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Applying Skolem sequences to gracefully label new families of triangular windmills

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Abstract

A function f is a graceful labelling of a graph $G = (V, E)$ with m edges if f is an injection $f : V \mapsto \{0, 1, 2, \ldots, m\}$ such that each edge $uv \in E$ is assigned the label $|f(u) - f(v)|$, and no two edge labels are the same. If a graph G has a graceful labelling, we say that G itself is graceful. In this paper, we prove any Dutch windmill with three pendant triangles is (near) graceful, which settles Rosa's conjecture for a new family of triangular cacti.

1 Introduction

Graceful labellings, much like Skolem sequences, are an attempt to transform a problem in graph decompositions into a problem of combinatorial arrangement. Historically, the root of the problem is an attempt to solve Ringel's conjecture [10], that is, given an arbitrary tree T on $m + 1$ vertices, it is possible to decompose the complete graph K_{2m+1} into copies of T. Rosa [11] introduced what he called a β-*valuation* (later renamed a *graceful labelling*) in an attempt to solve this problem, as any tree whose vertices were gracefully labelled led naturally to a cyclic decomposition of the complete graph. This has, in turn, inspired the Ringel-Kotzig Conjecture (reported by Rosa in [11]) that every tree is graceful.

A tremendous amount of work has been done in showing that many families of trees are graceful, and further that many non-trees are also graceful. A dynamic survey by Gallian [6] captures the state of the art in graceful labelling and several related labellings.

One particular family of graphs that has drawn some interest is that of triangular cacti, connected graphs whose blocks are all 3-cycles. Thus, triangular cacti are a sort of "triangular tree," and due to their tree-like structure, form a natural setting to consider gracefulness. Further, since a great deal is known about combinatorial triple systems, some labellings will follow directly from known triple system

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constructions. This latter fact was used by Bermond [2], who found necessary and sufficient conditions for the gracefulness of Dutch windmills, a subtype of triangular cacti where every 3-cycle shares a common vertex. (Following our tree analogy, these may be likened to the stars of the triangular cactus world.)

In [12], Rosa conjectured the following for triangular cacti, as well as verifying the conjectures for all cacti using up to five 3-cycles.

1. All triangular cacti with n blocks, $n \equiv 0,1 \pmod 4$, are graceful,
2. All triangular cacti with n blocks, $n \equiv 2,3 \pmod 4$, are near graceful.

(Being near graceful is a slight weakening of being graceful, and is defined in Section 2.) In 1989, Moulton [8] proved Rosa's conjecture for triangular snakes, a type of triangular cactus whose block cutpoint graph is a path. In 2012, Dyer, Payne, Shalaby, and Wicks [5] verified Rosa's conjecture for a new class of triangular cacti, Dutch windmills with at most two pendant triangles. The method was a variation of the one suggested by Bermond; namely, that Skolem sequences could be used to label these triples. Furthermore, Skolem-type sequences were used recently in [1] to prove the existence of (near) graceful labellings for variable windmills.

In this paper, we further develop the system by which we add pendant triangles to a family of gracefully labelled graphs to get new gracefully labelled families. In particular, we verify Rosa's conjecture for Dutch windmills of any possible order with three pendant triangles, proving Theorem 1.1, which will offer an extension of the results in [5].

Theorem 1.1 *Every Dutch windmill with exactly three pendant triangles that is not graceful is near graceful.*

2 Definitions and Preliminaries

In this section, we introduce some essential definitions and prior results needed to prove our results. The definitions for Skolem-type sequences in this section come from the Handbook of Combinatorial Designs [4] and will be extremely useful in Section 3.

A *Skolem sequence* of order n is a sequence $S = (s_1, s_2, \dots, s_{2n})$ of $2n$ integers satisfying the following conditions:

1. for every $k \in \{1, 2, \dots, n\}$, there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $S_4 = (1, 1, 4, 2, 3, 2, 4, 3)$ is a Skolem sequence of order 4. Equivalently, we sometimes just write the pairs of indices corresponding to identical terms, giving $\{(1, 2), (4, 6), (5, 8), (3, 7)\}$. We call j a *pivot* of a Skolem sequence $\{(a_j, b_j)\}_{j=1}^{n}$, if $b_j + j \leq 2n$. Skolem proved the following result for the existence of Skolem sequences.

Theorem 2.1 *[15] A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$.*

For $n \equiv 2, 3 \pmod 4$, the natural alternative is a hooked Skolem sequence. A *hooked Skolem sequence* of order n is a sequence $hS = (s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers satisfying the following conditions:

1. for every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in hS$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$;
3. $s_{2n} = 0$.

For example, $hS_2 = (1, 1, 2, 0, 2)$ is a hooked Skolem sequence of order 2. We call j a *pivot* of a hooked Skolem sequence $\{(a_j, b_j)\}_{j=1}^{n}$, if $b_j + j \leq 2n + 1$. O'Keefe proved the following result for the existence of hooked Skolem sequences.

Theorem 2.2 *[9] A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$.*

Skolem and hooked Skolem sequences have historically been used to approach several problems. For example, Heffter's first difference problem [7] asks: when can the set $\{1, \ldots, 3n\}$ be partitioned into n ordered triples (a_i, b_i, c_i), with $1 \leq i \leq n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n + 1}$? If such a partition is possible, then $\{\{0, a_i + n, b_i + n\} | 1 \leq i \leq n\}$ will be the base blocks of a cyclic Steiner triple system of order $6n + 1$, denoted CSTS$(6n + 1)$. The combined results of Skolem and O'Keefe gave a solution to Heffter's first difference problem as in Construction 2.3.

Construction 2.3 *[15] Consider the (hooked) Skolem sequence with pairs (a_i, b_i). The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, is a solution to the Heffter first difference problem. These triples yield the base blocks for a CSTS$(6n + 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$. Also, $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a CSTS$(6n + 1)$.*

For example, let $S_4 = (1, 1, 4, 2, 3, 2, 4, 3)$ be a Skolem sequence of order 4, yielding the pairs $(1, 2)$, $(4, 6)$, $(5, 8)$, $(3, 7)$. These pairs, in turn, yield the triples $(1, 5, 6)$, $(2, 8, 10)$, $(3, 9, 12)$, $(4, 7, 11)$, which form a solution to Heffter's first difference problem. These triples yield the base blocks for two CSTS$(25)s$:

1. $\{0, 5, 6\}, \{0, 8, 10\}, \{0, 9, 12\}$, and $\{0, 7, 11\} \pmod{25}$;
2. $\{0, 1, 6\}, \{0, 2, 10\}, \{0, 3, 12\}$, and $\{0, 4, 11\} \pmod{25}$.

Another useful pair of sequences are the Langford and hooked Langford sequences. A *Langford sequence* of defect d and order m, denoted L_d^m, is a sequence $L = (l_1, l_2, \ldots, l_{2m})$ which satisfies these conditions:

1. for every $k \in \{d, d+1, \ldots, d+m-1\}$, there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$;
2. if $l_i = l_j = k$, with $i < j$, then $j - i = k$.

A *hooked Langford sequence* of defect d and order m, denoted hL_d^m, is a sequence $hL = (l_1, l_2, \cdots, l_{2m+1})$ which satisfies these conditions:

1. for every $k \in \{d, d+1, \ldots, d+m-1\}$, there exist exactly two elements $l_i, l_j \in hL$ such that $l_i = l_j = k$;
2. if $l_i = l_j = k$, with $i < j$, then $j - i = k$;
3. $l_{2m} = 0$.

For example, $(4, 2, 3, 2, 4, 3)$ is a Langford sequence with $d = 2$ and $m = 3$ and $(8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, 0, 5)$ is a hooked Langford sequence with $d = 3$ and $m = 6$.

The necessary and sufficient conditions for the existence of (hooked) Langford sequences are given in Theorem 2.4.

Theorem 2.4 *[14] A Langford sequence of order m and defect d exists if and only if*

1. $m \geq 2d - 1$
2. $m \equiv 0, 1 \pmod 4$ *and d is odd, or*
3. $m \equiv 0, 3 \pmod 4$ *and d is even.*

Also, a hooked Langford sequence of order m and defect d exists if and only if

1. $m(m - 2d + 1) + 2 \geq 0$
2. $m \equiv 2, 3 \pmod 4$ *and d is odd, or*
3. $m \equiv 1, 2 \pmod 4$ *and d is even.*

Although Langford sequences can be thought of as a natural generalization of Skolem sequences, they have also been classically used to build new Skolem sequences. This is done by concatenating sequences of appropriate order. Further, hooked sequences may be interlaced, wherein one sequence is reversed so that the hook occurs in the second position (rather than the second-last), and then the new sequence is formed by taking the untouched hooked sequence up to third-last term, then the first term of the reversed sequence, then the final term of the untouched sequence, followed by the remainder of the reversed sequence. This classic technique is a much used method of constructing Skolem sequences and is stated in the following lemma.

Lemma 2.5 *[3] If a (hooked) Skolem sequence of order $d - 1$ exists, and a (hooked) Langford sequence of order m and defect d exists, then a (hooked) Skolem sequence of order $N = m + d - 1$ exists. In particular, a new Skolem sequence of order N is obtained by concatenating a Skolem sequence with a Langford sequence or by* *interlacing a hooked Skolem sequence and hooked Langford sequence. A new hooked Skolem sequence of order N is obtained by concatenating a Skolem sequence with a hooked Langford sequence or a hooked Skolem sequence and a Langford sequence so that the hook occurs in the second-last position.*

For example, let $hS_2 = (1,1,2,0,2)$ be a hooked Skolem sequence and $hL_3^6 = (8,4,7,3,6,4,3,5,8,7,6,0,5)$ be a hooked Langford sequence. By interlacing hS_2 and the reverse of hL_3^6, we obtain a new Skolem sequence $S_8 = (1,1,2,5,2,6,7,8,5,3,4,6,3,7,4,8)$ of order 8.

Having completed our review of Skolem-like sequences, we turn to the meat of this paper. We begin with the definitions of graceful and near graceful labellings, following [11].

Let $G = (V,E)$ be a graph with m edges. Let f be a labelling defined from $V(G)$ to $\{0,1,2,\ldots,m\}$ and let g be the induced edge labelling defined from $E(G)$ to $\{1,2,\ldots,m\}$ by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be *graceful*, if f is an injective mapping and g is a bijection. If a graph G has a graceful labelling, then it is *graceful*.

Let $G = (V,E)$ be a graph with m edges. Let f be a labelling defined from $V(G)$ to $\{0,1,2,\ldots,m+1\}$ and let g be the induced edge labelling defined from $E(G)$ to $\{1,2,\ldots,m-1,m\}$ or $\{1,2,\ldots,m-1,m+1\}$ by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be *near graceful*, if f is an injective mapping and g is a bijection. If a graph G has a near graceful labelling, then it is *near graceful*. Certainly, every graceful labelling is near graceful, and every graceful graph is also near graceful. Figure 1 is an example of a graceful graph (K_3) and near graceful graph that is not graceful (C_5).

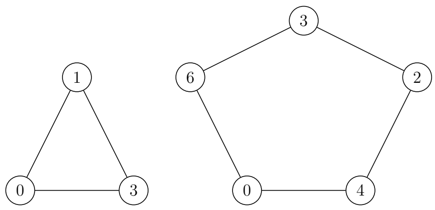

Figure 1: Graceful labelling of K_3 and near graceful labelling of C_5.

We apply the concept of gracefulness to triangular cacti, connected graphs whose blocks are all triangles (K_3). The *order* of a triangular cactus is the number of blocks in the cactus. The necessary conditions for gracefulness and near gracefulness of general triangular cacti are as follows.

Theorem 2.6 *[5] Let G be a triangular cactus of order n. Then,*

1. *if G is graceful, then $n \equiv 0,1 \pmod 4$, and,*
2. *if G is near graceful but not graceful, then $n \equiv 2,3 \pmod 4$.*

Proving that all triangular cacti are graceful or near graceful (Rosa's conjecture) remains an open and difficult problem. However, some subfamilies of triangular cacti have been shown to be graceful, including Dutch windmills. A Dutch windmill is a triangular cactus with the property of all its blocks having a common vertex, called the *central vertex*. In Dutch windmills and graphs related to Dutch windmills, blocks will be called *vanes*. We refer to Dutch windmill with k-vanes as a k-vane Dutch windmill. A *pendant triangle* is a block that is added to any triangular cacti. Thus, a Dutch windmill with one or more pendant triangles is another family of triangular cacti; the Dutch Windmills with three pendant triangles come in several different types, and we will introduce and discuss them in Section 3.

3 Dutch Windmills with three Pendant Triangles

In order to verify Rosa's conjecture for a new family of triangular cacti, namely Dutch windmills of any order with three pendant triangles, we will use Langford sequences to obtain Skolem and hooked Skolem sequences of considerable sizes. This technique was introduced in [5]. We categorize all such cacti into one of eleven types, called Type (a) through Type (k), and then gracefully or near gracefully label each type.

Figure 2 shows all eleven types of triangular cacti with exactly three pendant triangles using a 4-vane Dutch windmill as the base. By inspection, it can be seen that these eleven graphs represent all of the possible ways three pendant triangles can be attached to any Dutch Windmill up to isomorphism. For the remainder of this paper, we will refer to such triangular cacti as *Type* (λ) *Dutch windmills*, where (λ) is the letter from $\{a, b, \ldots, k\}$ that represents the attachment style of the three pendant triangles to the Dutch windmill. Thus, a Type (λ) Dutch windmill of order n is a $(n-3)$-vane Dutch windmill with three pendant triangles that are attached to it in the manner shown in Figure 2(λ).

Before proving that all Type (λ) Dutch windmills are graceful or near graceful, we need the following preliminary results. Theorem 3.1 and Theorem 3.2 and their proofs from [2] are included below for completeness.

Theorem 3.1 *[2] Let G be a Dutch windmill with n blocks. If there exists a Skolem sequence of order n, then G is graceful.*

Proof. Let G be a Dutch windmill with n blocks. Let S_n be a Skolem sequence of order n of the form (a_i, b_i) with $a_i < b_i$, for $i = 1, 2, \ldots, n$. These pairs give n base blocks which are $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$.

We can obtain three types of differences from $\{0, a_i+n, b_i+n\}$ as follows: let $A = \{(b_i + n) - (a_i + n)\} = \{1, 2, \ldots, n\}$, $B = \{(a_i + n) - 0\}$, and $C = \{(b_i + n) - 0\}$. Thus, $A \cup B \cup C = \{1, 2, \ldots, 3n\}$.

By labelling each block of G with a unique base block from the Skolem sequence, where the common vertex of the Dutch windmill is labelled 0, we obtain a graceful labelling of G. □

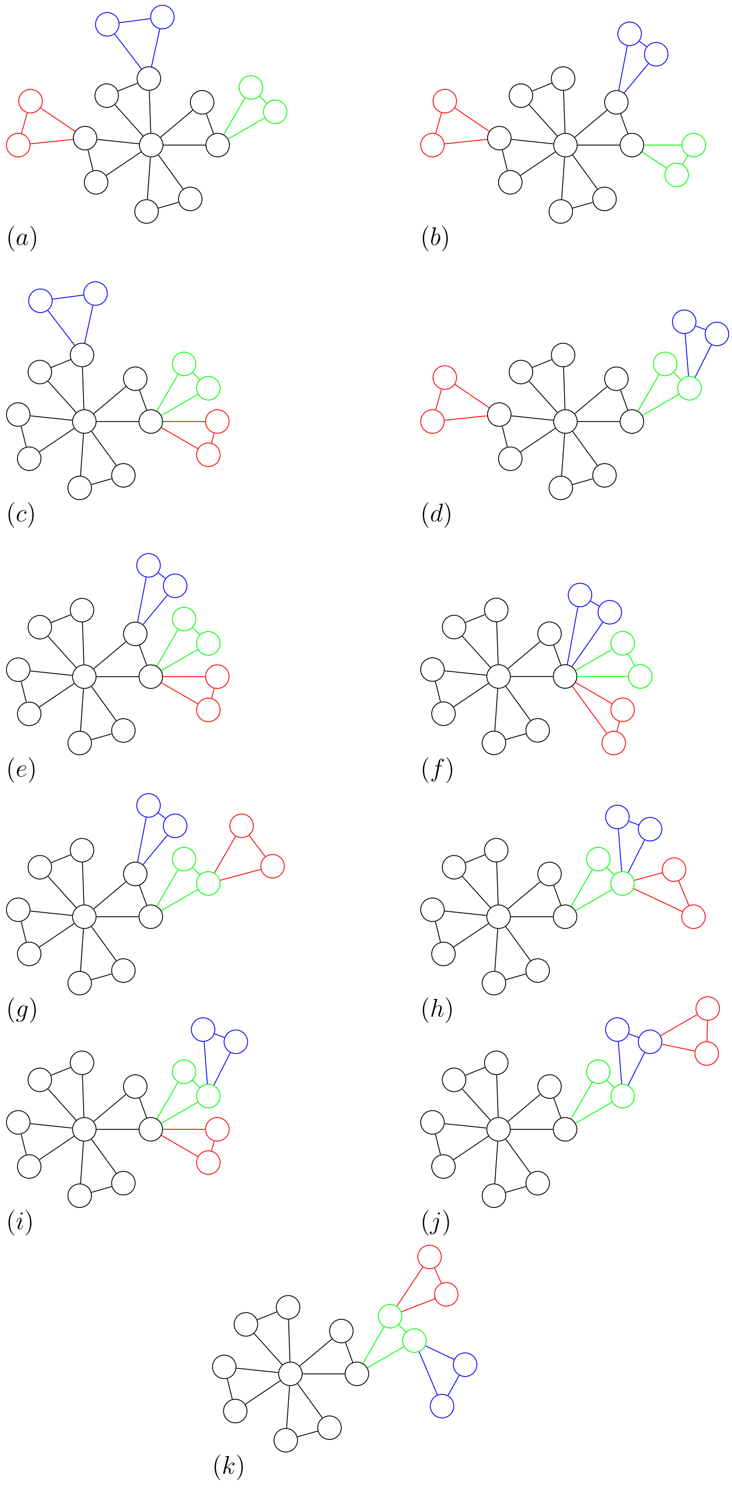

Figure 2: All triangular cacti of order 7 formed by attaching 3 pendant triangles to a 4-vane Dutch windmill.

Theorem 3.2 *[2] Let G be a Dutch windmill with n blocks. If there exists a hooked Skolem sequence of order n, then G is near graceful.*

Proof. Let G be a Dutch windmill with n blocks. Let hS_n be a hooked Skolem sequence of order n of the form (a_i, b_i) with $a_i < b_i$, for $i = 1, 2, \ldots, n$. These pairs give n base blocks which are $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$.

We can obtain three types of differences from $\{0, a_i+n, b_i+n\}$ as follows: let $A = \{(b_i + n) - (a_i + n)\} = \{1, 2, \ldots, n\}$, $B = \{(a_i + n) - 0\}$, and $C = \{(b_i + n) - 0\}$. Thus, $A \cup B \cup C = \{1, 2, \ldots, 3n - 1, 3n + 1\}$. (We know that $3n$ is omitted since, by construction of the hooked Skolem sequence, there is no entry in position $2n$.)

By labelling each block of G with a unique base block from the hooked Skolem sequence, where the common vertex of the Dutch windmill is labelled 0, we obtain a near graceful labelling of G. □

Note that in the (near) graceful labelling by (hooked) Skolem sequences, we can use the base blocks of the form $\{0, a_i + n, b_i + n\}_{i=1}^{n}$ or $\{0, i, b_i + n\}_{i=1}^{n}$. They give two different vertex labellings, but both use the same edge labels. For instance, taking the base blocks $\{0, 5, 6\}$, $\{0, 8, 10\}$, $\{0, 9, 12\}$, and $\{0, 7, 11\}$; or $\{0, 1, 6\}$, $\{0, 2, 10\}$, $\{0, 3, 12\}$, and $\{0, 4, 11\}$; from the example in Section 2 will give us the gracefully labelled Dutch windmill of order 4 shown in Figures 3(a) and (b).

The graceful labelling of the Dutch windmill is formed with base blocks of the form $\{0, a_i + n, b_i + n\}$ in Figure 3(a) and with base blocks of the form $\{0, i, b_i + n\}$ in Figure 3(b). Finally, in Figure 3(c) we use a mixed set of base blocks that come from both forms, where for each i we choose exactly one of $\{0, a_i + n, b_i + n\}$ or $\{0, i, b_i + n\}$. We will use this mixing technique in Section 3.11 to create graceful labellings of Type (k) Dutch windmills.

Since $x - y = (x + c) - (y + c)$ for any x, y, the following lemma is straightforward and classical in the literature. In the special case of having a triple of the form $(0, a_i + n, b_i + n)$, where i is a pivot, we call replacing that triple with the new triple $(i,\ a_i + i + n,\ b_i + i + n)$, *pivoting.*

Lemma 3.3 *If we add a constant c to each element of any triple $\{x, y, z\}$, which results in $\{x + c, y + c, z + c\}$, then the differences between the elements of $\{x + c, y + c, z + c\}$ will be the same.*

For example, consider the triples used in Figure 3(a). Add 2 to each element of the triple $\{0, 8, 10\}$ to obtain $\{2, 10, 12\}$. This gives the blocks $\{0, 5, 6\}$, $\{2, 10, 12\}$, $\{0, 9, 12\}$, and $\{0, 7, 11\}$. Consequently, we have obtained a graceful labelling for a 3-vane Dutch windmill with one pendant triangle.

We can also use pivoting on a hooked Skolem sequence. For example, let $hS_7 = (7, 4, 6, 3, 5, 4, 3, 7, 6, 5, 1, 1, 2, 0, 2)$ be a hooked Skolem sequence of order 7. This yields the pairs $\{(11, 12), (13, 15), (4, 7), (2, 6), (5, 10), (3, 9), (1, 8)\}$. This sequence gives the base blocks of the form $\{0, a_i + n, b_i + n\}$ as follows: $A = \{(0, 18, 19), (0, 20, 22), (0, 11, 14), (0, 9, 13), (0, 12, 17), (0, 10, 16), (0, 8, 15)\}$. The above sequence hS_7 has

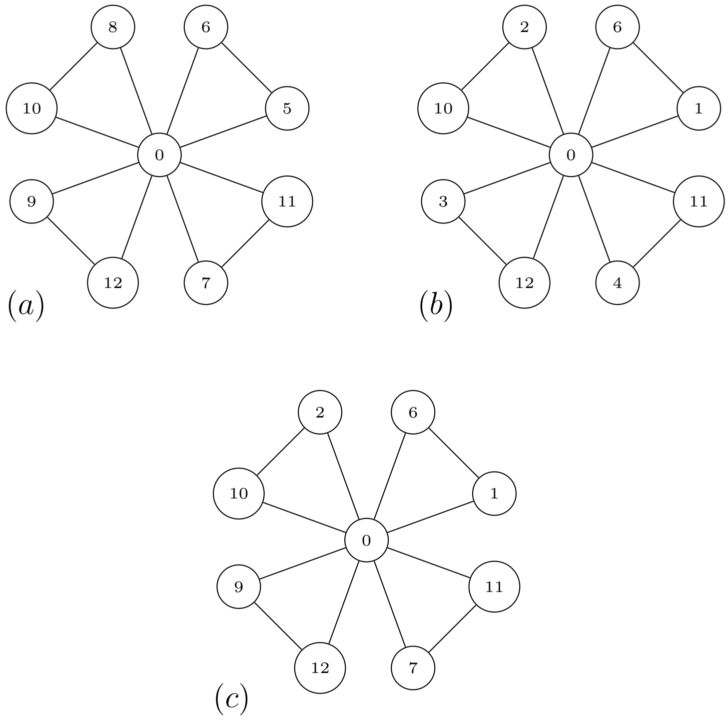

Figure 3: Three different gracefully labelled Dutch windmills of order 4.

six pivots, but we consider only three in particular, which are $1, 4$, and 7. We obtain a new set of base blocks by pivoting as follows: $A' = \{(1, 19, 20), (0, 20, 22), (0, 11, 14), (4, 13, 17), (0, 12, 17), (0, 10, 16), (7, 15, 22)\}$. (In this paper, if the sequence contains more than three pivots, we only consider the three distinguished pivots that allow us to find appropriate labels for the Dutch windmill with exactly three pendant triangles.)

We can use the set A' to label the graph in Figure 2(b) as follows: label the central vertex with 0. Then label all the vanes by these base blocks containing 0. Finally, label the pendant triangles with the triples corresponding to the pivots. The blocks $\{1, 19, 20\}$ and $\{7, 15, 22\}$ each intersect $\{0, 20, 22\}$ at a single distinct element, namely 20 for the first block and 22 for the second block. The block $\{4, 13, 17\}$ intersects $\{0, 12, 17\}$ at a single element, namely 17. This results in a near graceful labelling of the Type (b) Dutch windmill in Figure 2.

We can even extend this result to create arbitrarily long Skolem sequnces that contain "short" Skolem sequences with some configuration of pivots useful for labelling a "small" Dutch Windmill. We will do this by adding a Langford or hooked Langford sequence to our short Skolem sequence, preserving the configuration of the important pivots in the long Skolem sequence. Then, using the same basic construction method as in the previous examples (forming triples, then pivoting a select few of them) we will find appropriate labels for "big" Dutch windmills with the same pendant structure as the small windmill. This is exactly the method of the following corollary.

Corollary 3.4 *[5] Let S be a Skolem or hooked Skolem sequence of order n with m pivots. Suppose the pivots can be used to label some configuration of m pendant triangles. Then every Dutch windmill of order $3n+1$ or higher with that particular configuration of m pendant triangles is graceful or near graceful.*

In the remainder of this section, we provide the graceful and near graceful labellings for each of the eleven Type (λ) Dutch windmills. For Types (a) through (j), this will be through the application of Corollary 3.4, and for Type (k), it will be through an explicit construction.

3.1 Type (a)

Type (a) Dutch windmills of order n are formed by adding three pendant triangles to three distinct vanes of a $(n-3)$-vane Dutch windmill. For $n \leq 5$, there are not enough triangles to form a Type (a) Dutch windmill of order n. When $n = 6$, see the near graceful labelling in Figure 4.

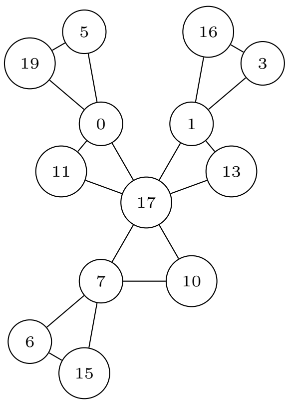

Figure 4: A near gracefully labelled Type (a) Dutch windmill with six blocks.

Lemma 3.5 *Any Type (a) Dutch windmill with at least six blocks is graceful or near graceful.*

Proof. The case with $n = 6$ blocks is given above. We begin by constructing a hooked Skolem sequence which we will use to label a Type (a) Dutch windmill of order 7. Consider the hooked Skolem sequence, $hS_7 = (3,4,7,3,2,4,2,5,6,7,1,1,5,0,6)$. This sequence gives triples of the form $(0, a_i + n, b_i + n)$ as follows: $(0,18,19)$, $(0,12,14)$, $(0,8,11)$, $(0,9,13)$,$(0,15,20)$, $(0,16,22)$, $(0,10,17)$. The above sequence hS_7 has four pivots (1, 2, 3, and 4), and we select three of them: $1,3$, and 4. We pivot the triples corresponding to those three values, and obtain the following: $(1,19,20)$, $(0,12,14)$, $(3,11,14)$, $(4,13,17)$, $(0,15,20)$, $(0,16,22)$, $(0,10,17)$. These triples give a near graceful labelling of a Type (a) Dutch windmill with 7 blocks.

By Theorem 2.4, a hooked Langford sequence of order m with defect d exists for $m \geq 15$, with m congruent to 1 or 2 (mod 4). Using the same hooked Skolem sequence hS_7, by Lemma 2.5 we obtain an associated Skolem sequence of order n. Note that 1, 3, and 4 will still be pivots of this Skolem sequence. Hence by Lemma 3.4, we can gracefully label a Type (a) Dutch windmill with n blocks for $n \geq 22$ with m congruent to 1 or 2 $\pmod 4$.

Likewise, by Theorem 2.4, a Langford sequence of order m with defect d exists for $m \geq 15$, with m congruent to 0 or 3 $\pmod 4$. Using the same hooked Skolem sequence hS_7, by Lemma 2.5 we obtain an associated hooked Skolem sequence of order n. Again, 1, 3, and 4 will still be pivots of this hooked Skolem sequence. Then by Lemma 3.4, we can near gracefully label a Type (a) Dutch windmill with n blocks for $n \geq 22$ with m congruent to 0 or 3 $\pmod 4$. Hence all Type (a) Dutch windmills with $n \geq 22$ blocks are graceful or near graceful.

Skolem and hooked Skolem sequences with three distinguished pivots of order $8 \leq n \leq 21$ are given in Table 1. By using the same pivoting technique as with hS_7 and the given pivots, we obtain (near) graceful labellings for all Type (a) Dutch windmills of order $8 \leq n \leq 21$. In this table and all subsequent tables, we will represent 10 by A, 11 by B, 12 by C, and so on.

Therefore any Type (a) Dutch windmill of order $n \geq 6$ is graceful or near graceful, as required. □

n	Skolem or hooked Skolem sequence	Pivots
8	4857411568723263	$1, 2$ and 4
9	759242574869311368	$2, 4$ and 5
10	A853113598A7426249706	$5, 8$ and A
11	B68527265A8B7941134A309	$2, 7$ and B
12	A8531135C8A6B9742624C79B	$3, 6$ and A
13	B97D4262479B6CA8D5311358AC	$3, 7$ and B
14	CA75311357$EACDB$864292468EBD09	$4, 6$ and C
15	FDB9753EC3579BDF6A84CE64118A202	$4, 9$ and F
16	9$FDBG$864292468$BDFECAG$75311357ACE	$3, 4$ and F
17	FDB9H64282469BDF8$GECAH$75311357$ACEG$	$4, 5$ and 9
18	GEC9753113579$ICEGHFDA$8642B2468$AIDFH$0B	$4, 6$ and 9
19	$IGEC$9753113579$JCEGIHFDA$8642B2468$AJDFH$0B	$7, 9$ and C
20	$JHFDB$9753CK3579$BDFHJICEGA$86411K468A2E2IG	$1, 6$ and 9
21	$KIGEC$9753113579$LCEGIKAJHFDB$8642A2468$LBDFHJ$	$5, A$ and K

Table 1: Skolem and hooked Skolem sequences with three pivots for Type (a).

3.2 Type (b)

Type (b) Dutch windmills of order n are formed by adding one pendant triangle to some vane of an $(n-3)$-vane Dutch windmill, and two more pendant triangles to another vane, each to a distinct vertex (other than the central vertex). For $n \leq 4$, there are not enough triangles to form a Type (b) Dutch windmill of order n.

Lemma 3.6 *Any Type* (b) *Dutch windmill with at least five blocks is graceful or near graceful.*

Proof. For Type (b), consider $S_5 = (4, 1, 1, 5, 4, 2, 3, 2, 5, 3)$. This sequence has three pivots, which are $1, 2$, and 4. This gives us a graceful labelling of a Type (b) Dutch windmill with five blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with S_5 to obtain a (near) graceful labelling of a Type (b) Dutch windmill with n blocks for $n \geq 16$. Table 2 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (b) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
6	5611453643202	$1, 3$ and 5
7	746354376511202	$1, 4$ and 7
8	3723258476541186	$1, 3$ and 5
9	759242574869311368	$1, 4$ and 5
10	$2529115784A694738630A$	$2, 4$ and 5
11	$B68527265A8B7941134A309$	$5, 7$ and B
12	$A8531135C8A6B9742624C79B$	$5, 6$ and A
13	$B97D4262479B6CA8D5311358AC$	$1, 6$ and 7
14	$CA75311357EACDB864292468EBD09$	$3, 4$ and C
15	$FDB9753EC3579BDF6A84CE64118A202$	$6, 9$ and F

Table 2: Skolem and hooked Skolem sequences with three pivots for Type (b).

3.3 Type (c)

Type (c) Dutch windmills of order n are formed by adding one pendant triangle to some vane of a $(n-3)$-vane Dutch windmill and two more pendants to another vane, both at the same non-central vertex. For $n \leq 4$, there are not enough triangles to form a Type (c) Dutch windmill of order n.

Lemma 3.7 *Any Type* (c) *Dutch windmill with at least five blocks is graceful or near graceful.*

Proof. For Type (c), consider $S_5 = (3, 5, 2, 3, 2, 4, 5, 1, 1, 4)$. This sequence has three pivots, which are $1, 2$, and 3. This gives us a graceful labelling of a Type (c) Dutch windmill with five blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with S_5 to obtain a (near) graceful labelling of a Type (c) Dutch windmill with n blocks for $n \geq 16$. Table 3 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (c) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
6	6451146523203	$1, 5$ and 6
7	746354376511202	$4, 6$ and 7
8	3723258476541186	$1, 4$ and 5
9	372329687115649854	$2, 3$ and 7
10	$36232A768119574A85409$	$1, 2$ and 3
11	$35232B549A841167B98A607$	$1, 2$ and 3
12	$3A232C78119ABB7685C49654B$	$1, 2$ and 3
13	$CA8531135D8AC6B9742624D79B$	$2, 4$ and 6
14	$3B932A211DE9B6CA485647D5E8C07$	$4, 6$ and A
15	$ECA8642D2468ACEFB953D735119B70F$	$4, 5$ and E

Table 3: Skolem and hooked Skolem sequences with three pivots for Type (c).

3.4 Type (d)

Type (d) Dutch windmills of order n are formed by adding one pendant to some vane of an $(n-3)$-vane Dutch windmill, then adding one pendant to another vane, then adding a pendant to this latter pendant at a vertex not shared with the $(n-3)$-vane Dutch windmill. For $n \leq 4$, there are not enough triangles to form a Type (d) Dutch windmill of order n. For $n = 5$, a graceful labelling of the Type (d) 5-vane Dutch windmill appears in [12]. When $n = 6$, see the near graceful labelling in Figure 5.

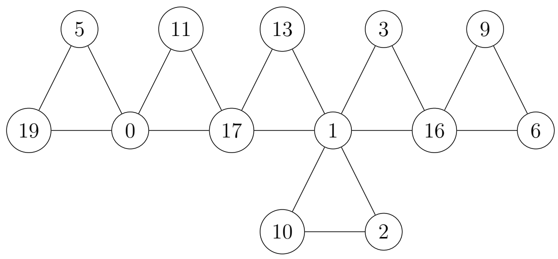

Figure 5: A near gracefully labelled Type (d) Dutch windmill with six blocks.

Lemma 3.8 *Any Type (d) Dutch windmill with at least five blocks is graceful or near graceful.*

Proof. For Type (d), consider $hS_7 = (3, 4, 7, 3, 2, 4, 2, 5, 6, 7, 1, 1, 5, 0, 6)$. This sequence has four pivots, and we select $1, 2$, and 3. This gives us a near graceful labelling of a Type (d) Dutch windmill with seven blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with hS_7 to obtain a (near) graceful labelling of Type (d) Dutch windmill with n blocks for $n \geq 22$. Table 4 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $8 \leq n \leq 21$, each of which gives a (near) graceful labelling of a Type (d) Dutch windmill with n blocks. $\square$

n	Skolem or hooked Skolem sequence	Pivots
8	4857411568723263	$1, 4$ and 5
9	759242574869311368	$1, 5$ and 7
10	$A853113598A7426249706$	$3, 8$ and A
11	$B68527265A8B7941134A309$	$2, 5$ and 7
12	$A8531135C8A6B9742624C79B$	$6, 8$ and A
13	$B97D4262479B6CA8D5311358AC$	$4, 6$ and 7
14	$CA75311357EACDB864292468EBD09$	$1, 3$ and A
15	$FDB9753EC3579BDF6A84CE64118A202$	$5, B$ and F
16	$9FDBG864292468BDFECAG75311357ACE$	$4, B$ and D
17	$FDB9H64282469BDF8GECAH75311357ACEG$	$2, 6$ and 9
18	$GEC9753113579ICEGHFDA8642B2468AIDFH0B$	$3, 6$ and 9
19	$IGEC9753113579JCEGIHFDA8642B2468AJDFH0B$	$1, 5$ and E
20	$JHFDB9753CK3579BDFHJICEGA86411K468A2E2IG$	$6, D$ and F
21	$KIGEC9753113579LCEGIKAJHFDB8642A2468LBDFHJ$	$6, E$ and G

Table 4: Skolem and hooked Skolem sequences with three pivots for Type (d).

3.5 Type (e)

Type (e) Dutch windmills of order n are formed by adding one pendant to some vane of an ($n-3$)-vane Dutch windmill, then adding two pendants to the same vane but both at the other non-central vertex. For $n \leq 3$, there are not enough triangles to form a Type (e) Dutch windmill of order n. For $n = 4$, a graceful labelling of the Type (e) 4-vane Dutch windmill appears in [12].

Lemma 3.9 *Any Type (e) Dutch windmill with at least four blocks is graceful or near graceful.*

Proof. For Type (e), consider $S_5 = (2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$. This sequence has four pivots, and we select $1, 2$, and 3. This gives us a graceful labelling of a Type (e) Dutch windmill with five blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with S_5 to obtain a (near) graceful labelling of a Type (e) Dutch windmill with n blocks for $n \geq 16$. Table 5 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (e) Dutch windmill with n blocks. □

3.6 Type (f)

Type (f) Dutch windmills of order n are formed by adding three pendant triangles to a single common non-central vertex of some vane of an ($n-3$)-vane Dutch windmill. For $n \leq 3$, there are not enough triangles to form a Type (f) Dutch windmill of order n. For $n = 4$, a graceful labelling of the Type (f) 4-vane Dutch windmill appears in [12].

n	Skolem or hooked Skolem sequence	Pivots
6	5611453643202	$3, 4$ and 5
7	746354376511202	$1, 6$ and 7
8	3723258476541186	$1, 2$ and 3
9	572825967348364911	$2, 3$ and 7
10	$36232A768119574A85409$	$2, 3$ and 6
11	$B68527265A8B7941134A309$	$4, 6$ and B
12	$A8531135C8A6B9742624C79B$	$4, 5$ and 6
13	$39B32D258A9C5B6784DA647C11$	$4, 8$ and B
14	$DB964E1146C9BDA8537E35C8A7202$	$3, 6$ and B
15	$CE3693BF262DC97EAB1187F4D5A4805$	$1, 7$ and B

Table 5: Skolem and hooked Skolem sequences with three pivots for Type (e).

Lemma 3.10 *Any Type* (f) *Dutch windmill with at least four blocks is graceful or near graceful.*

Proof. For Type (f), consider $S_5 = (2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$. This sequence has four pivots, and we select 1, 3, and 4. This gives us a graceful labelling of a Type (f) Dutch windmill with five blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with S_5 to obtain a (near) graceful labelling of a Type (f) Dutch windmill with n blocks for $n \geq 16$. Table 6 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (f) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
6	6451146523203	$2, 5$ and 6
7	746354376511202	$5, 6$ and 7
8	3723258476541186	$4, 5$ and 7
9	746394376825291158	$2, 6$ and 7
10	$A869117468A4973523205$	$2, 3$ and A
11	$378392A2768B5946A54110B$	$4, 5$ and 9
12	$2529115B86AC947684B3A73C$	$4, 6$ and 9
13	$9BD3753AC957B82D2A46C84116$	$2, 7$ and 9
14	$E7D6C5B47654A8EDCB2928A311309$	$5, 6$ and 7
15	$3C9382B2E1198CDFAB5647E546AD70F$	$4, 5$ and B

Table 6: Skolem and hooked Skolem sequences with three pivots for Type (f).

3.7 Type (g)

Type (g) Dutch windmills of order n are formed by adding two pendants to the two non-central vertices of some vane of an $(n-3)$-vane Dutch windmill, then adding one pendant to one of these pendants, at a vertex not shared with the $(n-3)$-vane Dutch windmill. For $n \leq 3$, there are not enough triangles to form a Type (g) Dutch windmill of order n. For $n = 4$, a graceful labelling of the Type (g) 4-vane Dutch windmill appears in [12].

Lemma 3.11 *Any Type* (g) *Dutch windmill with at least four blocks is graceful or near graceful.*

Proof. For Type (g), consider $S_5 = (3,4,5,3,2,4,2,5,1,1)$. This sequence has three pivots, which are $2,3$, and 4. This gives us a graceful labelling of a Type (g) Dutch windmill with five blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with S_5 to obtain a (near) graceful labelling of a Type (g) Dutch windmill with n blocks for $n \geq 16$. Table 7 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (g) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
6	6451146523203	$1,2$ and 4
7	746354376511202	$1,4$ and 5
8	4857411568723263	$1,2$ and 5
9	759242574869311368	$1,4$ and 7
10	$2529115784A694738630A$	$1,4$ and 5
11	$B68527265A8B7941134A309$	$2,5$ and B
12	$A8531135C8A6B9742624C79B$	$5,6$ and 8
13	$B97D4262479B6CA8D5311358AC$	$1,4$ and 6
14	$CA75311357EACDB864292468EBD09$	$1,3$ and C
15	$FDB9753EC3579BDF6A84CE64118A202$	$5,6$ and F

Table 7: Skolem and hooked Skolem sequences with three pivots for Type (g).

3.8 Type (h)

Type (h) Dutch windmills of order n are formed by adding one pendant to some vane of an $(n-3)$-vane Dutch windmill, then adding two pendants to a common vertex of that pendant that is not shared with the $(n-3)$-vane Dutch windmill. For $n \leq 3$, there are not enough triangles to form a Type (h) Dutch windmill of order n. For $n = 4,5$, a graceful labelling of the Type (h) 4-vane and 5-vane Dutch windmill appears in [12].

Lemma 3.12 *Any Type* (h) *Dutch windmill with at least four blocks is graceful or near graceful.*

Proof. For Type (h), consider $hS_6 = (4, 5, 3, 6, 4, 3, 5, 1, 1, 6, 2, 0, 2)$. This sequence has four pivots, and we select $1, 3$, and 4. This gives us a near graceful labelling of a Type (h) Dutch windmill with six blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, again by using Lemma 3.4 with hS_6 to obtain a (near) graceful labelling of a Type (h) Dutch windmill with n blocks for $n \geq 19$. Table 8 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $7 \leq n \leq 18$, each of which gives a (near) graceful labelling of a Type (h) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
7	746354376511202	$3, 4$ and 5
8	1157468543763282	$3, 4$ and 5
9	736931176845924258	$1, 3$ and 7
10	$5262854A674981137A309$	$1, 4$ and 6
11	$635A37659B117A842924B08$	$5, 6$ and 7
12	$52426549B86CA711983B73AC$	$4, 5$ and 6
13	$86272C568B75D9A11C34B394AD$	$1, 5$ and 8
14	$2726AD1176C4EBA458D935C3B8E09$	$4, 6$ and 7
15	$8C3473D48117ACEFB96D52A2659BE0F$	$2, 5$ and C
16	$637A3E6FC7D52A2G58BEC9FD48114B9G$	$2, 5$ and 7
17	$962D2G5649754EFHD78CAGB1138E3FACHB$	$4, 5$ and D
18	$5D9485E4F1198BDG6HIAEC6FB7232A3G7CH0I$	$8, 9$ and E

Table 8: Skolem and hooked Skolem sequences with three pivots for Type (h).

3.9 Type (i)

Type (i) Dutch windmills of order n are formed by adding two pendants to some common non-central vertex of a vane of an $(n-3)$-vane Dutch windmill, then adding one pendant to one of those pendants at a vertex not shared with the $(n-3)$-vane Dutch windmill. For $n \leq 3$, there are not enough triangles to form a Type (i) Dutch windmill of order n. For $n = 4, 5$ a graceful labelling of the Type (i) 4-vane and 5-vane Dutch windmill appears in [12]. When $n = 6$, see the near graceful labelling in Figure 6.

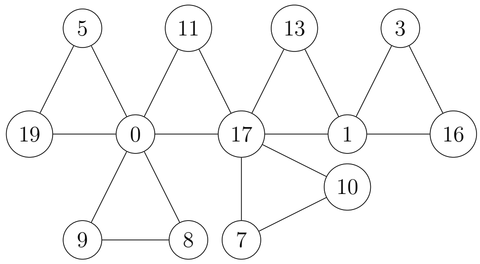

Figure 6: A near gracefully labelled Type (i) Dutch windmill with six blocks.

Lemma 3.13 *Any Type* (i) *Dutch windmill with at least four blocks is graceful or near graceful.*

Proof. For Type (i), consider $hS_7 = (7,4,6,3,5,4,3,7,6,5,1,1,2,0,2)$. This sequence has six pivots, of which we select $4,5$, and 6. This gives us a near graceful labelling of a Type (i) Dutch windmill with seven blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with hS_7 to obtain a (near) graceful labelling of a Type (i) Dutch windmill with n blocks for $n \geq 22$. Table 9 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $8 \leq n \leq 21$, each of which gives a (near) graceful labelling of a Type (i) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
8	7536835726248114	$1,6$ and 7
9	975386357946824211	$2,6$ and 7
10	$A853113598A7426249706$	$2,3$ and 8
11	$B68527265A8B7941134A309$	$4,8$ and B
12	$A8531135C8A6B9742624C79B$	$4,6$ and 8
13	$CA8531135D8AC6B9742624D79B$	$4,6$ and A
14	$DB964E1146C9BDA8537E35C8A7202$	$3,9$ and B
15	$ECA8642D2468ACEFB953D735119B70F$	$3,A$ and C
16	$9FDBG864292468BDFECAG75311357ACE$	$2,6$ and 9
17	$FDB9H64282469BDF8GECAH75311357ACEG$	$3,D$ and F
18	$IGECA8642H2468ACEGIFD9753BH357911DF0B$	$6,7$ and I
19	$IGEC9753113579JCEGIHFDA8642B2468AJDFH0B$	$2,5$ and E
20	$JHFDB9753CK3579BDFHJICEGA86411K468A2E2IG$	$6,F$ and J
21	$KIGEC9753113579LCEGIKAJHFDB8642A2468LBDFHJ$	$6,G$ and K

Table 9: Skolem and hooked Skolem sequences with three pivots for Type (i)

3.10 Type (j)

Type (j) Dutch windmills of order n are formed by adding one pendant to some vane of an $(n-3)$-vane Dutch windmill, then adding one pendant to that pendant (at a vertex not shared with the $(n-3)$-vane Dutch windmill) and then adding one pendant to that second pendant, at a vertex not shared with the first pendant. For $n \leq 3$, there are not enough triangles to form a Type (j) Dutch windmill of order n. For $n = 4$ a graceful labelling of the Type (j) 4-vane Dutch windmill appears in [12].

Lemma 3.14 *Any Type* (j) *Dutch windmill with at least four blocks is graceful or near graceful.*

Proof. For Type (j), consider $S_5 = (2,3,2,5,3,4,1,1,5,4)$. This sequence has three pivots, which are $1,2$, and 3. This gives us a graceful labelling of a Type (j) Dutch windmill with five blocks. The rest of the proof is analogous to the steps taken to prove Lemma 3.5, by using Lemma 3.4 with S_5 to obtain a (near) graceful labelling of a Type (j) Dutch windmill with n blocks for $n \geq 16$. Table 10 provides Skolem and hooked Skolem sequences with three distinguished pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (j) Dutch windmill with n blocks. □

n	Skolem or hooked Skolem sequence	Pivots
6	2326351146504	$1, 2$ and 3
7	232437546115706	$2, 3$ and 4
8	3753811576428246	$1, 3$ and 5
9	759242574869311368	$1, 2$ and 7
10	$A853113598A7426249706$	$1, 3$ and 8
11	$35232B549A841167B98A607$	$2, 4$ and 5
12	$A8531135C8A6B9742624C79B$	$3, 6$ and 8
13	$A8D536C358A6B97D42C2479B11$	$3, 4$ and A
14	$7A8C53E7358ADB9C6411E469BD202$	$3, 4$ and A
15	$DB964F1146E9BD7CA853F735E8AC202$	$4, 5$ and B

Table 10: Skolem and hooked Skolem sequences with three pivots for Type (j).

3.11 Type (k)

Type (k) Dutch windmills of order n are formed by adding one pendant to some vane of an $(n-3)$-vane Dutch windmill, then adding two pendants to this pendant, at distinct vertices not shared with the $(n-3)$-vane Dutch windmill.

Suppose we label all the triangles of a Type (k) Dutch windmill using the same method as in the previous types: with triples of the form $(0, a_i + n, b_i + n)$, excluding the three pendant triangles, where we use pivots to alter the labelling. Consider the portion of a Type (k) Dutch windmill shown in Figure 7, where triangles 2, 3, and 4 are the pendant triangles. Let triangle 1 be labelled by $(0, a_j + n, b_j + n)$; we proceed under the assumption that the remaining triangles can be labelled by the pivoting technique. By pivoting, we label triangle 2 with $(k, a_k + n + k, b_k + n + k)$, where $b_j + n = a_k + n + k$ or $a_j + n = a_k + n + k$. We will consider $b_j + n = a_k + n + k$ at vertex c in Figure 7. Again, by pivoting we label triangle 3 with $(l, a_l + n + l, b_l + n + l)$. We will consider $b_k + n + k = a_l + n + l$, at vertex a, and hence vertex b is labelled k. Note that $1 \leq j, k, l \leq n$, and all are distinct. If we pivot triangle 4, we label it by the triple $(s, a_s + n + s, b_s + n + s)$. However $k \neq s$ and $\min(a_s + n + s, b_s + n + s) > n \geq k$. Since this is impossible, we must abandon the pivoting method.

Instead, while we label most of the vanes of the windmill using our standard technique, we will introduce a special construction for the pendant triangle, where we will use some triples of the form $(0, i, b_i + n)$ in lieu of the triples $(0, a_i + n, b_i + n)$, and one triple formed by the method of Lemma 3.3.

For this type, we will introduce the sequences and the corresponding triples to indicate the forms of the triples we use.

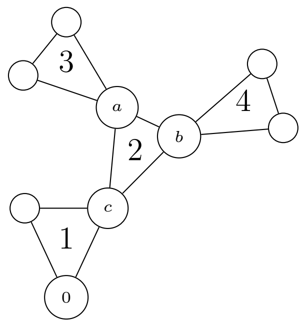

Figure 7: Illustration for Type (k) Dutch windmill labelling.

For $n \leq 3$, there are not enough triangles to form a Type (k) Dutch windmill of order n. For $n = 4$, a graceful labelling of the Type (k) 4-vane Dutch windmill appears in [12].

Lemma 3.15 *Any Type* (k) *Dutch windmill with at least five blocks is graceful or near graceful.*

Proof. Consider $S_5 = (2, 3, 2, 5, 3, 4, 1, 1, 5, 4)$. When $n = 5$, this sequence gives triples of the form $(0, a_i + n, b_i + n)$ as follows: $(0, 12, 13)$,$(0, 6, 8)$,$(0, 7, 10)$, $(0, 11, 15)$,$(0, 9, 14)$. We will use mixed forms of the triples to gracefully label a Type (k) Dutch windmill with five blocks. These triples are: $(1, 13, 14)$, $(1, 7, 9)$, $(3, 10, 13)$, $(0, 11, 15)$, $(0, 5, 14)$. The first and the third triples are formed by pivoting the elements, the second triple is formed by adding 1 to each element, the fourth triple is formed in our standard method, and the fifth triple is a block of the form $(0, i, b_i + n)$. Certainly, this gives us a graceful labelling of a Type (k) Dutch windmill with five blocks.

As in the previous lemmas, we use S_5 and a (hooked) Langford sequence of order $m \geq 11$ with $d = 6$ (which exists by Theorem 2.4). By Lemma 2.5, we obtain an associated Skolem sequence of order n, and then form triples of the form $(0, a_i + n, b_i + n)$. Note that for any triple which comes from the Langford sequence portion of the associated Skolem sequence, non-zero entries are at least $n + 11$. Further, these triples use the edges of lengths 6 to n, as well as those of length $n+11$ to $3n$. Finally, we consider the triples that correspond to S_5.

These triples are $(0, 7+n, 8+n)$, $(0, 1+n, 3+n)$, $(0, 2+n, 5+n)$, $(0, 6+n, 10+n)$, and $(0, 4+n, 9+n)$. We replace them in the same fashion as our earlier construction with S_5, namely with the triples $(1, 8 + n, 9 + n)$, $(1, 2 + n, 4 + n)$, $(3, 5 + n, 8 + n)$, $(0, 6 + n, 10 + n)$, and $(0, 5, 9 + n)$. We do not use any vertex label greater than $n + 10$, so there is no conflict with any of the Langford triples. Further, each of these triples uses exactly the same edge differences as the corresponding initial triple.

Thus, we use the edge lengths 1 to 5, and $n+1$ to $n+10$. Then these triples, along with the unchanged Langford triples, give us a graceful (or near-graceful) labelling of the Type (k) Dutch windmill of order $n \geq 16$.

Skolem and hooked Skolem sequences with corresponding triples of order $6 \leq n \leq 15$ are given in Table 11, where the triples in bold are formed in an unusual way (i.e pivoting, Lemma 3.3, or something totally ad hoc) and the rest of the triples are formed by the typical $(0, a_i + n, b_i + n)$ construction. By using a similar structure to S_5 with the given triples, we obtain (near) graceful labellings for Type (k) Dutch windmills of order $6 \leq n \leq 15$. Therefore, any Type (k) Dutch windmill of order $n \geq 5$ is graceful or near graceful, as required. □

n	Skolem or hooked Skolem sequence	Triples
6	2326351146504	**(1,14,15)**, **(1,8,10)**, **(3,11,14)**, (0, 15, 19), (0, 12, 17), **(0,6,16)**.
7	746354376511202	**(0,1,19)**, **(0,2,22)**, **(4,15,18)**, **(7,11,20)**, (0, 12, 17), (0, 10, 16), (7, 15, 22).
8	3723258476541186	(0, 21, 22), **(2,13,15)**, **(7,16,19)**, **(0,4,20)**, **(2,16,21)**, (0, 18, 24), (0, 10, 17), **(0,8,23)**.
9	736931176845924258	**(7,8,23)**, **(0,2,25)**, **(3,14,17)**, (0, 20, 24), (0, 21, 26), (0, 12, 18), **(7,17,24)**, (0, 19, 27), (0, 13, 22).
10	$5262854A674981137A309$	**(1,25,26)**, (0, 12, 14), (0, 26, 29), **(1,5,22)**, (0, 11, 16), **(6,19,25)**, (0, 20, 27), (0, 15, 23), **(0,9,31)**, (0, 18, 28).
11	$635A37659B117A842924B08$	(0, 22, 23), (0, 28, 30), (0, 13, 16), (0, 27, 31), **(7,12,26)**, **(6,18,24)**, **(7,24,31)**, **(0,8,34)**, (0, 20, 29), (0, 15, 25), (0, 21, 32).
12	$A85311135C8A6B9742624C79B$	(0, 17, 18), (0, 29, 31), (0, 16, 19), (0, 28, 32), **(6,11,26)**, **(6,30,36)**, (0, 27, 34), **(8,22,30)**, **(0,9,35)**, (0, 13, 23), (0, 25, 36), (0, 21, 33).
13	$3113692D286C7B9A5847D54CBA$	**(5,6,21)**, **(2,22,24)**, (0, 14, 17), (0, 32, 36), (0, 30, 35), **(6,24,30)**, (0, 26, 33), (0, 23, 31), (0, 19, 28), (0, 29, 39), (0, 27, 38), (0, 25, 37), **(0,13,34)**.
14	$B36A349C647BDAE9578C25211D80E$	(0, 38, 39), **(2,37,39)**, **(2,5,21)**, (0, 20, 24), (0, 31, 36), (0, 17, 23), (0, 25, 32), (0, 33, 41), **(0,9,30)**, (0, 18, 28), **(11,26,37)**, (0, 22, 34), (0, 27, 40), (0, 29, 43).
15	$11FB479242DE76B9CF86A35D3E85C0A$	**(2,3,19)**, (0, 23, 25), **(3,40,43)**, (0, 20, 24), (0, 38, 43), (0, 29, 35), (0, 21, 28), (0, 34, 42), **(9,31,40)**, (0, 36, 46), **(0,11,30)**, (0, 32, 44), (0, 26, 39), (0, 27, 41), (0, 18, 33).

Table 11: Skolem and hooked Skolem sequences with the triples for Type (k).

4 Some Remarks

In this paper, we proved, in Theorem 1.1, Rosa's conjecture for a new family of triangular cacti: Dutch windmills of any order with three pendant triangles. This result, combined with those of Dyer et al. [5], gives the following theorem.

Theorem 4.1 *Every Dutch windmill with at most three pendant triangles is graceful or near graceful.*

As we discussed earlier, Langford sequences have been classically used to build new Skolem sequences. In this paper, we use this technique to gracefully label Dutch windmills with three pendant triangles. However, in our construction, the Langford sequence implicitly contains a pivot, which, when pivoted, can gracefully label Dutch windmills with four pendant triangles. In Section 2, we recalled the definition of a pivot of a (hooked) Skolem sequence. Here we will define the pivot of a Langford sequence. We call j a *pivot* of a Langford sequence $\{(a_j, b_j)\}_{j=d}^{d+m-1}$, if $b_j + j \leq 2m$ or a *pivot* of a hooked Langford sequence $\{(a_j, b_j)\}_{j=1}^{d+m-1}$, if $b_j + j \leq 2m + 1$. Thus we can, in many cases, label the Dutch windmills with four pendant triangles for Dutch windmills of a large order. In this paper, we show that every Dutch windmill with at most three pendant triangles is graceful or near graceful. Consequently, we can find a (hooked) Skolem sequence of order n with three pivots. From (hooked) Langford sequences, we can construct a new (hooked) Skolem sequence as explained in Lemma 2.5. Furthermore, after examining all the patterns from [14], we confirm that each (hooked) Langford sequence has at least one pivot. Based on this, we can conclude the following result:

Theorem 4.2 *There exists M such that every Dutch windmill of order $m > M$ with exactly four pendant triangles, where one pendant is attached to a vane containing no other pendants, is graceful or near graceful.*

For example, let $S_8 = (4, 8, 5, 7, 4, 1, 1, 5, 6, 8, 7, 2, 3, 2, 6, 3)$ be a Skolem sequence and L_9^{17} = (24, 17, 21, 22, 18, 14, 11, 19, 25, 23, 10, 20, 9, 16, 13, 15, 12, 11, 17, 14, 10, 9, 18, 21, 24, 22, 19, 13, 12, 16, 15, 20, 23, 25) be a Langford sequence. Now, if we take the Skolem sequence S_8 and the Langford sequence L_9^{17}, then we obtain a new Skolem sequence S_{25} = (4, 8, 5, 7, 4, 1, 1, 5, 6, 8, 7, 2, 3, 2, 6, 3, 24, 17, 21, 22, 18, 14, 11, 19, 25, 23, 10, 20, 9, 16, 13, 15, 12, 11, 17, 14, 10, 9, 18, 21, 24, 22, 19, 13, 12, 16, 15, 20, 23, 25) of order 25. If we take that triples obtained from S_{25} and pivot the triples corresponding to 1, 2, 4 and 11 we can gracefully label the Dutch windmill of order 25 with four pendant triangles.

In 1989, Moulton [8] proved Rosa's conjecture for triangular snakes, a type of triangular cactus whose block cutpoint graph is a path. So, it is natural to pose the following question. Can we use Skolem type sequences to gracefully label triangular snakes? Further, can we use them to gracefully label triangular snakes with pendant triangles?

Finally, we note that while a great deal of work has been done on Dutch windmills consisting of a single type of cycle (such as triangles [2], 4-cycles [13], 5-cycles [16], etc.), nothing seems to be known about Dutch windmills with non-uniform cycle length. Thus, we pose the following question as a place to begin: for what m and n can we gracefully label a Dutch windmill consisting of m triangles and n 4-cycles?

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