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On colorings and orientations of signed graphs II

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Abstract

A classical theorem independently due to Gallai, Hasse, Roy, and Vitaver states that a graph G has a proper n-coloring if and only if G has an orientation without coherent paths of length n. We prove that a signed graph has a proper n-coloring if and only if it has an orientation without coherent paths or balloons of length n.

1 Introduction

Theorem 1 is a classical result in graph theory which simply and elegantly characterizes the existence of a proper n-coloring in terms of orientations. It was independently discovered in the 1960's by Gallai [1], Hasse [2], Roy [4], and Vitaver [6].

Theorem 1. *For a loopless graph G the following are equivalent.*

(1) G has a proper n-coloring.
(2) G has an acyclic orientation without coherent paths of length n.
(3) G has an orientation without coherent paths of length n.

Integer-valued vertex coloring of signed graphs was first defined by Zaslavsky [8]. It is an attractive generalization of vertex coloring of ordinary graphs in that it also generalizes additional aspects of and results on graph coloring; for instance, the connection between graph coloring and matroid theory though chromatic and Tutte polynomials [7]. In this note, we present Theorem 2 which generalizes Theorem 1 in the following sense: if we consider a graph to be a signed graph in which all edges are positive, then the set of graphs which satisfy Part(x) of Theorem 1 is properly contained in the set of signed graphs which satisfy Part(x) of Theorem 2. Thus we have another classical aspect of graph coloring which is generalized by signed-graph coloring.

The terms used in Theorem 2 are well known among those familiar with the theory of signed graphs. For others, a short introduction is provided in Section 2 where all relevant terms are defined. The proof is presented in Section 3.

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Theorem 2. *For a signed graph (G, σ) without positive loops the following are equivalent.*

(1) (G, σ) has a proper n-coloring.

(2) (G, σ) has an acyclic orientation without coherent paths or balloons of length n.

(3) (G, σ) has an orientation without coherent paths or balloons of length n.

In a previous publication we presented [5, Theorem 5.2] as an analogue of Theorem 1 for signed graphs. The result here is much nicer: it is more simply stated and similar to Theorem 1, it makes no reference to sign switching in its statement, and the implications $(2 \to 1)$ and $(3 \to 1)$ are stronger results than the corresponding implication in [5, Theorem 5.2].

2 Background

A *signed graph* is a pair (G, σ) in which G is a graph and $\sigma\colon E(G) \to \{+, -\}$. Let $M_{2k+1} = \{-k, \ldots, -1, 0, 1, \ldots, k\}$ and $M_{2k} = \{-k, \ldots, -1, 1, \ldots, k\}$. An *$n$-coloring* of a signed graph (G, σ) is a function $\kappa\colon V(G) \to M_n$. An n-coloring κ is *proper* when for each edge e in (G, σ) with endpoints u and v (possibly equal), $\kappa(u) \neq \sigma(e)\kappa(v)$. Evidently every signed graph (G, σ) without positive loops has a proper $2|V(G)|$-coloring and if (G, σ) has a proper n-coloring, then (G, σ) has a proper $(n+1)$-coloring. Thus it makes sense to define the *chromatic number* $\chi(G, \sigma)$ as the smallest n such that (G, σ) has a proper n-coloring. This formulation of the chromatic number of a signed graph is due to Máčajová, Raspaud, and Škoviera [3]. It is a more streamlined adaptation of coloring and chromatic numbers defined earlier by Zaslavsky [8].

In a graph G, an *incidence* is where an end of an edge meets a vertex. As such each edge (including a loop) has two distinct incidences. An incidence can be denoted by a pair (v, e) in which vertex v is an endpoint of edge e. Although this notation does not distinguish between the two distinct incidences of a loop, the notation can be modified as $(v, e)_1$ and $(v, e)_2$ in order to do so. Let $I(G)$ denote the set of incidences of G. A *bidirection* on G is a function $\beta\colon I(G) \to \{+, -\}$. Graphically, $\beta(v, e) = +$ is envisioned as an arrow at (v, e) pointing towards v. Similarly, $\beta(v, e) = -$ is envisioned as an arrow at (v, e) pointing away from v. Thus bidirections produce three types of edges: *extroverted*, *introverted*, and *directed*.

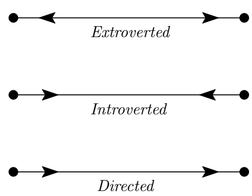

Figure 1: The three types of edges in a bidirected graph.

An *orientation* of a signed graph (G,σ) is a bidirection β satisfying $\beta(v,e)\beta(u,e) = -\sigma(e)$. As such, each negative edge is either introverted or extroverted and each positive edge has one of two possible directions. An *oriented* signed graph is a triple (G,σ,β) where β is an orientation of (G,σ). A vertex v in (G,σ,β) is a *sink* (or *source*) when all of the bidirectional arrows at v are directed towards (or away from) v. A vertex in v in (G,σ,β) is *singular* when it is either a source or a sink.

Let (G,σ,β) be an oriented signed graph. A path P in (G,σ,β) is *coherent* when every internal vertex of (P,σ,β) is non-singular. A cycle C in a signed graph (G,σ) is called *positive* (or *negative*) when the product of signs on its edges is positive (or negative). A *balloon* $B = C \cup P$ in (G,σ) consists of a negative cycle C and a path P (possibly of length zero) which intersects C at a single vertex only. The length of a balloon is the length of C plus twice the length of P. A balloon $B = C \cup P$ in (G,σ,β) is *coherent* when there is a unique singular vertex in (B,σ,β). When P has positive length, then this singular vertex must be the degree-1 vertex in B. When P has length zero, the singular vertex is any vertex on C. Figure 2 depicts a coherent path and three coherent balloons, each of which has length 7. When depicting signed graphs, positive edges are solid curves and negative edges are dashed curves. A circle around a vertex indicates that the vertex is singular.

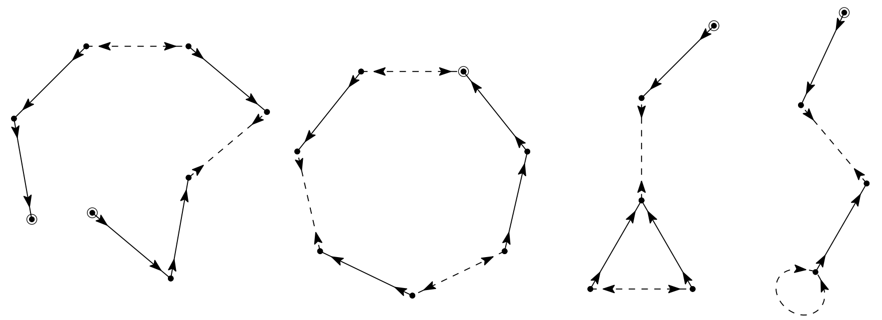

Figure 2: A coherent path and three coherent balloons, each of which has length 7. Positive edges are solid curves and negative edges are dashed curves. A circle around a vertex indicates that the vertex is singular.

A *circuit* in a signed graph (G,σ) is a subgraph which is either a positive cycle, two negative cycles which intersect in a single vertex (called a *tight handcuff*), or two vertex-disjoint negative cycles along with a minimal connecting path (called a *loose handcuff*). If C is a circuit in (G,σ), then C is *coherent* in the oriented signed graph (G,σ,β) when every vertex of C is non-singular in (C,σ,β). The reader can check that there are exactly two possibilities for a coherent orientation β of circuit C and, furthermore, if β is one of them, then $-\beta$ is the other. An oriented signed graph (G,σ,β) is *acyclic* when it contains no coherent circuit. Zaslavsky [9, Corollary 5.3] proved that if (G,σ,β) is acyclic, then (G,σ,β) has a singular vertex.

Given a signed graph (G,σ), a *switching function* is a function $\eta\colon V(G) \to \{+,-\}$. Define σ^η by $\sigma^\eta(e) = \eta(u)\sigma(e)\eta(v)$ in which u and v are the endpoints of e. (This includes the case for a loop.) The sets of circuits of (G,σ) and (G,σ^η)

are the same. If β is an orientation of (G,σ), then $\eta\beta$ is an orientation of (G,σ^η). One can think of $\eta\beta$ as being obtained from β by reversing the arrows at v when $\eta(v) = -$ and leaving the arrows at v the same when $\eta(v) = +$. Since a vertex is singular in (G,σ,β) if and only if it is singular in $(G,\sigma^\eta,\eta\beta)$, coherence of circuits, paths, and balloons is invariant under switching.

If κ is a proper n-coloring of (G,σ), then there is a natural orientation of (G,σ) *induced* by κ, call it β_κ, which is defined as follows. Given an edge e with ends (u,e) and (v,e), we have that $\kappa(u) \neq \sigma(e)\kappa(v)$; that is, $\kappa(u) - \sigma(e)\kappa(v) \neq 0$. Now because an orientation β must satisfy $\beta(u,e)\beta(v,e) = -\sigma(e)$, there is only one choice for $\beta_\kappa(u,e)$ and $\beta_\kappa(v,e)$ so that

$$\beta_\kappa(u,e)\kappa(u) + \beta_\kappa(v,e)\kappa(v) > 0.$$

An equivalent formulation of β_κ is as follows: If e is positive, then without loss of generality $\kappa(u) > \kappa(v)$. Thus e under β_κ is directed with head u and tail v. If e is negative, then $\kappa(u)+\kappa(v) \neq 0$. Thus e is extroverted under β_κ when $\kappa(u)+\kappa(v) > 0$ and e is introverted under β_κ when $\kappa(u)+\kappa(v) < 0$.

Note that β_κ is acyclic when κ is proper because on any subgraph of (G,σ,β_κ) a vertex v is singular when $|\kappa(v)|$ is maximum for that subgraph.

If η is a switching function for (G,σ) and κ a proper n-coloring, then $\eta\kappa$ is a proper n-coloring of (G,σ^η). In fact, $\kappa \mapsto \eta\kappa$ is a bijection between the collection of all proper n-colorings of (G,σ) and those of (G,σ^η). If κ is a proper n-coloring of (G,σ), then $\beta_{\eta\kappa} = \eta\beta_\kappa$.

A useful notion of normalizing colorings and acyclic orientations was explored in [5]. If κ is a proper n-coloring of (G,σ), then let η be the switching function defined by $\eta(v) = -$ when $\kappa(v) < 0$. Now $\eta\kappa$ is a non-negative proper n-coloring of (G,σ^η). The coloring $\eta\kappa$ is called the *normalization* of κ. Note now that every negative edge of $(G,\sigma^\eta,\eta\beta_\kappa)$ is extroverted.

If β is an acyclic orientation of (G,σ), then there is a partition $L_1, L_2, \ldots$ of $V(G)$ called the *canonical level decomposition* of (G,σ,β) which is defined iteratively as follows: L_1 is the set of singular and isolated vertices of (G,σ) and L_{i+1} is the set of singular and isolated vertices of $(G,\sigma)-(L_1\cup\cdots\cup L_i)$. Let η be the switching function for which $\eta(v) = -$ if v was accounted for as a source (rather than a sink or isolated vertex) during the construction of the canonical level decomposition. The acyclic orientation $\eta\beta$ of (G,σ^η) is called the *normalization* of β. The proof of Proposition 3 is easy.

Proposition 3 ([5, Proposition 4.1]). *Let β be a normalized acyclic orientation of (G,σ) and let $L_1,\ldots,L_m$ be the canonical level decomposition.*

(1) If e is a negative edge, then e is extroverted.

(2) If e is a positive edge with head in L_i and tail end in L_j, then $i < j$.

(3) If $v \in L_j$ for $j > 0$, then there is a positive edge e with v as its tail with $w \in L_{j-1}$ as its head.

3 The Proof of Theorem 2

($1 \to 2 \wedge 3$) Let $n \in \{2k, 2k+1\}$ and suppose that (G, σ) has a proper n-coloring κ. We may assume by switching that κ is normalized as coherence of paths and balloons is invariant under switching. Let $C_0, \ldots, C_k$ be the color classes of $V(G)$. Let P be a coherent path in $(G, \sigma, \beta_\kappa)$, then because each negative edge is extroverted, there can be at most one negative edge on P. Furthermore, if e is a positive edge with head v and tail u, then $\kappa(v) \geq \kappa(u) + 1$. If P has no negative edges, then the maximum possible length of P is $k-1$ when $n = 2k$ and k when $n = 2k+1$. If P has one negative edge, then the maximum possible length of P occurs when the colors of the endpoints of the negative edge are a minimum. If $n = 2k$, then this is when the endpoints of the negative edge both have color 1. In this case the maximum possible length is $1 + 2(k-1) = 2k-1 < n$, as required. If $n = 2k+1$, then this is when the endpoints of the negative edge have colors 0 and 1. In this case, the maximum possible length is $1 + k + (k-1) = 2k < n$, as required. If B is a coherent balloon in $(G, \sigma, \beta_\kappa)$, then because all negative edges are extroverted, there is exactly one negative edge in B and it is on the negative cycle of B. We can associate a coherent path with a coherent balloon of the same length as suggested in Figure 3. The same argument for paths now implies that the maximum possible length of B is less than n.

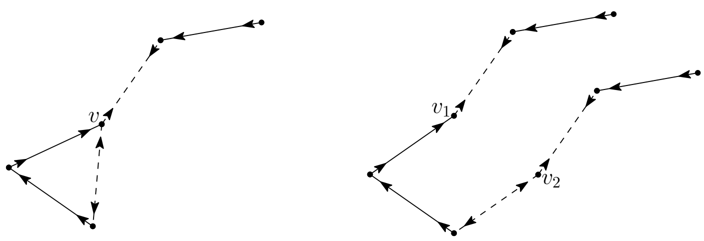

Figure 3: Break the cycle at v and append two copies of the path to the ends.

($2 \to 1$) Let (G, σ, β) be an acyclic oriented signed graph without coherent paths or balloons of length n. Since n-colorability is invariant under switching, we may assume that β is normalized. Let $L_1, \ldots, L_m$ be the canonical level decomposition of $V(G)$ given by β. Because there are no coherent paths of length n, Proposition 3(3) implies that $m \leq n$. If necessary, append copies of the empty set to the sequence of L_i's so that $m = n$. Consider two cases based on the parity of n.

Case 1 Assume that $n = 2k+1$. Relabel the sets $L_1, \ldots, L_{2k+1}$ respectively as $C_k, \ldots, C_0, \ldots, C_{-k}$. If e is a negative edge of (G, σ, β) whose endpoints are in C_i and C_j, then by Proposition 3, e is contained in a coherent path or balloon of length $1 + (k-i) + (k-j) < 2k+1$. Hence $i + j > 0$. Thus all edges in the induced subgraph $G[C_0 \cup \cdots \cup C_{-k}]$ are positive and there are no negative edges with endpoints in both C_i and C_{-i} for any $i \in \{0, \ldots, k\}$. Thus if we color the vertices in C_i with color i, then we have a proper $(2k+1)$-coloring of (G, σ), as required.

Case 2 Assume that $n = 2k$. Relabel the sets $L_1, \dots, L_{2k}$ respectively as $C_k, \dots, C_1, C_{-1}, \dots, C_{-k}$. If e is a negative edge with endpoints in C_i and C_j, then Proposition 3 implies that at least one of i and j is positive; furthermore, if i is positive and j is negative, then there is a coherent path or balloon of length $1 + (k - i) + (|j| - 1 + k) < 2k$. This again implies that $i + j > 0$, all edges in the induced subgraph $G[C_{-1} \cup \cdots \cup C_{-k}]$ are positive, and there are no negative edges with endpoints in both C_i and C_{-i} for any $i \in \{1, \dots, k\}$. Thus the coloring of (G, σ) in which vertices of C_i receive color i is a proper $2k$-coloring, as required.

($3 \to 1$) Let (G, σ, β) be an oriented signed graph without coherent paths or balloons of length n. We may assume that G is connected. Let (H, σ, β) be a maximal acyclic subgraph of (G, σ, β). Necessarily H is connected and spans G. Assume that a switching function is applied to (G, σ, β) so that (H, σ, β) is normalized.

Consider the proper n-coloring κ on (H, σ, β) constructed in the proof for ($2 \to 1$). We finish by showing that κ extends to all of (G, σ, β). So if $e \in E(G) - E(H)$, then $(H \cup e, \sigma, \beta)$ has a coherent circuit, call it C, using e. In Case 1 assume that e is positive and in Case 2 that e is negative.

Case 1 A coherent handcuff cannot have only extroverted negative edges, thus C is a positive cycle. Because all negative edges in (H, σ, β) are extroverted, $C - e$ is a coherent path consisting of positive edges only. Thus the endpoints of P must have different colors under κ and so edge e is properly colored by κ as well.

Case 2 A coherent circuit which contains negative edges cannot contain only extroverted edges. Thus e must be introverted. Denote the endpoints of e by x and y. (This includes the case that $x = y$.) If C is a positive cycle, then $C - e$ is a coherent path containing a single negative edge which is extroverted. Thus $C - e$ is as shown on the left in Figure 4 where either or both of the $y'y$- or $x'x$-paths may have length zero. Because $\kappa(y') + \kappa(x') > 0$, this implies that $\kappa(y) + \kappa(x) > 0$ and so e is properly colored by κ.

If C is a handcuff, then e is either on a negative cycle of C or on its connecting path. In the latter case, $C - e$ is as shown in the middle of Figure 4 where either or both of the v_1x- or v_2y-paths may have length zero. Since the sum of the colors of the endpoints of a negative edge in (H, σ, β) must be positive, $\kappa(v_1)$ and $\kappa(v_2)$ are both positive which implies the same for $\kappa(x)$ and $\kappa(y)$. Thus e is properly colored by κ. If e is on a negative cycle of C, then $C - e$ is as shown on the right of Figure 4 in which $1 \leq |\{v, v'x, y\}| \leq 4$. Again $\kappa(v) > 0$ which implies that $\kappa(x)$ and $\kappa(y)$ are positive and so e is properly colored by κ.

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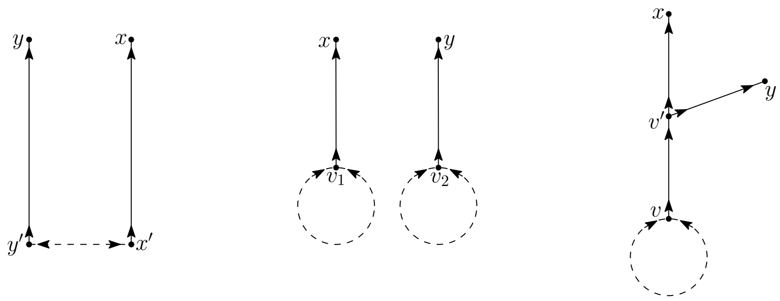

Figure 4: Possibilities for $C - e$ when e is introverted. Positive edges shown in this figure represent coherent paths of positive edges of any length, including zero. Negative loops shown in this figure represent negative cycles having a unique singular vertex and which consist of a single extroverted negative edge along with any number (including zero) of positive edges.

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