AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **91(1)** (2025), Pages 20–31

Switching m-edge-coloured graphs using non-abelian groups

Christopher Duffy

School of Mathematics and Statistics
University of Melbourne, Melbourne, Australia
`christopher.duffy@unimelb.edu.au`

Gary MacGillivray

Mathematics and Statistics
University of Victoria, Victoria, Canada
`gmacgill@math.uvic.ca`

Ben Tremblay

Combinatorics and Optimization
University of Waterloo, Waterloo, Canada
`ben.tremblay@uwaterloo.ca`

Abstract

Let G be a graph whose edges are each assigned one of the m-colours $1, 2, \ldots, m$, and let Γ be a subgroup of S_m. The operation of switching at a vertex x with respect to $\pi \in \Gamma$ permutes the colours of the edges incident with x according to π. There is a well-developed theory of switching when Γ is abelian. Much less is known for non-abelian groups. In this paper we consider switching with respect to non-abelian groups including symmetric, alternating and dihedral groups. We first consider the question of whether there is a sequence of switches using elements of Γ that transforms an m-edge-coloured graph $\hat{G}$ to an m-edge coloured graph $\hat{H}$. Necessary and sufficient conditions for the existence of such a sequence are given for each of the groups being considered. We then consider the question of whether an m-edge coloured graph can be switched using elements of Γ so that the transformed m-edge coloured graph has a vertex k-colouring, or a homomorphism to a fixed m-edge coloured graph $\hat{H}$. For the groups mentioned we establish dichotomy theorems for the complexity of these decision problems. These are the first dichotomy theorems to be established for colouring or homomorphism problems and switching with respect to any group other than S_2.

ISSN: 2202-3518

1 Introduction and Definitions

An *m-edge-coloured graph* is an ordered pair $\hat{G} = (G, \sigma)$, where G is a graph and $\sigma : E(H) \to \{1, 2, \ldots, m\}$ is its *signature*. The graph G is the *underlying graph* of $\hat{G}$. The vertices of $\hat{G}$ are the vertices of G. The edges of $\hat{G}$ are coloured edges of G, that is, edges $e \in E(G)$ together with their signature (or *colour*) $\sigma(e)$. We use $E_i(\hat{G})$ to denote the set of edges of $\hat{G}$ with colour i. An m-edge-coloured graph is *monochromatic of colour j* if all of its edges have colour j.

An m-edge coloured graph is called *simple* if its underlying graph is simple. Throughout this work we assume that all m-edge coloured graphs under consideration are simple, even though some of the results hold in a more general context.

Let $\hat{G}$ be an m-edge-coloured graph and let Γ be a subgroup of S_m. For $x \in V$ and $\pi \in \Gamma$, the operation of *switching at x with respect to* π transforms $\hat{G}$ into the m-edge-coloured graph $\hat{G}^{(x,\pi)}$ that has the same underlying graph as $\hat{G}$ and with the colours of the edges incident with x permuted according to π, that is, if $\sigma(\hat{G})(xy) = i$, then $\sigma(\hat{G}^{(x,\pi)})(xy) = \pi(i)$.

Let $\mathcal{S} = (x_1, \pi_1), (x_2, \pi_2), \ldots, (x_t, \pi_t)$ be a sequence of elements of $V(\hat{G}) \times \Gamma$. Recursively define

$$\hat{G}^{\mathcal{S}} = \hat{G}^{(x_1,\pi_1),(x_2,\pi_2),\ldots,(x_t,\pi_t)} = \left(\hat{G}^{(x_1,\pi_1)}\right)^{(x_2,\pi_2),\ldots,(x_t,\pi_t)}.$$

We call the sequence $\mathcal{S}$ a Γ-*switching sequence*, and say that it *transforms* $\hat{G}$ into $\hat{G}^{\mathcal{S}}$. Two m-edge-coloured graphs $\hat{G}$ and $\hat{H}$ are called Γ-*switch equivalent* when there exists a Γ-switching sequence $\mathcal{S}$ such that $\hat{G}^{\mathcal{S}} = \hat{H}$. In other words, $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent when there exists a Γ-switching sequence that transforms $\hat{G}$ into an m-edge coloured graph that is equal to $\hat{H}$. It is easy to see that Γ-switch equivalence defines an equivalence relation on the set of all m-edge coloured graphs on a fixed vertex set. The equivalence class of the m-edge-coloured graph $\hat{G}$ is denoted by $[\hat{G}]_\Gamma$.

Switching 2-edge coloured graphs with respect to S_2 first appears in the work of Abelson and Rosenberg in the context of behavioural science [1]. Switching 2-edge coloured graphs in which the colours are $\{+1, -1\}$ is integral to the study of signed graphs. These are different from 2-edge coloured graphs because the product of colours on each cycle is invariant under switching, which leads to the fundamental concept of *balance* of a cycle. Signed graphs have been extensively studied by Zaslavsky; for example see [13, 14]. The related concept of *pushing vertices* in oriented graphs is considered in [8]. Switching m-edge coloured graphs with respect to cyclic groups was first studied by Brewster and Graves [2]. Their results are extended to all abelian groups in [9].

After noting some preliminary information, in the first part of this paper we consider the question of when two m-edge-coloured graphs $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent when Γ is a symmetric, alternating or dihedral group. In each case we give necessary and sufficient conditions for two m-edge coloured graphs $\hat{G}$ and $\hat{H}$ to be Γ-switch equivalent. We believe these to be the first results on switch equivalence with respect to non-abelian groups. This follows previous work in this area that characterised switch equivalence for 2-edge-coloured graphs, for abelian groups and for a related concept in oriented graphs (see [2, 8, 9, 11]).

In the last part of the paper we consider colourings and homomorphisms of m-edge coloured graphs. Recall that a k-colouring of a graph G is a function $c : V(G) \to \{1, 2, \ldots, k\}$ such that if $xy \in E(G)$, then $c(x) \neq c(y)$. A *homomorphism* from a graph G to a graph H is a function $f : V(G) \to V(H)$ such that if $xy \in E(G)$, then $f(x)f(y) \in E(H)$.

Suppose there is a homomorphism, f, of the graph G to a graph H on k vertices. If the vertices of H are regarded as colours, then f is an assignment of these colours to the vertices of G such that adjacent vertices in G are assigned adjacent (hence different) colours. Thus a k-colouring of a graph G can equivalently be defined as a homomorphism of G to *some* graph H on k vertices. Defining a k-colouring in this way allows the idea of a (vertex) k-colouring to be extended to m-edge-coloured graphs, oriented graphs, and other types of graphs [10, 12].

Let $\hat{G}$ and $\hat{H}$ be m-edge-coloured graphs. A *homomorphism* of $\hat{G}$ to $\hat{H}$ is a function $f : V(\hat{G}) \to V(\hat{H})$ such that, for all $i \in \{1, 2, \ldots, m\}$, if $xy \in E_i(\hat{G})$ then $f(x)f(y) \in E_i(\hat{H})$. For an integer $k \geq 1$, a *vertex k-colouring* of an m-edge-coloured graph G is a homomorphism of G to some m-edge-coloured graph on k vertices.

Let Γ be a subgroup of S_m. An m-edge-coloured graph G has a Γ-*switchable homomorphism* to an m-edge-coloured graph H if some $\hat{G}' \in [\hat{G}]_\Gamma$ has a homomorphism to $\hat{H}$, that is, if $\hat{G}$ can be Γ-switched so that the transformed graph has a homomorphism to $\hat{H}$. For an integer $k \geq 1$, an m-edge-coloured graph $\hat{G}$ has a Γ-*switchable k-colouring* if it has a Γ-switchable homomorphism to some m-edge-coloured graph on k vertices.

We are interested in the complexity of deciding whether a given m-edge coloured graph $\hat{G}$ can be switched so it has a vertex k-colouring or a homomorphism to a fixed m-edge coloured graph $\hat{H}$. We are able to give dichotomy theorems for these problems with respect to the groups we consider. A dichotomy theorem for Γ-switchable k-colouring when Γ is abelian appears in [9]. Kidner has proved that for all groups Γ the problem of deciding whether an m-edge coloured graph $\hat{G}$ has a Γ-switchable k-colouring is solvable in polynomial time when $k \leq 2$ and is NP-hard when $k \geq 3$ [7] (also see [4, 9]). The dichotomy theorems for the homomorphism problem generalize the fundamental result of Hell and Nešetřil, and the dichotomy theorem for S_2 switchable homomorphism due to Brewster et al. [3] Related results for oriented graphs appear in [8].

We now describe a way to algorithmically determine whether two m-edge coloured graphs, $\hat{G}$ and $\hat{H}$, on the same vertex set, V, are Γ-switch equivalent with respect to an arbitrary group Γ. One can construct an auxiliary graph with vertex set equal to the set of all $(m+1)^{\binom{|V|}{2}}$ m-edge coloured graphs on V. There is an edge from F to F' when there exists a vertex x of F and $\pi \in \Gamma$ such that $F^{(x,\pi)} = F'$. Two m-edge coloured graphs are Γ-switch equivalent if and only if they belong to the same component of this auxiliary graph. Determining whether $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent using this procedure involves considering Γ-switching sequences of length at most $(m+1)^{\binom{|V|}{2}}$.

We can further restrict the length of such sequences when Γ is abelian. Suppose Γ is abelian. Then the same transformed graph arises from any rearrangement of a given Γ-switching sequence. Since there is a rearrangement so that the switches at each vertex occur consecutively, and the result of switching with respect to $(x,\alpha_1),(x,\alpha_2),\dots,(x,\alpha_k)$ is the same as the result of switching with respect to $(x,\alpha_1\alpha_2\dots\alpha_k)$, it suffices to consider Γ-switching sequences in which there is at most one switch at each vertex. Such a sequence has length at most $|V|$.

To see that the order of switches can matter when Γ is non-abelian, consider a 3-edge-coloured graph $\hat{G}$, the group $\Gamma = S_3$, and an edge xy of colour 1. Let $\alpha = (1\ 2)$ and $\beta = (2\ 3)$. For the switching sequence $\mathcal{S} = (x,\alpha)(y,\beta)(x,\alpha^{-1})$ we have $\sigma(\hat{G}^{\mathcal{S}})(xy) = 3$, whereas for the switching sequence $\mathcal{S}' = (x,\alpha)(x,\alpha^{-1})(y,\beta)$ we have $\sigma(G^{\mathcal{S}'})(xy) = 1$.

2 Properties $\mathcal{T}_{i,j}$ and $\mathcal{T}_j$

Consider a pair of m-edge-coloured graphs $\hat{G}$ and $\hat{H}$ such that $G = H$ and the edge colours in $\hat{G}$ and $\hat{H}$ differ only on a single edge. That is, consider $\hat{G} = (G,\sigma_G)$, $(\hat{H},\sigma_H)$ such that $G = H$ and there exists $e \in E$ and $i,j \in 1,2,\dots,m$ with $i \neq j$ such that $\sigma_G(e) = i, \sigma_H(e) = j$ and $\sigma_G(f) = \sigma_H(f)$ for all $f \neq e$. Whether $\hat{G}$ and $\hat{H}$ are switch equivalent with respect to some group $\Gamma \subseteq S_m$ depends on whether there is a sequence of switches whose only impact on the colours of the edges of $\hat{G}$ is to change the colour of a single edge from i to j.

Proposition 2.1. *Let $\hat{G}$ be an m-edge-coloured graph, where $m \geq 3$, and let $i,j \in \{1,2,\dots,m\}$ be such that $i \neq j$. If $xy \in E_i(\hat{G})$, then there exists $\hat{G}' \in [\hat{G}]_{S_m}$ such that $xy \in E_j(\hat{G}')$ and $\hat{G} - xy = \hat{G}' - xy$.*

Proof. Let $\alpha = (i\ j)$. Since $m \geq 3$, for any $k \in \{1,2,\dots,m\} \setminus \{i,j\}$ there exists $\beta = (j\ k)$. Consider the S_m-switching sequence (x,α), (y,β), (x,α), (y,β). We claim this sequence transforms $\hat{G}$ into $\hat{G}'$.

The only edges that change colour in the transformation are incident with x or y. It is given that the edge xy has colour i in $\hat{G}$. After the first, second, third and fourth switch, the edge xy has colour j,k,k,j, respectively, in the transformed graph. Any edge e incident with x and not y changes from its colour, c_e, to $\alpha(c_e)$ and then back to c_e. Similarly, any edge incident with y has the same colour as in G after switching. The result follows. $\square$

Let Γ be a subgroup of S_m. For $i,j \in \{1,2\dots,m\}$, we say Γ has *property $\mathcal{T}_{i,j}$* when there exist permutations $\alpha,\beta \in \Gamma$ such that α maps i to j and fixes some element k, and β maps j to k.

By applying the technique in the proof of the proposition above, we notice that if Γ has property $\mathcal{T}_{i,j}$ for a fixed pair (i,j), then Γ-switching can be used to transform an m-edge coloured graph into one where every edge that had colour i now has colour j.

Proposition 2.2. *Let* Γ *be a subgroup of* S_m *with property* $\mathcal{T}_{i,j}$. *If* $\hat{G}$ *is an* m*-edge-coloured graph, where* $m \geq 3$, *and* $xy \in E_i(\hat{G})$, *then there exists* $\hat{G}' \in [\hat{G}]_\Gamma$ *such that* $xy \in E_j(G')$ *and* $\hat{G} - xy = \hat{G}' - xy$.

Proof. The switching sequence $(x, \alpha), (y, \beta), (x, \alpha^{-1}), (y, \beta^{-1})$ transforms $\hat{G}$ to $\hat{G}'$. □

Let Γ be a subgroup of S_m. For $j \in \{1, 2 \dots, m\}$, we say Γ has *property* $\mathcal{T}_j$ when Γ has property $\mathcal{T}_{i,j}$ for every $i \in \{1, 2 \dots, m\} \setminus j$. If Γ has property $\mathcal{T}_j$ then for any edge xy in an m-edge coloured graph $\hat{G}$ there exists a Γ-switching sequence such that xy is of colour j, and the colour of every other edge of $\hat{G}$ is unchanged. It follows that $\hat{G}$ can be transformed to be monochromatic of colour j by changing the colour of one edge at a time.

Lemma 2.3. *Let* Γ *be a subgroup of* S_m *with property* $\mathcal{T}_j$. *For any* m*-edge coloured graph* $\hat{G}$ *there exists a switching sequence* $\mathcal{S}$ *such that* $\hat{G}^{\mathcal{S}}$ *is monochromatic of colour* j.

Theorem 2.4. *Let* Γ *be a subgroup of* S_m *with property* $\mathcal{T}_j$. *Two* m*-edge-coloured graphs* $\hat{G}$ *and* $\hat{H}$ *are* Γ*-switch equivalent if and only if* $G = H$.

Proof. By definition, two m-edge coloured graphs which are Γ-switch equivalent have the same underlying graph.

Now suppose $G = H$. By Lemma 2.3, both $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent to an m-edge-coloured graph that is monochromatic of colour j. Since Γ-switch equivalence is an equivalence relation, $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent. □

By Theorem 2.4, to check Γ-switch equivalence for a group Γ with property $\mathcal{T}_j$ it suffices to check for equality in the underlying graph. By observation, for $m \geq 3$, S_m has property $\mathcal{T}_1$ and for all $m \geq 4$, A_m has property $\mathcal{T}_1$.

Corollary 2.5. *For* $m \geq 3$, *two* m*-edge-coloured graphs* $\hat{G}$ *and* $\hat{H}$ *are* S_m*-switch equivalent if and only if* $G = H$.

Corollary 2.6. *For* $m \geq 4$, *two* m*-edge-coloured graphs* $\hat{G}$ *and* $\hat{H}$ *are* A_m*-switch equivalent if and only if* $G = H$.

3 The Dihedral Group

For $m \geq 3$ we denote by D_m the group of permutations of $\{1, 2, \dots, m\}$ corresponding to symmetries of the regular m-gon with vertices $1, 2, \dots, m$ in cyclic order. The cases m odd and m even are different. We consider the case of odd m first.

Proposition 3.1. *For any odd integer $m \geq 3$ and any $j \in \{1, 2, \ldots, m\}$, the group D_m has property $\mathcal{T}_j$.*

Proof. Let $j \in \{1, 2, \ldots, m\}$. To prove D_m has property $\mathcal{T}_j$ it suffices to prove D_m has property $\mathcal{T}_{i,j}$ for all $i \in \{1, 2, \ldots, m\} \setminus j$.

Since m is odd, either the least residue of $i - j$ modulo m is even, or the least residue of $j - i$ modulo m is even. Without loss of generality, the latter holds. Then there exists $k \in \{1, 2, \ldots, m - 1\}$ such that $j - i \equiv 2k \pmod{m}$, so that $j - k \equiv k + i \pmod{m}$, that is, k is the midpoint of the even-length path joining i and j. Let α be the permutation of $\{1, 2, \ldots, m\}$ which corresponds to flipping the m-gon over while fixing vertex k. Then α maps i to j and fixes k, so D_m has property $\mathcal{T}_{i,j}$ This completes the proof. □

Corollary 3.2. *For odd $m \geq 3$, two m-edge-coloured graphs $\hat{G}$ and $\hat{H}$ are D_m-switch equivalent if and only if $G = H$.*

We now consider the case of switching with respect to D_m when m is even. The following basic facts from group theory will be used.

Observation 3.3. *Suppose $m \geq 2$ is even. Let $\mathcal{E} = \{2, 4, \ldots, m\}$ and $\mathcal{O} = \{1, 3, \ldots, m - 1\}$. Then,*

1. *$\{\mathcal{O}, \mathcal{E}\}$ is a block system for the action of D_m on $\{1, 2, \ldots, m\}$;*
2. *Stabilizer$(\mathcal{E})$ = Stabilizer$(\mathcal{O})$ is a normal subgroup of D_m;*
3. *D_m/Stabilizer$(\mathcal{E}) \cong D_m$/Stabilizer$(\mathcal{O}) \cong S_2$;*
4. *Stabilizer$(\mathcal{E})$ has Property $\mathcal{T}_j$ for all $j \in \mathcal{E}$; and*
5. *Stabilizer$(\mathcal{O})$ has Property $\mathcal{T}_j$ for all $j \in \mathcal{O}$.*

Let $\hat{G} = (G, \sigma)$ be an m-edge-coloured graph, where $m \geq 2$ is an even integer. The 2-edge-coloured graph $\hat{G}_2$ is obtained from G by assigning each edge e colour 1 if $\sigma(e) \in \mathcal{O}$, and colour 2 if $\sigma(e) \in \mathcal{E}$. Notice that this is equivalent to regarding the edge colours of $\hat{G}_2$ to be $\mathcal{E}$ and $\mathcal{O}$, with the colour of an edge of $\hat{G}_2$ being the name of the block containing the colour of the corresponding edge in $\hat{G}$. The colours of the edges of $\hat{G}_2$ are naturally permuted by D_m/*Stabilizer*$(\mathcal{E}) \cong S_2$.

Theorem 3.4. *Let $\hat{G}$ and $\hat{H}$ be m-edge-coloured graphs, where $m \geq 2$ is an even integer. Then $\hat{G}$ and $\hat{H}$ are switch equivalent with respect to D_m if and only if $\hat{G}_2$ and $\hat{H}_2$ are switch equivalent with respect to S_2.*

Proof. Suppose $\hat{G}$ and $\hat{H}$ are switch equivalent with respect to D_m. Then there is a D_m-switching sequence $\mathcal{S} = (x_1, \pi_1), (x_2, \pi_2), \ldots, (x_t, \pi_t)$ that transforms $\hat{G}$ to $\hat{H}$. By Observation 3.3, each permutation $\pi_i \in D_m$ either maps $\mathcal{E}$ to $\mathcal{E}$ and $\mathcal{O}$ to $\mathcal{O}$, or maps $\mathcal{E}$ to $\mathcal{O}$ and vice-versa. Let $\mathcal{S}'$ be the subsequence of $\mathcal{S}$ consisting of the permutations that map $\mathcal{E}$ to $\mathcal{O}$. Replacing each of the permutations in this subsequence by the transposition (1 2) gives an S_2-switching sequence that transforms $\hat{G}_2$ to $\hat{H}_2$.

Now suppose $\hat{G}_2$ and $\hat{H}_2$ are switch equivalent with respect to S_2. Then, $G_2 = H_2$. Without loss of generality, assume $G_2 = H_2$. Let $\mathcal{A} = (x_1, \sigma_1), (x_2, \sigma_2), \ldots, (x_p, \sigma_p)$ be an S_2-switching sequence that transforms $\hat{G}_2$ to $\hat{H}_2$. Replacing each permutation $\sigma_i \in S_2$ by the m-cycle $(1\ 2\ \cdots\ m) \in D_m$ gives a D_m-switching sequence that transforms G to a graph G' in which the colour of edge belongs to the same block as the corresponding edge G. Since *Stabilizer*$(\mathcal{E})$ has Property $\mathcal{T}_j$ for all $j \in \mathcal{E}$, and *Stabilizer*$(\mathcal{O})$ has Property $\mathcal{T}_j$ for all $j \in \mathcal{O}$, the m-edge-coloured graph G' is D_m-switch equivalent to H (as in the proof of Proposition 2.2, edges other than the one whose colour is intended to change switches from their colour then back again). □

Zaslavsky proved that the 2-edge-coloured graphs $\hat{G}$ and $\hat{H}$ with the same underlying graph are switch equivalent with respect to S_2 if and only if they have the same collection of cycles for which the number of edges whose colour is in E_2 is odd [13]. Together with Theorem 3.4, this yields a similar result for D_m, where $m \geq 2$ is even.

Corollary 3.5. *Suppose $m \geq 2$ is even. Two m-edge-coloured graphs $\hat{G}$ and $\hat{H}$ with the same underlying graph are switch equivalent with respect to D_m if and only if they have the same collection of cycles for which the number of edges whose colour is in $\mathcal{E}$ is odd.*

4 Colourings and Homomorphisms

Our goal in this section is to present analogues of Theorems 4.1 and 4.2 below for Γ-switchable colourings and homomorphisms when Γ is a group with property $\mathcal{T}_j$ for some $j \in \{1, 2, \ldots, m\}$, or an even order dihedral group. Theorems such as these are known as *dichotomy theorems* because they exhibit a dichotomy for the complexity of a particular decision problem.

Theorem 4.1 ([5]). *For an integer $k \geq 1$, the problem of deciding whether a given graph G has a k-colouring is solvable in polynomial time when $k \leq 2$, and is NP-complete if $k \geq 3$.*

Theorem 4.2 ([6]). *If H is a fixed graph then the problem of deciding whether a given graph G has a homomorphism to H is solvable in polynomial time if H is bipartite, and is NP-complete if H is not bipartite.*

It follows from the definition of Γ-switchable homomorphism that if there exists a Γ-switchable homomorphism of an m-edge-coloured graph $\hat{G}$ to an m-edge-coloured graph $\hat{H}$, then there is a homomorphism from G to H. To see that the converse is false, let $\Gamma = S_2$, let $\hat{G}$ be the 2-edge coloured K_3 with two edges of colour 1 and one edge of colour 2 and let $\hat{H}$ be the 2-edge coloured K_3 with two edges of colour 2 and one edge of colour 1. There is a homomorphism of G to H but no S_2-switchable homomorphism of $\hat{G}$ to $\hat{H}$.

The following theorem from [9] is useful because it transforms the problem of deciding whether there is a Γ-switchable homomorphism of $\hat{G}$ to $\hat{H}$ to the problem of deciding the existence of a homomorphism (with no switching) to any m-edge coloured graph Γ switch equivalent to H.

Theorem 4.3 ([9]). *Let $\hat{G}$ and $\hat{H}$ be m-edge-coloured graphs and let Γ be a subgroup of S_m. Then there is a Γ-switchable homomorphism of $\hat{G}$ to $\hat{H}$ if and only if, for all $\hat{H}' \in [\hat{H}]_\Gamma$ there exists $\hat{G}' \in [\hat{G}]_\Gamma$ such that there is a homomorphism of $\hat{G}'$ to $\hat{H}'$.*

Theorem 4.4. *Let Γ be a subgroup of S_m that has Property $\mathcal{T}_j$ for some $j \in \{1, 2, \ldots, m\}$, and let $k \geq 1$ be an integer. If $k \leq 2$ then the problem of deciding whether a given m-edge-coloured graph has a Γ-switchable k-colouring is solvable in polynomial time. If $k \geq 3$ then the problem of deciding whether a given m-edge-coloured graph has a Γ-switchable k-colouring is NP-complete.*

Proof. By Corollary 2.5 every m-edge-coloured graph $\hat{F}$ is Γ-switch equivalent to an m-edge-coloured graph $\hat{F}'$ that is monochromatic of colour j. Thus by Theorem 4.3 there is a Γ-switchable homomorphism of $\hat{G}$ to an m-edge-coloured graph $\hat{H}$ on k vertices if and only if there is a homomorphism of $\hat{G}'$ to $\hat{H}'$, if and only if there is a homomorphism of G to H, if and only if G has a k-colouring. The result now follows from from Theorem 4.1. □

The proof of the corresponding result for Γ-switchable homomorphisms proceeds similarly, and is thus omitted.

Theorem 4.5. *Let Γ be a subgroup of S_m that has Property $\mathcal{T}_j$ for some $j \in \{1, 2, \ldots, m\}$. Let $\hat{H}$ be a fixed m-edge-coloured graph. If $\hat{H}$ is bipartite, then the problem of deciding whether a given m-edge-coloured graph has a Γ-switchable homomorphism to $\hat{H}$ is solvable in polynomial time. If $\hat{H}$ is not bipartite then the problem of deciding whether a given m-edge-coloured graph has a Γ-switchable homomorphism to $\hat{H}$ is NP-complete.*

Proof. By Corollary 2.5, the m-edge-coloured graphs $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent to m-edge-coloured graphs $\hat{G}'$ and $\hat{H}'$, which are monochromatic of colour j. It follows from Theorem 4.3 that there is a Γ-switchable homomorphism of $\hat{G}$ to $\hat{H}$ if and only if there is a homomorphism of $\hat{G}'$ to $\hat{H}'$, if and only if there is a homomorphism of G to H. The result now follows from Theorem 4.2. □

We have found dichotomy theorems for the complexity of the Γ-switchable k-colouring problem and the problem of deciding whether there exists a Γ-switchable homomorphism to a fixed m-edge coloured graph $\hat{H}$ when Γ is one of $S_m, m \geq 3$; $A_m, m \geq 4$; $D_m, m \geq 2$ and odd; any other group with property $\mathcal{T}_j$ for some j. Finally, we consider dihedral groups of even order. The following theorem will be useful.

Theorem 4.6 ([3])**.** *Let $\hat{H}$ be a 2-edge-coloured graph. If there is an S_2-switchable homomorphism of $\hat{H}$ to a monochromatic K_2, then the problem of deciding whether a given 2-edge-coloured graph $\hat{G}$ has an S_2-switchable homomorphism to $\hat{H}$ is solvable in polynomial time. If there is no S_2-switchable homomorphism of $\hat{H}$ to a monochromatic K_2, then the problem of deciding whether a given 2-edge-coloured graph $\hat{G}$ has an S_2-switchable homomorphism to $\hat{H}$ is NP-complete.*

As an aside, we note that it is easy to test whether a 2-edge coloured graph has a S_2-switchable homomorphism to a monochromatic K_2. By [2] such an S_2-switchable homomorphism exists if and only if there is a homomorphism (without switching) of H to a 4-cycle where the edge colours alternate. The latter condition can be tested in polynomial time. Without loss of generality H is connected. By symmetry the image of any vertex can be chosen without loss of generality. Since each vertex is adjacent with exactly one edge of each colour, there is only one choice to extend the mapping to a neighbouring vertex. Successively doing so either leads to the desired homomorphism or to a contradiction in which some vertex is forced to have two different images.

Theorem 4.7. *Let $k \geq 1$ be an integer, and let $m \geq 2$ be an even integer. The problem of deciding whether a given m-edge-coloured graph has a D_m-switchable k-colouring is solvable in polynomial time if $k \leq 2$ and is NP-complete if $k \geq 3$.*

Proof. Let $\hat{G}$ be an m-edge coloured graph. It is clear that $\hat{G}$ has a D_m-switchable 1-colouring if and only if it has no edges.

Suppose $k = 2$. By definition, $\hat{G}$ has a D_m-switchable 2-colouring if and only if there exists j such that it has a D_m-switchable homomorphism to a K_2 of colour j. Thus, by Theorem 4.3, $\hat{G}$ has a D_m-switchable 2-colouring if and only if it is bipartite and there exists $\hat{G}' \in [\hat{G}]_{D_m}$ such that $\hat{G}'$ is monochromatic of colour j.

Without loss of generality j is odd. By Theorem 3.4 the m-edge coloured graph $\hat{G}'$ exists if and only if $\hat{G}_2$ (as in Theorem 3.4) is S_2-switch equivalent to $\hat{G}'_2$. Since $\hat{G}'_2$ is bipartite and switchable homomorphisms compose [9], this is equivalent to $\hat{G}_2$ having an S_2-switchable homomorphism to a K_2 of colour 1, which is decidable in polynomial time by Theorem 4.6.

Now suppose $k \geq 3$. The transformation is from the problem of deciding whether a given graph G has a k-colouring. Suppose a graph G is given. We claim that G has a k-colouring if and only if the m-edge-coloured graph $\hat{G}$, whose underlying graph is G, and is monochromatic of colour j has a D_m-switchable k-colouring. Clearly $\hat{G}$ can be constructed in polynomial time.

Suppose $\hat{G}$ has a D_m-switchable k-colouring. By definition, such a mapping gives a k-colouring of G. Suppose G has a k-colouring. Then there is a homomorphism of G to K_k. Therefore there is a homomorphism of $\hat{G}$ to a copy of K_k which is monochromatic of colour j. Thus $\hat{G}$ has a D_m-switchable k-colouring.

It now follows that D_m-switchable k-colouring is NP-complete. □

The proof of the following lemma is very similar to the proof of Theorem 3.4.

Lemma 4.8. *Let $\hat{G}$ and $\hat{H}$ be m-edge coloured graphs, and $m \geq 2$ be an even integer. There is a D_m-switchable homomorphism of $\hat{G}$ to $\hat{H}$ if and only if there is an S_2-switchable homomorphism of $\hat{G}_2$ to $\hat{H}_2$.*

Proof. Suppose first that there is a D_m-switchable homomorphism of $\hat{G}$ to $\hat{H}$. Then there is a D_m-switching sequence $\mathcal{S} = (x_1, \pi_1), (x_2, \pi_2), \ldots, (x_t, \pi_t)$ that transforms $\hat{G}$ to $\hat{G}' \in [\hat{G}]_{D_m}$ for which there is a homomorphism of $\hat{G}'$ to $\hat{H}$. By Observation 3.3, each permutation $\pi_i \in D_m$ either maps $\mathcal{E}$ to $\mathcal{E}$ and $\mathcal{O}$ to $\mathcal{O}$, or maps $\mathcal{E}$ to $\mathcal{O}$ and vice-versa. Let $\mathcal{S}'$ be the subsequence of $\mathcal{S}$ consisting of the permutations that map $\mathcal{E}$ to $\mathcal{O}$. Replacing each of the permutations in this subsequence by the transposition (1 2) gives an S_2-switching sequence that transforms $\hat{G}_2$ to a 2-edge coloured graph $\hat{G}_2'$ that has a homomorphism to $\hat{H}_2$. Therefore there is an S_2-switchable homomorphism of $\hat{G}_2$ to $\hat{H}_2$.

Suppose there is an S_2-switchable homomorphism of $\hat{G}_2$ to $\hat{H}_2$. We claim that $\hat{G}$ and $\hat{H}$ are D_m-switch equivalent to $\hat{G}_2$ and $\hat{H}_2$, respectively. Since *Stabilizer*$(\mathcal{E})$ has property $\mathcal{T}_j$ for all $j \in \mathcal{E}$, any edge of $\hat{G}$ whose colour is in $\mathcal{E}$ can be D_m-switched to have colour 2. As in the proof of Proposition 2.2, edges other than the one whose colour is intended to change switch from their colour then back again. Similarly, any edge of $\hat{G}$ whose colour is in $\mathcal{O}$ can be D_m-switched to have colour 1. Thus $\hat{G}$ is D_m-switch equivalent to $\hat{G}_2$. Similarly $\hat{H}$ is D_m-switch equivalent to $\hat{H}_2$, and the claim is proved.

The same function that gives a homomorphism of the 2-edge coloured graph $\hat{G}_2$ to the 2-edge coloured graph $\hat{H}_2$ is also a homomorphism of the m-edge coloured graph $\hat{G}_2$ to the m-edge coloured graph $\hat{H}_2$. It now follows from Theorem 4.3 that there is a D_m-switchable homomorphism of $\hat{G}$ to $\hat{H}$. □

Theorem 4.9. *Let $\hat{H}$ be an m-edge coloured graph, and $m \geq 2$ be an even integer. If there is a homomorphism of $\hat{H}_2$ to a monochromatic K_2, then the problem of deciding whether a given m-edge-coloured graph $\hat{G}$ has a D_m-switchable homomorphism to $\hat{H}$ is solvable in polynomial time. If there is no homomorphism of $\hat{H}_2$ to a monochromatic K_2, the problem of deciding whether a given m-edge-coloured graph $\hat{G}$ has a D_m-switchable homomorphism to $\hat{H}$ is NP-complete.*

Proof. Let $\hat{H}$ be an m-edge coloured graph, and $m \geq 2$ be an even integer.

Suppose first there is a homomorphism of $\hat{H}_2$ to a monochromatic K_2. Let $\hat{G}$ be an m-edge coloured graph. Then, by Theorem 4.6, it can be decided in polynomial time whether $\hat{G}_2$ has a $\hat{S}_2$-switchable homomorphism to $\hat{H}_2$. Since $\hat{G}_2$ can be constructed in polynomial time, Lemma 4.8 implies it can be decided in polynomial time whether $\hat{G}$ has a D_m-switchable homomorphism to $\hat{H}$.

Now suppose there is no homomorphism of the 2-edge coloured graph $\hat{H}_2$ to a monochromatic K_2. We show the problem of deciding whether a given m-edge coloured graph has a D_m-switchable homomorphism to H is NP-complete. The transformation is from the problem of deciding whether a given 2-edge coloured graph $\hat{F}$ has an S_2-switchable homomorphism to $\hat{H}_2$, which is NP-complete by Theorem 4.6.

Suppose such a 2-edge-coloured graph $\hat{F}$ is given. The transformed instance of the problem is the m-edge coloured graph $\hat{F}'$ in which every edge has the same colour as in $\hat{F}$. Then $\hat{F}'_2 = \hat{F}$, and the result follows from Lemma 4.8. $\square$

5 Concluding Remarks

Zaslavsky's original work on switching defined switch equivalence in the same manner as herein. One may consider a more general version, switch isomorphism, in which $\hat{G}$ and $\hat{H}$ are considered as equivalent when there is a switching sequence $\mathcal{S}$ such that $\hat{G}^S$ is isomorphic to $\hat{H}$. For any group Γ, two m-edge-coloured graphs that are both monochromatic of colour j are Γ-switch isomorphic if and only if their underlying graphs are isomorphic. And so, deciding whether two m-edge coloured graphs are equivalent under this model is at least as hard as deciding if they are isomorphic.

For an abelian group Γ and an m-edge coloured graph $\hat{G}$ there is a graph $P_\Gamma(\hat{G})$ such that $\hat{G}$ and $\hat{H}$ are Γ-switch isomorphic if and only if $P_\Gamma(\hat{G}) \cong P_\Gamma(\hat{H})$. However, under Zaslavsky's original definition of switch equivalence, $\hat{G}$ and $\hat{H}$ are Γ-switch equivalent if and only if $P_\Gamma(\hat{G})$ and $P_\Gamma(\hat{H})$ are equal after first being converted to a canonical form [9]. No similar result is known to hold when Γ is non-abelian.

Acknowledgements

Research of the first two authors supported by NSERC.

References

[1] R. P. Abelson and M. J. Rosenberg, Symbolic psycho-logic: A model of attitudinal cognition, *Behavioral Sci.* **3**(1958), 1–13.

[2] R. C. Brewster and T. Graves, Edge-switching homomorphisms of edge-coloured graphs, *Discrete Math.* **309** (2009), 5540–5546.

[3] R. C. Brewster, F. Foucaud, P. Hell and R. Naserasr, The complexity of signed graph and 2-edge-colored graph homomorphisms, *Discrete Math.* **340** (2017), 223–235.

[4] R. C. Brewster, A. Kidner and G. MacGillivray, k-colouring (m, n)-mixed graphs with switching, Manuscript, 2022, `https://arxiv.org/abs/2203.08070`.

[5] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H. Freeman & Co, San Francisco, 1979.

[6] P. Hell and J. Nešetřil, On the complexity of H-colouring, *J. Combin. Theory Ser. B* **48** (1990), 92–110.

[7] A. Kidner, Γ-Switchable 2-Colourings of (m, n)-Mixed Graphs, *Master's Thesis* (2021), University of Victoria.

[8] W. F. Klostermeyer and G. MacGillivray, Homomorphisms and oriented colorings of equivalence classes of oriented graphs, *Discrete Math.* **274** (2004), 161–172.

[9] E. Leclerc, G. MacGillivray and J. M. Warren, Switching (m, n)-mixed graphs with respect to abelian groups, Manuscript, 2021.

[10] J. Nešetřil and A. Raspaud, Colored homomorphisms of colored mixed graphs, *J. Combin. Theory Ser. B* **80** (2000), 147–155.

[11] S. Sen, A contribution to the theory of graph homomorphisms and colorings, *Ph.D. Thesis* (2015), L'Université Bordeaux 1, France.

[12] E. Sopena, The chromatic number of oriented graphs, *J. Graph Theory* **25** (1997), 191–205.

[13] T. Zaslavsky, Signed graphs, *Discrete Applied Math.* **4** (1982), 47–74.

[14] T. Zaslavsky, A Mathematical Bibliography of Signed and Gain Graphs and Allied Areas. *Electron. J. Combin.* Dynamic Survey #DS8, Eighth edition, 2012.

(Received 26 July 2022; revised 16 Nov 2024)