r,s})$ when $DS_{r,s}$ does not admit an exact k-unique coloring.

The upper bound in Lemma 4.5 is achieved through a combination of a reduction method, inspired by the method in [12], and the Erdős-Sós Conjecture [4]. Proposed in 1963 by Vera Sós, the Erdős-Sós Conjecture is based on the observation that the traditional Turán number is the same for paths and stars.

Conjecture 3.1. *For any tree T with t edges,* $\mathrm{ex}(n, T) \leq \frac{(t-1)n}{2}$.

In the early 2000's, Ajtai, Komlós, Simonovits, and Szemerédi announced a proof of the conjecture; this has not yet appeared. There have been specific cases of the Erdős-Sós Conjecture for which results have been published. Of particular note is a result by McLennan [14] which proves the Erdős-Sós conjecture for trees with diameter at most 4. The double star, short brooms, and caterpillars fall under the above diameter constraint.

Theorem 4.7 uses the same methods to give an upper and lower bound bound on $\mathrm{ex}^\star(n, DS_{r,s})$. In Theorems 4.11, 4.8, and 4.10 we use the previous upper bound, but construct graphs which provide better lower bounds for the rainbow Turán numbers of $DS_{2,2}$ and $DS_{1,2k+1}$. Our result in Theorem 4.11 matches a result of Johnston and Rombach in [11], however we use a different method to achieve the matching upper bound.

In Theorem 4.12, 4.13, and 4.14 we extend our reduction method to cover caterpillars and k-ary trees. In general, these trees do not fall under the result from [14]. In this case, we assume the Erdős-Sós Conjecture is true for all trees and proceed by creating an algorithm to augment the caterpillars and k-ary trees respectively in order to reduce the rainbow Turán problem into a traditional Turán problem.

4 Proofs

Theorem 4.1. *Let F be a graph with a proper edge coloring with color classes $L_1, \ldots, L_r$, ordered so that $|L_1| \geq |L_2| \geq \cdots \geq |L_r|$. If $|L_1| \geq 3$ and $|L_r| \geq 2$, then F has full spectrum, that is* $\mathrm{Spec}(F) = \{0, 1, \ldots, \|F\| - 2, \|F\|\}$.

Proof. Let F be a graph with a proper edge-coloring with color classes $L_1, \ldots, L_r$ with $|L_i| = l_i$, ordered so that $l_1 \geq l_2 \geq \cdots \geq l_r$ and let $l_1 \geq 3$ and $l_r \geq 2$. Label the edges of F by $e_{i,j}$ with $1 \leq j \leq l_i$ such that $L_i = \{e_{i,1}, \ldots, e_{i,l_i}\}$. Define a new proper edge coloring $\phi_R : E(F) \to \|F\|$ where each edge is assigned a distinct color, with the restriction that $\phi_R(e_{i,1}) = i$.

Under ϕ_R, each edge of F has a distinct color, so $\|F\| \in \text{Spec}(F)$. Recoloring $e_{1,2}$ to color 1 gives a $(\|F\| - 2)$-unique coloring, as these edges become the only edges with matching colors. Recoloring the additional edges $e_{1,j}$ to color 1 will decrease the number of distinctly colored edges by one because $e_{1,1}$ is no longer the only edge in its color class. Doing so sequentially gives k-unique colorings for $(\|F\| - l_1) \leq k \leq (\|F\| - 3)$, and so each of those values must belong in $\text{Spec}(F)$.

We know $l_1 \geq 3$, but it is possible that $l_i = 2$ for all $i > 1$. Additionally, if we simply recolor $e_{i,2}$ with color i, the number of distinctly colored edges will decrease by two as it did for $e_{1,2}$ and L_1. The following color-switching operation addresses both of these problems simultaneously.

Start with the coloring in which each of the l_1 edges $e_{1,j}$ are color 1 and all remaining edges in F are colored by ϕ_R. Then simultaneously recolor e_{1,l_1} to color $\phi_R(e_{1,l_1})$ and $e_{2,2}$ to color 2. By assigning e_{1,l_1} a distinct color for this step, a $(\|F\| - l_1 - 1)$-unique coloring is constructed. To construct a $(\|F\| - l_1 - 2)$-unique coloring, change the color of e_{1,l_1} back to color 1. If $l_2 > 2$, each remaining edge $e_{2,j}$ can be reassigned to color 2 one at a time. As before, this process constructs κ-unique colorings of F with $(\|F\| - l_1 - 3) \leq k \leq (\|F\| - l_1 - l_2)$. If $l_2 = 2$, there are no more edges $e_{2,j}$.

Repeating this color switching process using e_{1,l_1} and $e_{i,2}$ for all $2 \leq i \leq r$ constructs a set of colorings that show $\text{Spec}(F)$ is complete. ■

Corollary 4.2. *Every cycle C_k with $k \geq 6$ has full spectrum. For cycles with less than 6 vertices, we have* $\text{Spec}(C_5) = \{1, 3, 5\}$, $\text{Spec}(C_4) = \{0, 2, 4\}$, *and* $\text{Spec}(C_3) = \{3\}$.

Proof. The result for C_k with $k \geq 6$ follows directly from Theorem 4.1. For C_k with $k < 6$, we proceed by cases. Every proper edge coloring of C_3 is equivalent to the rainbow coloring, so $\text{Spec}(C_3) = \{3\}$.

Every proper edge coloring of C_4 which uses two colors is equivalent. Further, each color class has cardinality two, so no edges are assigned a unique color and so $0 \in \text{Spec}(C_4)$. In each proper edge coloring using three colors, there is a single color class with cardinality two and two color classes with cardinality one. For every proper edge coloring on C_4 with three colors, there are exactly two edges assigned a distinct color, and $2 \in \text{Spec}(C_4)$. Finally, we consider the edge coloring in which each edge is assigned a distinct color. There are only 4 edges, so we have $4 \in \text{Spec}(C_4)$. Then $\text{Spec}(C_4) = \{0, 2, 4\}$.

The odd cycle C_5 has a chromatic index of three, so we need not consider edge colorings using fewer than 3 colors. In each proper edge coloring of C_5 using three colors, we have two color classes of cardinality two and one color class with cardinality one. These colorings each assign a single edge to a unique color, so $1 \in \mathrm{Spec}(C_5)$. In proper edge colorings using precisely 4 colors, there is a single color class with cardinality two and three color classes with cardinality one. Thus $3 \in \mathrm{Spec}(C_5)$. The coloring which assigns each edge a distinct color uses 5 colors, and thus $5 \in \mathrm{Spec}(C_5)$. Then, we find that $\mathrm{Spec}(C_5) = \{1, 3, 5\}$. ■

Corollary 4.3. *Every path P_k of length k with $k \geq 5$ has full spectrum. For paths of length less than 5, we have* $\mathrm{Spec}(P_4) = \{0, 2, 4\}$*,* $\mathrm{Spec}(P_3) = \{1, 3\}$*,* $\mathrm{Spec}(P_2) = \{2\}$*, and* $\mathrm{Spec}(P_1) = \{1\}$*.*

Proof. The result for P_k with $k \geq 5$ follows directly from Theorem 4.1. For P_k with $k < 5$ we proceed by cases. P_1 and P_2 each have a single proper edge coloring (up to a renaming of the colors), and they both assign each edge a distinct color, so $\mathrm{Spec}(P_1) = \{1\}$ and $\mathrm{Spec}(P_2) = \{2\}$.

For P_3, we can trivially find a coloring with one color class of cardinality two and one color class with cardinality one. Thus $1 \in \mathrm{Spec}(P_3)$. Now we consider the edge coloring which assigns each edge to a distinct color. This gives $3 \in \mathrm{Spec}(P_3)$, and thus $\mathrm{Spec}(P_3) = \{1, 3\}$.

There are three kinds of non-equivalent proper edge colorings of P_4. Every proper edge coloring of P_4 which uses exactly two colors is equivalent, all with two color classes with cardinality two. In each, no edge is assigned to a distinct color so $0 \in \mathrm{Spec}(P_4)$. For P_4 with precisely three colors, there are two non-equivalent proper edge colorings. In each there are two color classes with cardinality one and one color class with cardinality two. Thus $2 \in \mathrm{Spec}(P_4)$. Finally, the edge coloring which assigns each edge a distinct color gives $4 \in \mathrm{Spec}(P_4)$, and we have $\mathrm{Spec}(P_4) = \{0, 2, 4\}$. ■

Lemma 4.4. *For a double star $DS_{r,s}$ with $r \leq s$ we have*

$$\mathrm{Spec}(DS_{r,s}) = \{s - r + 1 + 2l : 0 \leq l \leq r\}.$$

Proof. Since yx is incident to every other edge, its color cannot appear on any other edge of $DS_{r,s}$. Let j be the number of edges in $DS_{r,s}$ that have a distinct color under any proper edge coloring; we note we trivially have $j = s - r + 1$. We refer to edges incident to x or y but not both as *pendants*, since these will play the role of the leaves in our double star. In any proper edge coloring of $DS_{r,s}$, repeated colors must appear as pairs of pendants, one each from y and x. We call two pendants that share a color partners. Then, the number of repeated colors is at most r. If every color that appears on y also appears on a pendant from x, the remaining $s - r$ pendants from x are distinctly colored (giving us $j = s - r + 1$ distinctly colored edges from these pendants and the dominating edge). Now, by recoloring l pendants of y with colors not appearing elsewhere in the graph, we obtain a coloring with $j + 2l$ distinctly colored edges since each y pendant and its partner x pendant now have distinct colors (since y has only r pendants, we require that $l \leq r$). ■

Theorem 4.5. *For a double star $DS_{r,s}$ with $r \leq s$ which admits an exactly $(j+2l)$-unique-coloring, we have*

$$\frac{(s+l-1)n}{2} + o(n) \leq \mathrm{ex}_{j+2l}(n, DS_{r,s}) \leq \frac{(r+s+l)n}{2}$$

where $j = s - r + 1$ and $0 \leq l \leq r$.

Proof. From Lemma 4.4, we have $DS_{r,s}$ colored with $s+l-1$ colors is $(j+2l)$-unique. By Vizing's theorem, we know that a graph G with $\Delta(G) = s+l-1$ can be properly edge colored with $s+l$ colors. When $(s+\ell-1)n$ is even, we let G be a $(s+\ell-1)$-regular graph; in the case that $(s+\ell-1)n$ is odd, we let G be a graph with maximum degree $(s+\ell-1)$ and as many edges as possible. Then G can be properly colored so there is no rainbow $DS_{r,s}$ subgraph. This gives $\frac{(s+\ell-1)n}{2} + o(n) \leq \mathrm{ex}_{j+2\ell}(n, DS_{r,s})$.

By the Erdős-Sós Conjecture, a graph with $\frac{(r+s+\ell)n}{2}$ edges must contain a copy of $DS_{r,s+\ell}$. As noted in the proof of the previous lemma, any repeated colors in $DS_{r,s+\ell}$ must appear as pairs of pendants incident to y and x. Then there are $j+2\ell$ distinctly colored edges in any proper edge coloring of $DS_{r,s+\ell}$ when $j = s-r+1$. There must be a $(j+2\ell)$-unique copy of $DS_{r,s}$ in a properly edge colored $DS_{r,s+\ell}$, and the upper bound is achieved. ■

The reduction method, as illustrated in the proof of the upper bound in Theorem 4.5, begins with an observation from [12] that rainbow Turán problems can be reduced to traditional Turán problems. Consider P_3, the path on 3 edges. As shown in Figure 2, there is a proper edge coloring of P_3 that is not rainbow.

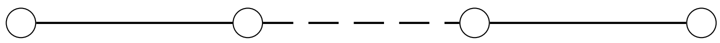

Figure 2: P_3 has a proper edge coloring that is not rainbow.

However, if we add one edge, as illustrated in Figure 3, the resulting graph contains a rainbow P_3 subgraph. We call the new graph, in this case, $DS_{r,s_{1,2}}$, the augmented graph. Although only one proper edge coloring of the augmented graph is shown, there is a rainbow P_3 in every proper edge coloring of $DS_{r,s_{1,2}}$.

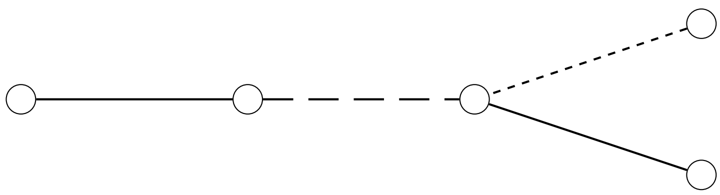

Figure 3: $DS_{r,s_{1,2}}$ always contains a rainbow P_3

Forbidding the augmented graph forbids rainbow P_3 as well as some additional edges. Likewise, a graph containing a $DS_{r,s_{1,2}}$ always has a rainbow P_3 subgraph. Then, the traditional Turán number of the augmented graph is an upper bound on the rainbow Turán number of P_3. In general, we refer to the forbidden graph as F (or T when the forbidden graph is specifically a tree), and an augmented graph that necessarily contains a rainbow F as F' (or T').

The main idea of the method used to find upper bounds for $\mathrm{ex}^\star(n,T)$ is the combination of the reduction method as described above in conjunction with the bound from the Erdős-Sós Conjecture in 3.1.

Lemma 4.6. *Let k be given and set k' to be the smallest $k' \geq k$ for which $k' \in \mathrm{Spec}(F)$. Then $\mathrm{ex}_{k'}(n,F) = \mathrm{ex}_k(n,F)$.*

Proof. Consider the largest graph on n vertices G which can be colored to avoid a k-unique copy of F. Since no exactly k''-unique coloring is possible with $k < k'' < k'$ (since such values of k'' are not in the spectrum), this k-unique copy must have at least k' unique colors. ∎

Theorem 4.7. *For all double stars $DS_{r,s}$ with $r \leq s$, we have*

$$\frac{(s+r-1)n}{2} + o(1) \leq \mathrm{ex}^\star(n, DS_{r,s}) \leq \frac{(s+2r)n}{2}.$$

Proof. As we saw in the proof of Theorem 4.5, any properly edge colored $DS_{r,s+r}$ must contain a rainbow $DS_{r,s}$. Therefore the Erdős-Sós bound for $\mathrm{ex}(n, DS_{r,s})$ gives the following upper bound:

$$\mathrm{ex}^\star(n, DS_{r,s}) \leq \mathrm{ex}(n, DS_{r,s+r}) \leq \frac{(s+2r)n}{2}.$$

The lower bound is shown by a simple application of Vizing's theorem, as in Theorem 4.5. In order for every proper edge coloring of some graph G to force a rainbow copy of $DS_{r,s}$, it is necessary that every proper coloring of G uses at least $s+r+1$ colors. By Vizing's theorem, a graph with $\Delta(G)+1 = s+r$ can be properly colored with $s+r$ colors to avoid a rainbow $DS_{r,s}$ subgraph. This gives a lower bound of $\mathrm{ex}^\star(n, DS_{r,s}) \geq \frac{(s+r-1)n}{2} + o(1)$. ∎

Theorem 4.8. *Let $\phi : E(K_6) \to [5]$ be an edge coloring of K_6 where every vertex is adjacent to exactly one edge of each color. Then ϕ is rainbow $DS_{2,2}$-free.*

Proof. Let $\phi : E(K_6) \to [5]$ be an edge coloring of K_6 where every vertex is adjacent to exactly one edge of each color, and suppose that K_6 with ϕ contains a rainbow $DS_{2,2}$. Without loss of generality, choose any vertex to act as y in a $DS_{2,2}$. Under ϕ, this vertex must be adjacent to five different colored edges. In Figure 4 below, the center vertex is y, and the solid black edges are assigned colors 1 through 5.

We arbitrarily select a vertex adjacent to y to be x and choose which vertices will be its leaves as shown by the vertex labels in Figure 4. In order for this $DS_{2,2}$ to have each edge assigned to a distinct color, edges xx_1 and xx_2 (shown below with big dashes) must be assigned colors 5 and 2 respectively. The remaining edges

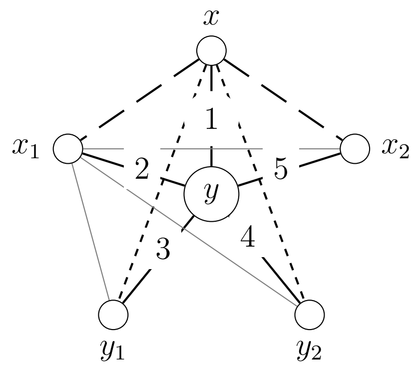

Figure 4: $DS_{2,2}$ embedded in K_6.

adjacent to x, specifically xy_1 and xy_2 (shown below with small dashed edges) must be assigned colors 4 and 3 respectively.

Then we consider how colors can be assigned to the uncolored edges incident to vertex x_1 (shown in grey). The three colors 1, 3, and 4 have not met vertex x_1. However neither edge x_1y_1 nor x_1y_2 may be assigned color 3 or 4. This is a contradiction, since in ϕ every vertex is incident to an edge of every color. Thus there is no rainbow $DS_{2,2}$ in K_6 with edge coloring ϕ. ■

Theorem 4.9. *Any proper edge coloring of* K_6 *which is not rainbow contains an exactly* 3*-unique* $DS_{2,2}$.

Proof. Consider any proper edge coloring $\phi : E(K_6) \to \mathbb{N}$ which is not rainbow. Then two edges must assigned to the same color. Label the vertices such that v_1v_2 and v_4v_5 are one of these pairs as in Figure 5.

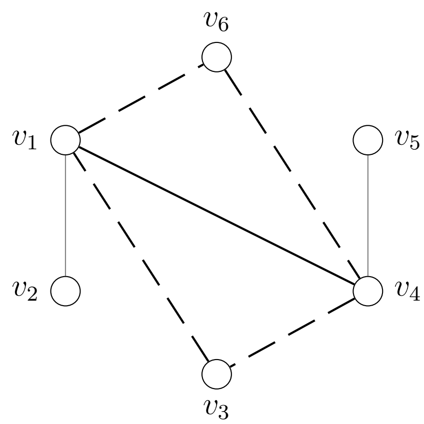

Figure 5: $DS_{2,2}$ embedded in K_6 with repeated edge color.

Let v_1v_4 be the dominating edge of a double star. From the two remaining vertices v_3 and v_6, one must be a leaf incident to v_1 and one must be a leaf incident to v_4.

Note the C_4: $v_1v_6v_4v_3$. Regardless of how ϕ assigns colors to these four edges, since the coloring is proper we are able to choose our two leaves so the edges have distinct colors, thus creating an exactly 3-unique $DS_{2,2}$ subgraph. ■

Theorem 4.10. *For the double star $DS_{2,2}$, we have the following inequality:*

$$\frac{5n}{2} \leq \text{ex}^{\star}(n, DS_{2,2}) \leq \frac{6n}{2}.$$

Proof. The lower bound follows by using disjoint K_6's (and possibly one smaller clique when $n \not\equiv 0 \mod 6$) colored as given in Theorem 4.8. For the upper bound, we apply the result from Theorem 4.7. Since $r = s = 2$, we find $\text{ex}^{\star}(n, DS_{2,2}) \leq \frac{6n}{2}$. ■

Theorem 4.11. *For $DS_{1,2s+1}$ for any $s \in \mathbb{N}$,*

$$\text{ex}^{\star}(n, DS_{1,2s+1}) = \frac{(2s+3)n}{2} + o(1).$$

Proof. The upper bound is an application of Theorem 4.7 which gives

$$\text{ex}^{\star}(n, DS_{1,2s+1}) \;\leq\; \frac{(2s+3)n}{2}.$$

We prove the lower bound by showing that it is possible to properly edge color K_{2s+4} in a way that avoids a rainbow $DS_{1,2s+1}$; the extremal construction then comes from disjoint copies of this (along with possibly one smaller clique, according to divisibility). Note that $|K_{2s+4}| = |DS_{1,2s+1}|$. Further, since $2s+4$ is even, K_{2s+4} admits an edge coloring with $2s+3$ colors in which each vertex meets each color exactly once. We call this edge coloring ϕ. Assume K_{2s+4} with ϕ contains a rainbow $DS_{1,2s+1}$. Up to renaming the colors, this $DS_{1,2s+1}$ appears as in Figure 6, with backbone xy and edge colors as labeled (without loss of generality).

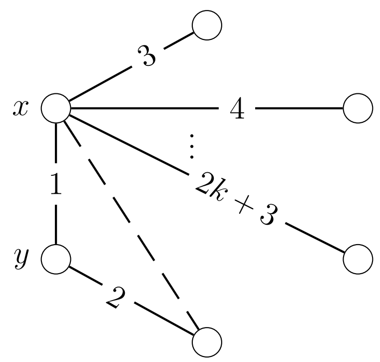

Figure 6: $DS_{1,s2+1}$ embedded in K_{2s+4}.

However, the dotted edge between x and a shared neighbor of y is then incident to edges of colors $1, 2, \ldots, 2k+3$; as ϕ only uses $2k+3$ colors, it is impossible to properly color this edge and so such a coloring ϕ cannot exist. From this, we have that $\text{ex}^\star(n, DS_{1,2s+1}) \geq \frac{(2s+3)n}{2} + o(1)$ with equality when $(2s+3)$ divides n. ■

A *broom* $B_{k,r}$ is a path on k vertices with r leaves appended to one endpoint. Every graph in Theorem 4.11 is also a broom. However, there is a more general family of graphs, *caterpillars* $C_{c_1,\ldots,c_k}$, consisting of a path on k vertices labeled $x_1, \ldots x_k$ with c_i pendants attached at vertex x_i. This means every broom $B_{k,r}$ is also a caterpillar $C_{0,\ldots,0,r}$, where r appears in the k^{th} position. With this in mind, the following theorem also applies to all brooms, although Theorem 4.11 gives a better result when $k = 3$.

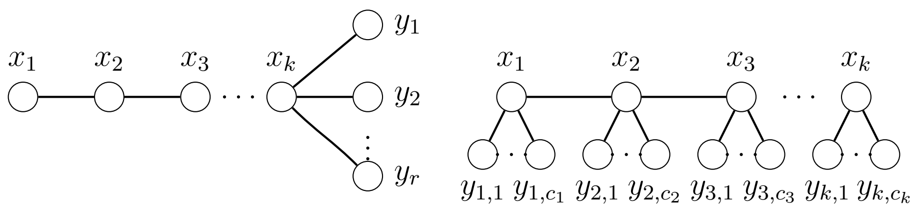

Figure 7: Broom $B_{k,r}$ (left) and caterpillar $C_{c_1,\ldots c_k}$ (right)

Theorem 4.12. *Let $c_1, \ldots, c_k$ be non-negative integers. Then, contingent on the Erdős-Sós conjecture, we have*

$$\text{ex}^\star(n, C_{c_1,\ldots c_k}) \leq [3c_1 + 2c_2 + c_3 + 3 + \sum_{j=3}^{k} P_j(L_j + 1)]\frac{n}{2}$$

where $P_j = P_{j-1}\sum_{i=4}^{j-2}(c_i + 1)$, $P_1 = c_1$, and $L_j = j - 1 + \sum_{i=1}^{j} c_i$.

Proof. We label the pendant vertices of $C_{c_1,\ldots,c_k}$ such that the children of x_i are $v_{i,j}$ for $1 \leq j \leq c_i$. We will construct an augmented caterpillar $C_{c'_1,\ldots,c'_k}$ which contains a rainbow $C_{c_1,\ldots,c_k}$ under every proper edge-coloring. In order to do this, we first describe such a rainbow-forcing augmented graph for caterpillars with three backbone vertices, and then explain how to iteratively add to this graph in order to find our longer rainbow caterpillars. As we do so, we will need to add enough new neighbors to the previous backbone end that we can avoid all previously seen colors. Then, we add so many children to each of these that whichever is selected to play the role of x_{k+1} can have c_{k+1} distinctly colored children while still avoiding all previously seen colors.

In the case of C_{c_1,c_2,c_3}, the two backbone edges will be assigned distinct colors under any proper edge coloring. The pendants adjacent to x_1 will all be assigned distinct colors as well, and by adding an extra pendant, we can ensure that there will be at least c_1 edges which do not share a color with either of the backbone edge. Then

$c_1' = c_1 + 1$ in $C_{c_1',c_2',c_3'}$. The pendant edges adjacent to x_2 will all be assigned colors distinct from each other and distinct from both backbone edges. In order to ensure that there are c_2 of them which do not share a color with the pendants adjacent to x_1 we add another c_1 pendants. Thus $c_2' = c_2 + c_1$. Finally we consider the pendant edges adjacent to x_3. There are c_3 of them in C_{c_1,c_2,c_3}, and they will all be assigned colors distinct from each other and distinct from the edge x_2x_3. To guarantee that there are c_3 of them which do not share a color with the other backbone edges or any of the other pendant edges, we can simply add $c_1 + c_2 + 1$ extra pendants. Then $c_3' = c_1 + c_2 + c_3 + 1$. This new graph, $C_{c_1',\dots,c_k'}$, has $3c_1 + 2c_2 + c_3 + 4$ edges in total.

When $k = 3$, the augmented graph that must contain a rainbow C_{c_1,c_2,c_3} subgraph is still a caterpillar. For higher values of k, this will no longer be the case. As we generalize this method to cover larger k, we refer to the augmented graph of $C_{c_1,\dots,c_k}$ as $C'_{c_1,\dots,c_k}$.

We think of $C_{c_1,\dots,c_k}$ with $k = 4$ as a copy of C_{c_1,c_2,c_3}, augmented by an extra backbone edge (and its children). Similarly, we view $C'_{c_1,\dots,c_4}$ as an augmentation of $C_{c_1',c_2',c_3'}$. In order to ensure that there is an edge in the c_i' graph which can act as the image of the new backbone edge x_3x_4 in the rainbow subgraph, we add $c_1 + c_2 + 1$ more edges adjacent to x_3, for a total of $c_1 + c_2 + 2$ non-pendant edges separate from x_2x_3. Each of these edges will get $c_1 + c_2 + c_3 + 3$ new pendant edges adjacent to them. This is large enough to ensure that as we find our rainbow subgraph we can choose an edge to serve as x_3x_4 and still have the necessary c_4 pendants while avoiding colors already in our subgraph (since there are at most $c_1 + c_2 + c_3 + 3$ such colors to avoid).

We continue in this way for higher values of k; the next added backbone branches will be adjacent to x_{k-1} and not any of the $v_{k-1,j}$, and we add enough children to avoid the colors in all previous edges in our rainbow subgraph. With this in mind, the previous argument for $C_{c_1,\dots,c_4}$ generalizes into the following parts: P_j the number of backbone vertices which can act as x_j in a rainbow subgraph, and L_j the number of leaves adjacent to each of the P_j parent vertices. The sum of all c_i up to and including j plus $(j-1)$ non-adjacent backbone edges is the number of leaves at level j. Then there must be $L_j = j - 1 + \sum_{i=1}^{j} c_i$ pendants adjacent to each parent vertex. Since the backbone branches at every new vertex, we multiply the number of previous parents by the sum of all previous children to determine the number of parent vertices at each level. Then $P_j = P_{j-1} \sum_{i=4}^{j-2} (c_i + 1)$. In this case we start branching at $i = 4$ since we use the argument for C_{c_1,c_2,c_3} before then. Thus the total number of edges in $C'_{c_1,\dots,c_k}$ is $3c_1 + 2c_2 + c_3 + 5 + \left[\sum_{j=3}^{k} P_j(L_j + 1)\right]$. Applying the bound from the Erdős-Sós Conjecture gives the result stated in the theorem. ■

Let a *k-ary tree* be a rooted tree with every non-leaf vertex having exactly k children, and let a *perfect k-ary tree* be a k-ary tree with all leaves at the same depth, or distance from the rooted vertex. Then a perfect binary tree with a depth of 2 has one vertex of degree 2, two vertices with degree 3, and 4 leaves. We use $T(k, d)$ to denote a perfect k-ary tree with depth d. Applying our reduction method to $T(k, d)$ gives the following results.

Theorem 4.13. *For a perfect binary tree with depth d,*

$$\mathrm{ex}^\star(n, T(2,d)) \leq \Big(\sum_{j=2}^{d}[2\prod_{i=2}^{j}(2^i - 3)] + 1\Big)\frac{n}{2}.$$

Proof. Summing the 2^i vertices at each level of the tree, we find a total of $\sum_{i=0}^{d} 2^i = 2^{d+1} - 1$ vertices in $T(2,d)$, and thus $2^{d+1} - 2$ edges. We label the vertices and edges in $T(2,d)$ according to the following system, an example of which is shown in Figure 8. The vertices are labeled $v_{i,j}$, for $1 \leq j \leq 2^i$, where i is the distance to the root vertex, and with vertices $v_{i+1,2j}$ and $v_{i+1,2j-1}$ as the children of $v_{i,j}$. We will identify edges using the indices of the endpoint furthest from $v_{0,1}$; e.g. edge $v_{1,2}v_{2,4}$ would be labeled $e_{2,4}$. In the vein of the reduction method used earlier, we will construct a tree $T'(2,d)$ which contains a rainbow $T(2,d)$ subgraph under every proper edge coloring. For clarity, we do this first for $T(2,2)$ and then $T(2,d)$.

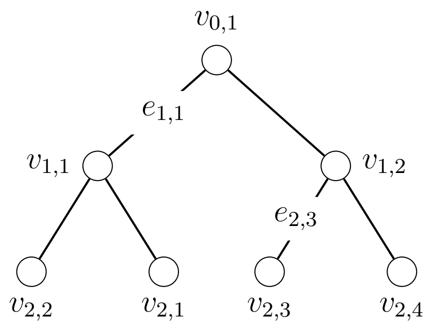

Figure 8: Perfect binary tree $T(2,2)$

In $T(2,2)$, $e_{1,1}$ and $e_{1,2}$ will be assigned distinct colors under any proper edge coloring. We need to ensure that we can always find a pair of pendants for each of $v_{1,1}$ and $v_{1,2}$ whose colors are different from those already in our rainbow subgraph. Without loss of generality, we focus on the children of $v_{1,1}$. The two associated edges ($e_{2,1}$ and $e_{2,2}$) will always be assigned colors distinct from each other, and distinct from $e_{1,1}$. In order to ensure there will be two edges with colors distinct from the rest of the leaves, we append two more leaves to the corresponding vertex in $T'(2,2)$. The addition of one more leaf ensures that there is also an edge with a color unique from $e_{1,2}$. Then the total number of leaves adjacent to $v_{1,1}$ in $T'(2,2)$ is the total number of leaves in $T(2,2)$ plus the number of edges in the subtree above it (minus one); in general, when considering $T(k,d)$, we will find $T(2,d-1) - 1$ leaves. This gives $2^{d+1} - 3$ leaves attached to each parent vertex. Since there are two vertices at depth one, the total number of leaves in $T'(2,2)$ is 10. Including the edges $e_{1,1}$ and $e_{1,2}$, there are 12 edges in $T'(2,2)$. Applying the bound from the Erdős-Sós Conjecture to $T'(2,2)$ gives $\mathrm{ex}^\star(n, T(2,2)) \leq \mathrm{ex}(n, T'(2,2)) \leq \frac{11n}{2}$, which is the bound claimed in the theorem.

The argument above generalizes neatly to cover all $T(2,d)$. Having appended $2^{d+1} - 3$ leaf edges each $v_{j-1,j}$ guarantees that even if the colors of the other $2^d - 2$ leaves and the colors of the $2^d - 1$ non-adjacent edges in the the $T(2, d-1)$ subgraph are all repeated, there are still two edges with distinct colors adjacent to $v_{d-1,j}$. The number of **parents of leaves** in $T'(2,d)$ is the same as the **number of leaves** in $T'(2, d-1)$. Continuing in this way, we find the number of vertices at depth $d-2$ in $T'(2,d)$ is the same as the number of leaves in $T'(2, d-2)$. As in the previous example, at depth 1 there are two single edges which will always be assigned distinct colors. This gives $2\prod_{i=2}^{k}(2^i - 3)$ as the total number of leaves in $T'(k,d)$. The sum $(2 + \sum_{j=2}^{d}(\text{number of leaves in } T'(2,j))$ provides the total number of edges in $T'(2,d)$. Applying the bound from the Erdős-Sós Conjecture to this number gives $\text{ex}^{\star}(n, T(2,d)) \leq \text{ex}(n, T'(2,d)) \leq (\sum_{j=2}^{d}[2\prod_{i=2}^{j}(2^i - 3)] + 1)\frac{n}{2}$. ■

Theorem 4.14. *For a perfect k-ary tree with depth d,*

$$\text{ex}^{\star}(n, T(k,d)) \leq \Big(k - 1 + \sum_{j=2}^{d}\big[k\prod_{i=2}^{j}(k^i + \frac{k^i - 1}{k-1} - 2)\big]\Big)\frac{n}{2}.$$

Proof. Generalizing the geometric series in Theorem 4.13 shows the number of vertices in $T(k,d)$ is $\frac{k^{d+1}-1}{k-1}$, and number of edges is $\frac{k^{d+1}}{k-1} - 1$. Following the same argument as in Theorem 4.13, each vertex $v_{d-1,j}$ in $T'(k,d)$ must be adjacent to $k^d + \frac{k^d-1}{k-1} - 2$ leaves to guarantee there are enough to choose k many pendants with distinct colors in a rainbow $T(k,d)$ subgraph. The number of parents of leaves in $T'(k,d)$ is the same as the number of leaves in $T'(k, d-1)$. This process gives $k\prod_{i=2}^{j}(k^i + \frac{k^i-1}{k-1} - 2)$ as the total number of leaves in $T'(k,d)$. Each vertex $v_{i,j}$ except for $v_{0,1}$ is associated with exactly one edge $e_{i,j}$. Then counting the number of vertices at each depth $i \geq 2$ gives $\sum_{j=2}^{d}[k\prod_{i=2}^{j}(k^i + \frac{k^i-1}{k-1} - 2)]$, which is the same as the number of edges in $T'(k,d) - v_{0,1}$. By adding k more edges to account for those adjacent to vertex $v_{0,1}$, we find the total number of edges in $T'(k,d)$. Applying the bound from the Erdős-Sós Conjecture to this edge count gives an upper bound of

$$\text{ex}^{\star}(n, T(k,d)) \leq \text{ex}(n, T'(k,d) \leq (k - 1 + \sum_{j=2}^{d}[k\prod_{i=2}^{j}(k^i + \frac{k^i - 1}{k-1} - 2)])\frac{n}{2}.$$

■

5 Conclusion

As we see in Theorem 4.10 and Theorem 4.11, when using the reduction method in combination with the Erdős-Sós Conjecture gives us a tree with a few edges more than the original forbidden subgraph, but which guarantees a rainbow copy of the forbidden graph, then we get upper and lower bounds which are quite close. At first glance it seems as though the small diameter of such graphs results in good upper bounds, but consider $DS_{20,20}$. In order to guarantee a rainbow subgraph, 20 edges must be appended to the original, and the upper bound achieved through the reduction method and Erdős-Sós Conjecture is seemingly further than what we expect the true value to be. On the contrary for the graph $DS_{1,39}$, which has the same diameter and number of edges as $DS_{20,20}$, the methods here give matching upper and lower bounds on $\text{ex}^\star(n, DS_{1,39})$. This implies a more complicated set of criteria for which this combination method provides good bounds; more investigation is needed.

There are many trees not covered by the methods in this paper. Subdivided stars, and others with a simple branching structure should also lend themselves nicely to the reduction method as outlined in this paper. In particular, the Erdős-Sós Conjecture has been proved for subdivided stars in [6], and so these are a natural next step.

In general, there may be ways to improve the reduction method to give better results for those trees with large size or diameter. For example, if we know that a host graph has a certain connectivity then how does that impact the rainbow subgraphs it may contain? Aside from just the connectivity, knowing more about the properties/invariants of the extremal graphs for specified forbidden subgraphs may provide a method to reduce the number of added edges needed when implementing the reduction method. Further, we may wish to use augmented graphs which are not trees, if the forbidden subgraph F already contains a cycle. Here, Erdős-Sós does not apply, and the extremal numbers for such graphs are much larger (either by Erdős-Stone [3], in the case of odd cycles, or by Bondy-Simonovits [1] in the case of even cycles. It is not clear how this affects the reduction method described in this paper.

This manuscript only gives a brief introduction into the k-unique Turán numbers. There is a lot more work that can be done here. Additionally, we define the generalized k-unique Turán problems, as a natural extension of generalized rainbow Turán problems as defined in [8]. What is the maximum number of copies of F in a properly edge-colored graph on n vertices without a k-unique copy of F? It is important to note that using the reduction method for the k-unique Turán numbers may provide useful information about this, but only for upper bounds. Progress on lower bounds will require different methods, and improvement of the lower bounds for k-unique Turán numbers provide one such approach.

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