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# Poincaré series of Coxeter groups and multichains in Eulerian posets

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**Abstract**

It is observed that the reciprocal of the Poincaré series (or growth series) of a finite rank Coxeter system with respect to its Coxeter generators is a specialization of the Hilbert series, in fine grading, of the face ring (or Stanley-Reisner ring) of the barycentric subdivision of the nerve, in which the indeterminate attached to each finite rank standard parabolic subgroup is specialized to a corresponding signed monomial determined by the subgroup's rank and longest element. This result is part of a close correspondence (described in detail in this note but not conceptually explained) between certain properties of reciprocals of Poincaré series of Coxeter groups and properties of Hilbert series of face rings of order complexes of lower Eulerian posets. In both settings, some new formulae (or novel reformulations of old formulae) are given which are analogous to well-known formulae in the other setting.

## Introduction

Let $(W, S)$ be a Coxeter system of finite rank $|S|$ (standard references are [2, 3, 7, 9]). Denote the standard length function of $(W, S)$ as $l\colon W \to \mathbb{N}$. The *Poincaré series* (or *growth series*) of $(W, S)$ is the formal power series $P_W := \sum_{w\in W} t^{l(w)}$ in $\mathbb{Z}[[t]]$. It has been extensively studied because of its significant applications in or to invariant theory, Lie groups, algebra, representation theory, combinatorics etc. (see [5] for a recent paper which surveys the main known facts and some of the applications).

It is well known that $P_W$ is rational (i.e. it is the power series expansion at 0 of some rational function). This follows directly from a formula

$$\frac{1}{P_W(t)} = \sum_{J\in\mathcal{N}} \frac{\epsilon_J}{P_{W_J}(t^{-1})}$$

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of Steinberg from [17], where $W_J := \langle J \rangle$ is the standard parabolic subgroup generated by $J \subseteq S$, $\epsilon_J := (-1)^{|J|}$ and $\mathcal{N} := \{ J \subseteq S \mid |W_J| < \infty \}$. As is also well known, the polynomial $P_{W_J}$ for $J$ in $\mathcal{N}$ is given by a formula (due to Solomon [13]) involving degrees of the basic invariants of $W_J$. Rationality of $P_W$ (and of Poincaré series of other subsets of $W$) can now also be established using finite state automata arising in the study of an automatic structure on $W$ ([4]).

The main purpose of this note is to record some analogies and relations between reciprocals of Poincaré series of Coxeter groups and standard generating functions attached to simplicial complexes (especially, order complexes of lower Eulerian posets). Steinberg's formula implies the following explicit formula for $P_W$:

$$\frac{1}{P_W} = \sum_{n\in\mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n \in \mathcal{N}} \frac{\epsilon_{J_1}\cdots\epsilon_{J_n} t^{m(J_1)+\cdots+m(J_n)}}{(1-\epsilon_{J_1}t^{m(J_1)})\cdots(1-\epsilon_{J_n}t^{m(J_n)})} \tag{$*$}$$

where for $J$ in $\mathcal{N}$, $w(J)$ is the longest element of $W_J$ and $m(J) := l(w(J))$.

Although this formula is not particularly efficacious for computation of $P_W$ in examples (cf. [2] for some computationally more convenient formulae), it does suggest some interesting structural features of these Poincaré series in general, as follows. For each subset $J$ of $S$, let $x_J$ be an indeterminate. The right-hand side of ($*$) is obtained by specializing each $x_J$ with $J \in \mathcal{N} \setminus \{\emptyset\}$ to $\epsilon_J t^{m(J)}$ in the left-hand side of the identity

$$\sum_{n\in\mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n \in \mathcal{N}} \frac{x_{J_1}\cdots x_{J_n}}{(1-x_{J_1})\cdots(1-x_{J_n})} = \sum_{n\in\mathbb{N}} \sum_{\emptyset = J_0 \subsetneq J_1 \subseteq \cdots \subseteq J_n \in \mathcal{N}} x_{J_1}\cdots x_{J_n}.$$

The right-hand side of the identity may be regarded as a generating function for multichains in the (inclusion-ordered) poset $\mathcal{N} \setminus \{\emptyset\}$. In other terms, $\mathcal{N}$ is the set of simplexes of an abstract simplicial complex on vertex set $S$, called the *nerve* of $(W, S)$, and the last generating function is the Hilbert series, in fine grading, of the face ring (or Stanley-Reisner ring, see [14]) of the barycentric subdivision of the nerve (i.e. the order complex of $\mathcal{N} \setminus \{0\}$). In this note, it is observed that many of the known general facts about reciprocals of Poincaré series of Coxeter groups are specializations of identities involving such multivariate rational functions attached to simplicial complexes.

We now discuss some previously known results related to formula ($*$). An elementary fact [14, Ch II, Theorem 7.1] on fine Hilbert series of face rings of simplicial complexes, applied to the barycentric subdivision of the closed simplex on vertex set $S$, gives the following identity:

$$\begin{aligned} &\sum_{n\in\mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n \subseteq S} \frac{x_{J_1}\cdots x_{J_n}}{(1-x_{J_1})\cdots(1-x_{J_n})} \\ &\quad = \sum_{n\in\mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n = S} \frac{(-1)^n \epsilon_S}{(1-x_{J_1})\cdots(1-x_{J_n})}. \end{aligned} \tag{$\dagger$}$$

Although ($*$) itself does not seem to appear in previous literature, specializing ($\dagger$) by $x_J \mapsto 0$ for $J \subseteq S$ with $J \notin \mathcal{N}$ and $x_J \mapsto \epsilon_J t^{m(J)}$ for $J \in \mathcal{N}$ gives an alternative ("dual") form of $(*)$ which is in [18], at least for crystallographic $W$. This dual form follows directly from a well-known algorithm for computing $P_W$ which is (mostly) implicit already in [3] and is described completely explicitly in [9]. We also mention that in the case of right-angled Coxeter groups, [7, Proposition 17.4.2] gives a formula for $\frac{1}{P_W}$ in terms of the $h$-polynomial of $\mathcal{N}$, and [8] gives a formula for $P_W$ as a specialization of a multivariate rational function (which also specializes to the growth series of $W$ for "automatic" generators $(w(J))_{J\in\mathcal{N}\setminus\{\emptyset\}}$) which is attached to $\mathcal{N}$; for $(A_1)^n$, the function is $1+\sum_{\emptyset\neq J\subseteq S} z_J$ with specialization $z_J \mapsto t^{m(J)}$.

Section 1 of this note recalls basic properties of (reciprocals of) Poincaré series of Coxeter groups and then proves $(*)$ and its dual form. Section 2 recalls some rudimentary properties of simplicial complexes and posets and then shows that the properties listed in Section 1, such as Steinberg's formula, are specializations of (known or easily proved) general facts about Hilbert series of face rings of order complexes of lower Eulerian posets. We state also some other formulae which specialize to apparently new formulae for $\frac{1}{P_W}$: notably, we give determinantal formulae (consequences of [15]) for the rational function in (†) and for its numerator, and observe that the coefficient of each (necessarily squarefree) monomial occurring in the numerator is the reduced Euler characteristic of the corresponding order complex (up to sign). We remark that the results of this note are very elementary, and the exposition has been kept as simple and self-contained as possible. In particular, no essential use is made of face rings or of topological or homological aspects of the theory of simplicial complexes.

As suggested above, the Poincaré series $P_W$ of a Coxeter system $(W, S)$ has close connections to many interesting objects naturally associated to $W$; for instance, for crystallographic (respectively, arbitrary) $W$, $P_W$ gives the non-trivial factor in the Hilbert-Poincaré series of the $T$-equivariant cohomology ring of an associated flag variety for a Kac-Moody group (respectively, its algebraic analogue, the dual nil Hecke ring; see [11]). As another example, the values of the rational function $\frac{1}{P_W}$ at a positive real number is the Euler characteristic for the correspondingly weighted $L^2$-cohomology theory of the Davis complex of $W$ (see [7]). There are many situations where (positive) Hilbert-Poincaré series are inverse to one another up to systematic change of some signs; for instance, this occurs for Koszul dual algebras (see [1]), although the nature of the sign changes here is different. Finally, the class of simplicial complexes arising as nerves of finite rank Coxeter systems is very extensive; this is an important fact in [7], where it is observed that it includes the barycentric subdivisions of all finite polytopal complexes. For such reasons, it would be desirable to have a more structural explanation for the correspondence of results between Sections 1 and 2 than the one through generating function manipulations in this note.

# 1 Reciprocals of Poincaré series of Coxeter groups

## 1.1

Let $(W, S)$ be a Coxeter system of finite rank $|S|$ (see [2, 3, 7, 9] as general references for basic properties of Coxeter groups used in this note). Let $l\colon W \to \mathbb{N}$ be the standard length function of $(W, S)$.

Let $\mathbb{Z}[t]$ and $\mathbb{Z}[[t]]$ denote respectively the polynomial ring and formal power series ring over $\mathbb{Z}$ in the indeterminate $t$. Let $\mathbb{Q}(t)$ and $\mathbb{Q}((t)) = (\mathbb{Q}[[t]])[t^{-1}]$ be their respective fields of fractions. We regard $\mathbb{Z}[t]$, $\mathbb{Z}[[t]]$ and $\mathbb{Q}(t)$ as subrings of $\mathbb{Q}((t))$, and say that a formal power series in the intersection $\mathbb{Q}(t) \cap \mathbb{Z}[[t]]$ is *rational*. Let $\tau\colon \mathbb{Q}(t) \to \mathbb{Q}(t)$ be the field automorphism given by $f(t) \mapsto f(\frac{1}{t})$.

For $w \in W$ define $X_w := t^{l(w)}$ in $\mathbb{Z}[[t]]$ (in particular, $X_{1_W} = 1_{\mathbb{Z}[t]}$). (The whole development also applies to certain multivariate Poincaré series as considered in [12], in which the $X_w$ are certain monomials of total degree $l(w)$ in a set of variables in bijection with the set of conjugacy classes of reflections; our notation is intended to make the corresponding generalization of the results here obvious for readers familiar with such length functions, but we do not give details.)

For any subset $Z$ of $W$, define its *Poincaré series* (or *growth series*) $P_Z$ in $\mathbb{Z}[[t]]$ to be the formal power series

$$P_Z := \sum_{w \in Z} X_w = \sum_{w \in Z} t^{l(w)}. \tag{1}$$

If $1 \in Z$, then $P_Z$ has constant term 1 and is invertible in $\mathbb{Z}[[t]]$.

The analogy between the results in this section and those in Section 2 would be highlighted if all formulae we give involving $P_Z$ with $1 \in Z$ were written instead in terms of $P_Z^{-1}$, but we do not do that if the resulting formula looks less natural here.

## 1.2

For $J \subseteq S$, let $W_J$ denote the *standard parabolic subgroup* $W_J = \langle\, J \,\rangle$ generated by $J$. Write $\epsilon_J := (-1)^{|J|}$. Let

$$W^J = \{\, w \in W \mid l(wx) = l(w) + l(x) \text{ for all } x \in W_J \,\} \tag{2}$$

be the set of *shortest left coset representatives* for $W_J$ in $W$. Define the set

$$\mathcal{N} := \{\, J \subseteq S \mid |W_J| < \infty \,\}, \tag{3}$$

called the *nerve* of $(W, S)$, and partially order $\mathcal{N}$ by inclusion. For $J \in \mathcal{N}$, let $w(J)$ denote the longest element of $W_J$ and set $x_J := \epsilon_J X_{w(J)}$. We set $x_J := 0$ for $J \subseteq S$ such that $J \notin \mathcal{N}$. Also, for $J$ in $\mathcal{N}$, let

$$\mu(J, \infty) := -\sum_{\substack{K \in \mathcal{N} \\ K \supseteq J}} \epsilon_J \epsilon_K \tag{4}$$

(the integers $\mu(J,\infty)$ are certain values of the Möbius function for a poset $\mathcal{N}\dot{\cup}\{\infty\}$ obtained by adjoining a maximum element $\infty$ to $\mathcal{N}$; see Section 2).

The theorem below collects some of the main formal facts on Poincaré series of Coxeter groups which apply "uniformly" across all (finite and infinite, crystallographic and non-crystallographic) types. Certain well-known and interesting formulae for the Poincaré series in terms of exponents for finite Coxeter groups ([13]) or root heights for finite and affine Weyl groups ([12]) are therefore not listed.

**Theorem 1.1.** *(a) If $J \subseteq S$, then $P_W = P_{W^J} P_{W_J}$.*

*(b) If $J$ is in $\mathcal{N}$, then $P_{W_J} = \epsilon_J x_{w_J} \tau(P_{W_J})$.*

*(c) In general,*

$$\sum_{J \subseteq S} \frac{\epsilon_J}{P_{W_J}} = x_S \frac{\epsilon_S}{P_W}.$$

*(d) One has*

$$\frac{1}{P_W} = \sum_{J \in \mathcal{N}} \frac{\epsilon_J}{\tau(P_{W_J})}.$$

*(e) One has*

$$\frac{1}{P_W} = -\sum_{J \in \mathcal{N}} \frac{\mu(J,\infty)}{P_{W_J}}.$$

*(f) $P_W$ is rational, and $\frac{1}{\tau(P_W)}$ lies in $\mathbb{Z}[[t]]$ with constant term $-\mu(\emptyset,\infty)$.*

*(g) Suppose there is an integer $N$ such $\mu(J,\infty) = (-1)^N \epsilon_J$ for all $J$ in $\mathcal{N}$. Then $\tau(P_W) = (-1)^{N+1} P_W$.*

*Proof.* The first results are in [3] and [9], while [2], [7] and [5] contain developments proving many of the facts listed. Specifically, see [3, Ch IV, §1, Ex 26] for (a), (b) and (c), [9, 5.12] for (a) and (c), [2, 7.1] for (a)–(c) and (e), and [7, Ch 17] for (a)–(g) (note that the results in [7] are proved where possible for the multivariate Poincaré series from [12] and notation there conflicts with that in this note). We provide below sketches of proofs and more references for some of these results.

(a) This holds since each element $w$ of $W$ can be uniquely written in the form $w = xy$ with $x \in W^J$ and $y \in W_J$ and $l(xy) = l(x) + l(y)$.

(b) This holds since the map $x \mapsto x\, w(J)$ is a bijection $W_J \to W_J$ satisfying $l(x\, w(J)) = l(w_J) - l(x)$.

(c) The proof is essentially the same as in the special case of finite $W$, which is from [13]. To sketch it, set $D(w) := \{\, s \in S \mid l(ws) < l(w) \,\}$ for $w$ in $W$. By (a),

$$\sum_{J \subseteq S} \frac{\epsilon_J P_W}{P_{W_J}} = \sum_{J \subseteq S} \epsilon_J P_{W^J} = \sum_{J \subseteq S} \sum_{w \in W^J} \epsilon_J X_w = \sum_{w \in W} \Bigg( \sum_{J \subseteq S \setminus D(w)} \epsilon_J \Bigg) X_w$$

where the inner sum is zero unless $D(w) = S$ (i.e. unless $W$ is finite and $w = w(S)$), when it is 1. The result now follows (recall $x_S := 0$ if $S \notin \mathcal{N}$ i.e if $W$ is infinite).

(d) For finite $W$, the result is equivalent to that of (c), by (b). It was proved in general in Steinberg [17]. One uses the fact that $D(w)$ is in $\mathcal{N}$ for all $w$, with $D(w) = \emptyset$ if and only if $w = 1$. By (a) and (b),

$$\sum_{J \in N} \frac{\epsilon_J P_W}{\tau(P_{W_J})} = \sum_{J \in N} \epsilon_J P_{W^J} X_{w(J)} = \sum_{J \in N} \sum_{\substack{w \in W \\ D(w) \supseteq J}} \epsilon_J X_w = \sum_{\substack{w \in W \\ J \subseteq D(w)}} \epsilon_J X_w = 1$$

where the second equality uses the fact (which follows easily from standard properties of $W^J$ and $w(J)$) that $W^J w(J) = \{ w \in W \mid D(w) \supseteq J \}$.

(e) This was proved in [6] (see also [2, Proposition 7.1.7]). It may be proved by writing $\frac{1}{\tau(P_{W_J})} = \sum_{K \subseteq J} \frac{\epsilon_K}{P_{W_K}}$ on the right in (d) (where this formula is proved by applying $\tau$ to (d) for $W_J$ instead of $W$) and reversing the order of summation.

(f) Rationality of $P_W$ and the fact that $\frac{1}{\tau(P_W)}$ is in $\mathbb{Z}[[t]]$ (with integer constant term) are in [3, Ch IV, §1, Ex 26(g)]; they are both clear from (d). The value of the constant term is from [6]; it may be proved by applying $\tau$ in (e) (or (d)) and examining the constant term.

(g) This is also from [6], where the hypothesis is expressed more geometrically in terms of the notion of "Euler spheres" (see also [7]). It may be proved by applying $\tau$ in (d) and comparing with (e). □

Now we state and prove the main result of this section.

**Theorem 1.2.** *(a) One has*

$$\frac{1}{P_W} = \sum_{n \in \mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n = S} \frac{(-1)^n \epsilon_S}{(1 - x_{J_1}) \cdots (1 - x_{J_n})}$$

*where $x_J = 0$ for $J \subseteq S$ with $J \notin \mathcal{N}$.*

*(b) One has*

$$\frac{1}{P_W} = \sum_{n \in \mathbb{N}} \sum_{\emptyset = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n \in \mathcal{N}} \frac{x_{J_1} \cdots x_{J_n}}{(1 - x_{J_1}) \cdots (1 - x_{J_n})}.$$

*Proof.* Before giving the proof, we remark that the results as formulated use standard conventions (which we impose also in Section 2) that empty sums have value 0 and empty products have value 1. In particular, the inner sum in (a) is empty if $n = 0$ and $S \neq \emptyset$ and has one term, the empty product, if $n = 0$ and $S = \emptyset$, while for $n = 0$, the inner sum in (b) has one term, the empty product.

(a) We prove that

$$\frac{1}{P_{W_K}} = \sum_{n \in \mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n = K} \frac{(-1)^n \epsilon_K}{(1 - x_{J_1}) \cdots (1 - x_{J_n})} \tag{5}$$

for all $K \subseteq S$ by induction on $|K|$. The result holds if $K = \emptyset$. Assume $K \neq \emptyset$. Applying Theorem 1.1(c) to $W_K$ instead of $W$ and solving for $P_{W_K}$ gives

$$\frac{1}{P_{W_K}} = \frac{-\epsilon_K}{1 - x_K} \sum_{J \subsetneq K} \frac{\epsilon_J}{P_{W_J}}.$$

Then (a) follows immediately by applying the induction hypothesis to each term in the sum on the right.

(b) Suppose in (a) that $K$ is in $\mathcal{N}$. Then all the terms $x_{J_i}$ (with $i \neq 0$) appearing in (5) are invertible in $\mathbb{Q}(t)$, so multiplying that equation by $\epsilon_K$ and applying $\tau$ shows that

$$\frac{\epsilon_K}{\tau(P_{W_K})} = \sum_{n \in \mathbb{N}} \sum_{\emptyset = J_0 \subsetneq \cdots \subsetneq J_n = K} \frac{x_{J_1} \cdots x_{J_n}}{(1 - x_{J_1}) \cdots (1 - x_{J_n})}$$

since $\frac{-1}{1-x_J^{-1}} = \frac{x_J}{1-x_J}$. Substituting into $\frac{1}{P_W} = \sum_{K \in \mathcal{N}} \frac{\epsilon_K}{\tau(P_{W_K})}$ (from Theorem 1.1(d)) proves (b). □

*Remark.* As noted in the introduction, (a) and its proof appear for crystallographic $W$ in [18, Section 7, (second) Proposition 3], while most of the argument for (a) is implicit in [3, Ch IV, §1, Ex 26(g)] and the whole argument for (a) (although not the formula there) is completely explicit in [9, Proof of Proposition 5.12(b)]. I do not know of any occurrence of (b) in the literature. Equivalence of (a) and (b) follows from the rational function identity (†) in the introduction.

## 2 Rational generating functions attached to order complexes

### 2.1

Recall that an *abstract simplicial complex* on a (finite) vertex set $V$ is, by definition, a collection $\Sigma$ of subsets of $V$, called *simplexes* of $\Sigma$, such that $\{v\} \in \Sigma$ for all $v$ in $V$, and $\sigma \subseteq \tau \in \Sigma$ implies that $\sigma$ is in $\Sigma$. (See [16] and [14] as general references for this section.) An abstract simplicial complex which is important in the study of Coxeter groups is the *nerve* of a Coxeter system $(W, S)$, which has vertex set $S$ and $\mathcal{N} = \{ J \subseteq S \mid |W_J| < \infty \}$ as its set of simplexes.

### 2.2

Fix an abstract simplicial complex $\Sigma$ on the finite vertex set $V$. A *subcomplex* of $\Sigma$ is a subset $\Sigma'$ of $\Sigma$ such that $\sigma \subseteq \tau \in \Sigma$ implies that $\sigma$ is in $\Sigma$; it is an abstract simplicial complex on vertex set $\cup_{\sigma \in \Sigma} \sigma$. The *link* in $\Sigma$ of a subset $\sigma$ of $V$ is the subcomplex $\mathrm{lk}(\sigma) = \mathrm{lk}_\Sigma(\sigma) := \{ \tau \in \Sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma \}$. Note the link is non-empty if and only of $\sigma$ is a simplex of $\Sigma$. For a simplex $\sigma$ in $\Sigma$, the corresponding *closed simplex* is the subcomplex $\overline{\sigma} = \{\tau \mid \tau \subseteq \sigma\}$.

A simplex $\sigma$ (of $\Sigma$) which is of cardinality $d$ is said to be of *dimension* $\dim(\sigma) = d - 1 = |\sigma| - 1$. The dimension of a non-empty simplicial complex is the maximum of the dimensions of the simplexes it contains.

The *reduced Euler characteristic* of $\Sigma$ is defined to be the integer

$$\widetilde{\chi}(\Sigma) := \sum_{\sigma\in\Sigma} (-1)^{\dim(\sigma)}.$$

Given abstract simplicial complexes $\Sigma_i$ for $i = 1, \dots, n$ on disjoint vertex sets $V_1, \dots, V_n$ their *join* is the abstract simplicial complex $\Sigma_1 * \cdots * \Sigma_n$ on vertex set $\dot\cup_{i=1}^n V_i$, with the subsets of the form $\dot\cup_i \sigma_i$, where each $\sigma_i$ is a simplex of $\Sigma_i$, as simplexes. Since $|\dot\cup_{i=1}^n \sigma_i| - 1 = n - 1 + \sum_{i=1}^n (|\sigma_i| - 1)$, one has

$$\widetilde{\chi}(\Sigma_1 * \cdots * \Sigma_n) = (-1)^{n-1}\, \widetilde{\chi}(\Sigma_1) \cdots \widetilde{\chi}(\Sigma_n). \tag{6}$$

## 2.3

Attach to each vertex $v$ in $V$ an indeterminate $y_v$. Define the polynomial ring $R := \mathbb{Z}[(y_v)_{v\in V}]$ and formal power series ring $\widehat{R} = \mathbb{Z}[[(y_v)_{v\in V}]]$. The field of fractions $\widetilde{R}$ of $\widehat{R}$ contains the field of rational functions $R_0 = \mathbb{Q}((y_v)_{v\in V})$ (i.e. the field of fractions of $R$) as a subring. We also regard $R \subseteq \widehat{R} \subseteq \widetilde{R}$ as subrings.

There is an automorphism $\phi_0$ of $R_0$ given by $y_v \mapsto x_v := \frac{y_v}{1+y_v}$ for $v$ in $V$, with inverse given by $y_v \mapsto \frac{y_v}{1-y_v}$ for all $v$. The map $y_v \mapsto x_v$ for $v$ in $V$ also extends to a (continuous) automorphism of $\widehat{R}$ given by $f((y_v)_{v\in V}) \mapsto f((x_v)_{v\in V})$ (where the right-hand side is the result of formally substituting the formal power series $x_v = \sum_{n\in\mathbb{N}} (-1)^n y_v^{n+1}$ for the variable $y_v$, for all $v$, in the formal power series $f = f((y_v)_{v\in V})$. This automorphism of $\widehat{R}$ extends to an automorphism $\phi$ of $\widetilde{R}$ which restricts to $\phi_0$ on $R_0$ (but does not map $R$ into itself if $V \neq \emptyset$).

One has $y_v = \phi^{-1}(x_v) = \frac{x_v}{1-x_v} = \sum_{n\in\mathbb{N}} x_v^{n+1}$ and $x_v = \sum_{n\in\mathbb{N}} (-1)^n y_v^{n+1}$. From above, one may when convenient regard $\widehat{R}$ as the formal power series ring $\widehat{R} = \mathbb{Z}[[(x_v)_{v\in V}]]$, $\widetilde{R}$ as its field of fractions, and $R_0$ as the function field $R_0 = \mathbb{Q}((x_v)_{v\in V})$ of indeterminates $x_v$ for $v$ in $V$.

There is also a ring automorphism of $R$ determined by $y_v \mapsto -y_v - 1$ for all $v$ in $V$. Denote its extension to an automorphism of the field $R_0$ as $\theta$ (it does not extend naturally to $\widehat{R}$ in general). One readily checks that $\theta$ is determined by the conditions $\theta(x_v) = x_v^{-1}$ for all $v$ in $V$; also, $\theta^2 = \mathrm{Id}_{R_0}$.

## 2.4

For any subset $\sigma = \{v_1, \dots, v_n\}$ of $V$, where the $v_i$ are distinct, define a corresponding monomial $Y_\sigma := y_{v_1} \cdots y_{v_n}$ in $R$. In particular, one has $Y_\emptyset = 1_R$. For any subset $\Gamma$ of the power set of $V$ define $F(\Gamma) := \sum_{\sigma\in\Gamma} Y_\sigma$. Note that both $F(\Gamma)$ and $\theta(F(\Gamma))$ are in the polynomial ring $R$, although they will usually be regarded as elements of $\widetilde{R} \cap R_0$, i.e. as rational formal power series, in the indeterminates $(x_v)_{v\in V}$.

In particular, $F(\Gamma)$ is defined for any simplicial complex with vertex set contained in $V$, and, as an element of the formal power series ring $\widehat{R}$, it has constant term 1 (and hence is invertible in $\widehat{R}$ and $R_0$) if $\Gamma$ is non-empty. It is easy to see that if $\Gamma_1$ and $\Gamma_2$ are complexes on disjoint vertex sets contained in $V$, then

$$F(\Gamma_1 * \Gamma_2) = F(\Gamma_1) * F(\Gamma_2). \tag{7}$$

If $\sigma$ above is a simplex of a simplicial complex $\Sigma$, one has

$$\theta(F(\{\sigma\}) = (-1)^n(y_{v_1}+1)\cdots(y_{v_n}+1) = (-1)^n \sum_{\tau\subseteq\sigma} Y_\tau = (-1)^{\dim(\sigma)+1}F(\overline{\sigma})$$

and hence $\theta(F(\Sigma)) = \sum_{\sigma\in\Sigma}(-1)^{\dim(\sigma)+1}F(\overline{\sigma})$. In terms of the monomial basis,

$$\theta(F(\Sigma)) = \sum_{\sigma\in\Sigma}\sum_{\tau\subseteq\sigma}(-1)^{|\sigma|}Y_\tau = \sum_{\tau\in\Sigma}\Bigg(\sum_{\substack{\sigma\in\Sigma\\ \sigma\supseteq\tau}}(-1)^{|\sigma\setminus\tau|-1}\Bigg)(-1)^{|\tau|-1}Y_\tau.$$

In the inner right sum, $\sigma\setminus\tau$ runs over $\mathrm{lk}(\tau)$ and $|\sigma\setminus\tau|-1 = \dim(\sigma\setminus\tau)$. Hence

$$\theta(F(\Sigma)) = \sum_{\tau\in\Sigma}(-1)^{\dim(\tau)}\,\widetilde{\chi}(\mathrm{lk}_\Sigma(\tau))Y_\tau. \tag{8}$$

Applying $\theta$ again gives the expression $F(\Sigma) = -\sum_{\tau\in\Sigma}\widetilde{\chi}(\mathrm{lk}_\Sigma(\tau))F(\overline{\tau})$ for $F(\Sigma)$ in terms of closed simplexes. Interpreting (8) as an equation in $\widehat{R} = \mathbb{Z}[[(x_v)_{v\in V}]]$ shows (see the proof of [14, Ch II,Theorem 1.4]) that it is equivalent to [14, Ch II, Theorem 7.1] on the Hilbert series in fine grading of the face ring of $\Sigma$, although we shall not use this fact (the proof here is essentially the same as in loc. cit., except expressed in terms of variables $(y_v)$ instead of $(x_v)$).

Our concern here is the following consequence of (8): if $\Sigma$ is non-empty, then

$$\begin{aligned}\theta(F(\Sigma)) = \pm F(\Sigma) &\iff \theta(F(\Sigma)) = (-1)^{\dim(\Sigma)+1}F(\Sigma)\\ &\iff \widetilde{\chi}(\mathrm{lk}_\Sigma(\tau)) = (-1)^{\dim(\Sigma)-\dim(\tau)-1} \text{ for all } \tau \text{ in } \Sigma.\end{aligned} \tag{9}$$

Indeed, $\theta(F(\Sigma)) = \epsilon F(\Sigma)$ with $\epsilon$ in $\{\pm 1\}$ if and only if $(-1)^{\dim(\tau)}\,\widetilde{\chi}(\mathrm{lk}_\Sigma(\tau)) = \epsilon$ for all simplexes $\tau$ of $\Sigma$, and then one must have $\epsilon = (-1)^{\dim(\Sigma)+1}$ by taking $\tau$ of maximal dimension in $\Sigma$ (since then $\dim(\tau) = \dim(\Sigma)$, $\mathrm{lk}(\tau) = \{\emptyset\}$ and $\widetilde{\chi}(\mathrm{lk}(\tau)) = -1$). The final condition in 9 may be phrased in terms of Euler spheres (cf. the proof of Theorem 1.1(g) for a reference to the definition).

## 2.5

We now specialize to the case of order complexes of finite posets. For any poset $\Omega$ and any $x \le y$ in $\Omega$, we let $[x,y] = [x,y]_\Omega = \{\, z \in \Omega \mid x \le z \le y \,\}$ and $(x,y)_\Omega = \{\, z \in \Omega \mid x < z < y \,\}$ denote the corresponding closed and open interval, respectively. It is frequently convenient to adjoin to $\Omega$ a maximum element, which we denote as $\infty = \infty_\Omega$, and a minimal element, which we denote as $-\infty = -\infty_\Omega$, to form a new poset $\widehat{\Omega} = \Omega \mathbin{\dot\cup} \{-\infty,\infty\}$ such that $\Omega$ has the order induced as subposet of $\widehat{\Omega}$, $-\infty < \infty$ and $(-\infty,\infty)_{\widehat{\Omega}} = \Omega$. An *order ideal* of $\Omega$ is a subset $\Lambda$ of $\Omega$, given the induced order from $\Omega$, such that $x \le y$ in $\Omega$ and $y$ in $\Lambda$ imply that $x$ is in $\Lambda$.

**2.6**

Fix a finite poset $\Omega$. Recall that a chain of $\Omega$ is a subset of $\Omega$ which is totally ordered in the induced order from $\Omega$. A chain $\{v_1, \ldots, v_n\}$ with $v_1 < \cdots < v_n$ is said to have length $n-1$ and, if $n > 0$, it is said to be a chain from $v_1$ to $v_n$. The *order complex* $\Delta(\Omega)$ of $\Omega$ is defined to be the abstract simplicial complex with vertex set $\Omega$ and with the chains of $\Omega$ as simplexes. One says a chain is a maximal chain of $\Omega$ if it is inclusion-maximal as a simplex of $\Delta(\Omega)$ i.e. if it is not properly contained in any other chain of $\Omega$.

We take $V := \Omega$ in 2.3–2.4, so $F(\Delta)$ is a rational power series in $(x_v)_{v\in V}$ for any complex with vertices in $V$. From the definitions, one has

$$\begin{aligned} F(\Delta(\Omega)) &= \sum_{n\in\mathbb{N}} \sum_{v_1<\cdots<v_n} y_{v_1}\cdots y_{v_n} = \sum_{n\in\mathbb{N}} \sum_{v_1<\cdots<v_n} \frac{x_{v_1}\cdots x_{v_n}}{(1-x_{v_1})\cdots(1-x_{v_n})} \\ &= \sum_{n\in\mathbb{N}} \sum_{\substack{v_1<\cdots<v_n \\ k_1,\ldots,k_n\in\mathbb{N}_{>0}}} x_{v_1}^{k_1}\cdots x_{v_n}^{k_n} = \sum_{n\in\mathbb{N}} \sum_{v_1\leq\cdots\leq v_n} x_{v_1}\cdots x_{v_n} \end{aligned} \tag{10}$$

where the $v_i$ run over vertices of $\Delta(\Omega)$ (i.e. elements of $\Omega$). Note that this formula shows that $F(\Delta(\Omega))$ may be regarded as a generating function for the set of chains $v_1 < \cdots < v_n$ in $\Omega$ (when expressed in terms of the elements $y_v$ of $R$ for $v$ in $\Omega$) or as a generating function for the set of multichains $v_1 \leq \cdots \leq v_n$ in $\Omega$ (when expressed in terms of the elements $x_v$ of $\widehat{R}$).

**2.7**

Write $F'(\Delta(\Omega)) = F(\Delta(\Omega))\prod_{v\in\Omega}(1-x_v)$, so that $F(\Delta(\Omega)) = \frac{F'(\Delta(\Omega))}{\prod_{v\in\Omega}(1-x_v)}$. It is clear from (10) that $F' \in \mathbb{Z}[(x_v)_{v\in V}]$ is a polynomial with integral coefficients in the indeterminates $x_v$, and we shall now determine them explicitly using a known determinantal formula from [15] for $F(\Delta(\Omega))$ as an element of $\mathbb{Z}[(y_v)_{v\in\Omega}]$.

Let $A = A_\Omega$ be the $\Omega \times \Omega$ matrix with entries in $R_0$ given by

$$A_{v,v'} = \begin{cases} 1, & \text{if } v \not\leq v' \\ 0, & \text{if } v \leq v' \end{cases} \tag{11}$$

for all $v, v' \in \Omega$. Let $D$ (respectively, $D'$) be the $\Omega \times \Omega$ diagonal matrix over $R_0$ with entries given by $D_{v,v'} = \delta_{v,v'} y_v$ (respectively, $D'_{v,v'} = \delta_{v,v'} x_v$) where $\delta$ denotes the Kronecker delta. Finally, let $I = \mathrm{Id}_\Omega$ denote the $\Omega \times \Omega$ identity matrix over $R_0$.

**Theorem 2.1.** *(a)* $F(\Delta(\Omega)) = \mathrm{Det}(I + D(I + A))$.

*(b) Set* $B := (I - D')(I + D(I + A))$. *Then* $B = I + D'A$ *and* $F'(\Delta(\Omega)) = \mathrm{Det}(B)$.

*(c)* $F'(\Delta(\Omega)) = \sum_{\Lambda\subseteq\Omega}(-1)^{|\Lambda|+1}\,\widetilde{\chi}(\Delta(\Lambda))x_\Lambda$ *where* $x_\Lambda := \prod_{v\in\Lambda} x_v$.

*Proof.* (a) This is stated, with a sketch of a proof, in [16, Ch 3, Exercise 22]. A more general result appears in [15], and is itself a special case of [10, Lemma 3.12]. We sketch a proof following [16]. Let $z$ be an indeterminate over $R_0$. It will suffice to show that

$$\mathrm{Det}(I + zD(I + A)) = \sum_{\sigma\in\Delta(\Omega)} z^{|\sigma|}Y_\sigma \tag{12}$$

in $R_0[z]$. This is shown using the fact that for any $\Omega\times\Omega$ matrix $M$ over $R_0$, one has $\mathrm{Det}(I + zM) = \sum_{\sigma\subseteq\Omega} z^{|\sigma|}\,\mathrm{Det}(M[\sigma])$ where $M[\sigma]$ is the $\sigma\times\sigma$-principal submatrix of $M$ (i.e. $(M[\sigma])_{v,v'} = M_{v,v'}$ for $v, v'\in\sigma$) and by convention, the $\emptyset\times\emptyset$ matrix has determinant 1. Using this with $M = D(I + A)$ shows that

$$\mathrm{Det}(I + zD(I + A)) = \sum_{\sigma\subseteq\Omega} z^{|\sigma|}\,\mathrm{Det}(D[\sigma])\,\mathrm{Det}((I + A)[\sigma]).$$

Now $\mathrm{Det}(D[\sigma]) = \prod_{v\in\sigma} y_v = Y_\sigma$ and $A[\sigma] = A_\sigma$ where $\sigma$ is regarded as poset in the induced order, so it suffices to show that

$$\mathrm{Det}(I + A) = \begin{cases} 1, & \text{if } \sigma\in\Delta(\Omega) \text{ i.e. } \sigma \text{ is a chain in } \Omega\\ 0, & \text{otherwise.}\end{cases} \tag{13}$$

Choose $v_1,\dots,v_n$ in $\Omega$ recursively so that $v_i$ is the maximum element of the poset $\Omega\setminus\{v_1,\dots,v_{i-1}\}$, and $n\in\mathbb{N}$ is maximal subject to this. If $n = |\Omega|$ then $\Omega$ is a chain $v_n < \cdots < v_1$. In this case, $(I + A)_{v_i,v_j}$ is equal to 1 if $i\le j$ and 0 otherwise, so $\det(I + A) = 1$. Otherwise, one has $n < |\Omega|$ and $\Omega\setminus\{v_1,\dots,v_n\}$ has at least two maximal elements, say $v'$ and $v''$. One has $(I + A)_{v',v_i} = (I + A)_{v'',v_i} = 0$ for $i = 1,\dots,n$ and $(I+A)_{v',v} = (I+A)_{v'',v} = 1$ for all $v\in\Omega\setminus\{v_1,\dots,v_n\}$. In this case, $\Omega$ is not a chain and the $v'$-th and $v''$-th rows of $I + A$ are equal, so $\det(I + A) = 0$.

(b) Since $(1 - x_v)y_v = (1 - x_v)\frac{x_v}{1-x_v} = x_v$, one has $(I - D')D = D'$ and so $(I - D')(I + D) = I$. Hence $B = (I - D')((I + D) + DA)) = I + D'A$. Since $\det(I - D') = \prod_{v\in\Omega}(1 - x_v)$, one has $\det(B) = \bigl(\prod_{v\in\Omega}(1 - x_v)\bigr)F(\Delta(\Omega)) = F'(\Delta(\Omega))$ by (a) and the definitions.

(c) It suffices to show that

$$\mathrm{Det}(I + zD'A) = \sum_{\Lambda\subseteq\Omega}(-1)^{|\Lambda|+1}z^{|\Lambda|}\,\widetilde{\chi}(\Delta(\Lambda))x_\Lambda. \tag{14}$$

As in (a), the left-hand side is

$$\sum_{\Lambda\subseteq\Omega} z^{|\Lambda|}\,\mathrm{Det}(D'[\Lambda])\,\mathrm{Det}(A[\Lambda]) = \sum_{\Lambda\subseteq\Omega} z^{|\Lambda|}x_\Lambda\,\mathrm{Det}(A[\Lambda])$$

and it suffices to show that $\mathrm{Det}(A[\Lambda]) = (-1)^{|\Lambda|+1}\,\widetilde{\chi}(\Delta(\Lambda))$. Since $A[\Lambda] = A_\Lambda$, we may assume $\Lambda = \Omega$. But specializing $z\mapsto -1$ and $y_v\mapsto 1$ for all $v$ in $V$ in the identity (12) shows that $(-1)^{|\Omega|}\,\mathrm{Det}(A) = \mathrm{Det}(-A) = \sum_{\sigma\in\Delta(\Omega)}(-1)^{|\sigma|} = -\,\widetilde{\chi}(\Delta(\Omega))$ as required. There is also a simple direct proof of (c) from (10) which we leave to the interested reader. □

**2.8**

We now recall some salient properties of the Möbius function $\mu = \mu_\Omega$ of the finite poset $\Omega$. Recall that the Möbius function may be regarded as the unique function $\mu\colon \{\, (x,y) \in \Omega \times \Omega \mid x \leq y \,\} \to \mathbb{Z}$ such that the following conditions hold:

(i) $\mu(x,x) = 1$ for all $x$ in $\Omega$.

(ii) $\sum_{y \in [x,z]} \mu(x,y) = 0$ if $x < z$ in $\Omega$.

(iii) $\sum_{y \in [x,z]} \mu(y,z) = 0$ if $x < z$ in $\Omega$.

(iv) $\mu(x,y) = \widetilde{\chi}(\Delta((x,y)_\Omega)$ if $x < y$ in $\Omega$.

It is well known and easily seen (see [16]) that $\mu$ is uniquely determined by (i) together with any one of the other three conditions (ii)–(iv) above.

Let $\sigma = \{v_1, \dots, v_n\}$ be any simplex of $\Delta(\Omega)$, where $n$ is in $\mathbb{N}$ and $v_1 < \cdots < v_n$ in $\Omega$. Set $v_0 := -\infty$ and $v_{n+1} := \infty$. One sees at once that

$$\mathrm{lk}_{\Delta(\Omega)}(\sigma) = \Delta((v_0, v_1)_\Omega) * \Delta((v_1, v_2)_\Omega) * \cdots * \Delta((v_n, v_{n+1})_\Omega) \tag{15}$$

and hence, by (6), one has

$$\widetilde{\chi}(\mathrm{lk}_{\Delta(\Omega)}(\sigma)) = (-1)^{\dim(\sigma)+1} \prod_{i=1}^{n+1} \mu(v_{i-1}, v_i). \tag{16}$$

By (9), $\theta(F(\Delta(\Omega))) = (-1)^{\dim(\Delta(\Omega))+1} F(\Delta(\Omega))$ holds if and only if one has $\prod_{i=1}^{n+1} \mu(v_{i-1}, v_i) = (-1)^{\dim(\Delta(\Omega))}$ for all $n$ in $\mathbb{N}$ and $v_1 < \cdots < v_n$ in $\Omega$, where $v_0 := -\infty$ and $v_{n+1} := \infty$.

**2.9**

A (finite) poset $\Lambda$ is said to be *lower Eulerian* if it has the following properties:

(i) It has a minimum element $m = m_\Lambda$, and for any $x$ in $\Lambda$, all maximal chains from $m$ to $x$ (i.e. the maximal chains of $[m, x]$) have the same length $r(x)$.

(ii) For any $x \leq y$ in $\Lambda$, one has $\mu(x,y) = (-1)^{r(y)-r(x)}$.

The function $r\colon \Lambda \to \mathbb{N}$ is called the *rank function* of $\Lambda$. One says that $\Lambda$ is *Eulerian* if it is a lower Eulerian poset with a maximum element $M = M_\Lambda$. Note that in any lower Eulerian poset $\Lambda$, and for any $x \leq y$ in $\Lambda$, any maximal chain from $x$ to $y$ has length $r(y) - r(x)$ and $[x, y]$ is an Eulerian poset (in the induced order from $\Lambda$).

For example, it is well known and easily seen that any finite Boolean interval (i.e. a poset isomorphic to the inclusion-ordered power set $\mathcal{P}(X)$ of a finite set $X$) is an Eulerian poset. It follows that for any non-empty finite simplicial complex, with vertex set $V$, the poset of its simplexes, ordered by inclusion, is lower Eulerian. In fact, it is an order ideal in the (Eulerian) poset $\mathcal{P}(V)$.

**2.10**

For the rest of this section, $\Omega'$ denotes a lower Eulerian poset with minimal element $m := m_{\Omega'}$. From 2.12 onwards, it will also be assumed that $\Omega'$ is an order ideal in a specified Eulerian poset $\widetilde{\Omega}$. Set $\Omega := \Omega' \setminus \{m\}$ and identify $\widehat{\Omega} = \Omega \,\dot\cup\, \{-\infty, \infty\}$ with the poset $\Omega' \dot\cup \{\infty\}$ (obtained by adjoining a maximum element $\infty$ to $\Omega'$) by identifying $-\infty = m$. For $v$ in $\Omega'$, let $r(v)$ denote the maximal length of a chain in $\widehat{\Omega}$ from $m$ to $v$, and set $\epsilon_v := (-1)^{r(v)}$. For any $v$ in $\widehat{\Omega}$, write $\Omega_{<v} := \{\, x \in \Omega \mid x < v \,\}$ and $\Omega_{\leq v} := \{\, x \in \Omega \mid x \leq v \,\}$. The construction of 2.3–2.4 will be applied to various (order) complexes $\Sigma$ on vertex sets contained in $V := \Omega$, regarding the resulting elements $F(\Sigma)$ as rational power series in $\widehat{R} = R[[(x_v)_{v \in \Omega}]]$. Also set $x_m := 1_{\widehat{R}}$ for convenience in certain formulae below.

Theorem 2.2 below states analogs in the context here of several of the main results from Section 1 (see 2.11). The equality of the rational functions $F_\infty$ in 2.2(a) and $F(\Delta(\Omega_{\leq v}))$ in 2.2(f) in the case $v$ is a maximal element of $\Omega$ generalizes the identity (†) of the introduction. These identities (and more generally, [14, Theorem 7.1]) have many interesting (known) specializations, which it would be too much of a digression to list here.

Summation indices $v_i$ below run over $\Omega'$ unless otherwise indicated. For convenience, Table 1 provides a dictionary of notational correspondences between §1 and §2; some of the notation it lists is introduced in the following subsections.

| Section §1 | §2 | Section §1 | §2 |
|---|---|---|---|
| $\mathcal{N}$ | $\Omega'$ | $(P_W)^{-1}$ | $F_\infty := F(\Delta(\Omega))$ |
| $\mathcal{N}' = \mathcal{N} \setminus \{\emptyset\}$ | $\Omega = \Omega' \setminus \{m\}$ | $((P_{W_J})^{-1})_{J \in \mathcal{N}}$ | $(F(\Delta(\Omega_{\leq v})))_{v \in \Omega'}$ |
| $\emptyset \in \mathcal{N}$ | $m = m_{\Omega'} \in \Omega'$ | $((1 - x_J) P_{W_J}^{-1})_{J \in \mathcal{N}'}$ | $(F(\Delta(\Omega_{<v})))_{v \in \Omega}$ |
| $\mathcal{N} \dot\cup \{\infty\}$ | $\widehat{\Omega} = \Omega' \dot\cup \{\infty\}$ | $\mathcal{P}(S)$ | $\widetilde{\Omega}$ |
| $(\mu(J, \infty))_{J \in \mathcal{N}}$ | $(\mu(v, \infty))_{v \in \Omega'}$ | $S \in \mathcal{P}(S)$ | $M = M_{\widetilde{\Omega}} \in \widetilde{\Omega}$ |
| $\mathbb{Z}[t]$, $\mathbb{Z}[[t]]$, $\mathbb{Q}(t)$ | $R$, $\widehat{R}$, $R_0$ | $(x_J = 0)_{J \in \mathcal{P}(S) \setminus \mathcal{N}}$ | $(x_v = 0)_{v \in \widetilde{\Omega} \setminus \Omega'}$ |
| $\tau$ | $\theta$ | $(\lvert J \rvert)_{J \in \mathcal{P}(S)}$ | $(r(v))_{v \in \widetilde{\Omega}}$ |
| $(\epsilon_J)_{J \in \mathcal{N}}$ | $(\epsilon_v)_{v \in \Omega'}$ | $(\epsilon_J)_{J \in \mathcal{P}(S)}$ | $(\epsilon_v)_{v \in \widetilde{\Omega}}$ |
| $(x_J)_{J \in \mathcal{N}}$ | $(x_v)_{v \in \Omega'}$ | $((P_{W_J})^{-1})_{J \in \mathcal{P}(S)}$ | $(F_q)_{q \in \widetilde{\Omega}}$ |
| $x_\emptyset = 1$ | $x_m = 1$ | $((P_{W^J})^{-1})_{J \in \mathcal{P}(S)}$ | $(F^q)_{q \in \widetilde{\Omega}}$ |

Table 1: Table of correspondence of notations between §1 and §2. Correspondences involving $\widetilde{\Omega}$ require additional assumptions as in 2.12. Note $F_q := F(\Delta(\Omega_{\leq q}))$ and $F^q := (F_q)^{-1} F_\infty$ for $q \in \widetilde{\Omega}$.

**Theorem 2.2.** *(a) One has*

$$F_\infty := F(\Delta(\Omega)) = \sum_{n \in \mathbb{N}} \sum_{m = v_0 < \cdots < v_n} \frac{x_{v_1} \cdots x_{v_n}}{(1 - x_{v_1}) \cdots (1 - x_{v_n})}.$$

*(b) If* $\widehat{\Omega}$ *is Eulerian, then* $\theta(F_\infty) = (-1)^{\dim(\Delta(\Omega))+1} F_\infty$.

*(c) For any* $v$ *in* $\Omega$*, one has* $\theta(F(\Delta(\Omega_{<v}))) = -\epsilon_v F(\Delta(\Omega_{<v}))$.

*(d) If* $v$ *is in* $\Omega$*, then* $F(\Delta(\Omega_{<v})) = (1 - x_v) F(\Delta(\Omega_{\leq v}))$.

*(e) For all* $v$ *in* $\Omega'$*, one has* $\theta(F(\Delta(\Omega_{\leq v}))) = \epsilon_v x_v F(\Delta(\Omega_{\leq v}))$.

*(f) If* $v$ *is in* $\Omega'$*, then*

$$F(\Delta(\Omega_{\leq v})) = \sum_{n \in \mathbb{N}} \sum_{m=v_0<\cdots<v_n=v} \frac{(-1)^n \epsilon_v}{(1 - x_{v_1}) \cdots (1 - x_{v_n})}.$$

*(g)* $F_\infty = \sum_{v \in \Omega'} \epsilon_v \theta(F(\Delta(\Omega_{\leq v})))$.

*(h)* $F_\infty = -\sum_{v \in \Omega'} \mu_{\widehat{\Omega}}(v, \infty) F(\Delta(\Omega_{\leq v}))$.

*Proof.* (a) This is part of (10) in the special case here.

(b) This follows from the last paragraph of 2.8. One needs only to note that for $v_0, \dots, v_{n+1}$ as there, one has

$$\begin{aligned} \prod_{i=1}^{n+1} \mu(v_{i-1}, v_i) &= \prod_{i=1}^{n+1} (-1)^{r(v_i)-r(v_{i-1})} = (-1)^{r(v_{n+1})-r(v_0)} = (-1)^{r(\infty)-r(m)} \\ &= (-1)^{r(\infty)} = (-1)^{r(\infty)-2} = (-1)^{\dim(\Delta(\Omega)))}. \end{aligned}$$

(c) This may be proved by applying (b) with $\widehat{\Omega}$ replaced by the Eulerian poset $[m, v] = \Omega_{\leq v} \cup \{m\}$ (with the maximum element $v$ playing the role of $\infty$), noting $\dim(\Delta(\Omega_{<v})) = r(v) - 2$.

(d) One way to prove this is to note that $\Delta(\Omega_{\leq v})$ is the cone on vertex $v$ over $\Delta(\Omega_{<v})$ (i.e. it is the join of $\Delta(\Omega_{<v})$ and the closed simplex $\overline{\{v\}}$) and use (7). Alternatively, note that the subsum on the right from the formula in (a) (taking $\Omega = \Omega_{\leq v}$) with $v_n < v$ is $F(\Delta(\Omega_{<v}))$ and that with $v_n = v$ is $\frac{x_{v_n}}{1-x_{v_n}} F(\Delta(\Omega_{<v}))$.

(e) Both sides are 1 if $v = m$. If $v \neq m$, the result follows from (c)–(d) since $\theta(x_v) = x_v^{-1}$.

(f) If $v = m$, both sides are equal to 1. Otherwise, one has

$$\begin{aligned} F(\Delta(\Omega(\leq v))) &= \frac{1}{1 - x_v} F(\Delta(\Omega_{<v})) = \frac{-\epsilon_v}{1 - x_v} \theta(F(\Delta(\Omega_{<v}))) \\ &= \frac{-\epsilon_v}{1 - x_v} \sum_{n \in \mathbb{N}} \sum_{m=v_0<\cdots<v_n<v} \frac{x_{v_1}^{-1} \cdots x_{v_n}^{-1}}{(1 - x_{v_1}^{-1}) \cdots (1 - x_{v_n}^{-1})} \\ &= \sum_{n \in \mathbb{N}} \sum_{m=v_0<\cdots<v_n<v} \frac{(-1)^{n+1} \epsilon_v}{(1 - x_{v_1}) \cdots (1 - x_{v_n})(1 - x_v)} \end{aligned}$$

using (d), (c) and (a) for the first, second and third equality respectively. The final expression is equal to the right-hand side of the equation in (f), because the inner sum there is zero if $n = 0$ since $v \neq m$.

(g) By (f), the right-hand side of (g) is equal to

$$\sum_{v \in \Omega'} \sum_{n \in \mathbb{N}} \sum_{m=v_0<\cdots<v_n=v} \frac{(-1)^n}{(1-x_{v_1}^{-1})\cdots(1-x_{v_n}^{-1})} \\ = \sum_{n \in \mathbb{N}} \sum_{v=v_0<\cdots<v_n} \frac{x_{v_1}\cdots x_{v_n}}{(1-x_{v_1})\cdots(1-x_{v_n})}$$

and the desired equality follows by (a).

(h) This is proved by the computation

$$F_\infty = \sum_{v \in \Omega'} \epsilon_v \theta(F(\Delta(\Omega_{\leq v}))) = \sum_{v \in \Omega'} \Bigg( \sum_{\substack{v'' \in \Omega' \\ v'' \geq v}} \sum_{\substack{v' \in \Omega' \\ v \leq v' \leq v''}} \mu(v', v'') \Bigg) \epsilon_v \theta(F(\Delta(\Omega_{\leq v})))$$
$$= \sum_{v' \in \Omega'} \Bigg( \sum_{\substack{v'' \in \Omega' \\ v'' \geq v'}} \mu(v', v'') \Bigg) \Bigg( \sum_{\substack{v \in \Omega' \\ v \leq v'}} \epsilon_v \theta(F(\Delta(\Omega_{\leq v}))) \Bigg)$$
$$= \sum_{v' \in \Omega'} -\mu(v', \infty) F(\Delta(\Omega_{\leq v'}))$$

in which the first equality holds by (g), the second is from 2.8(iii), the third follows by change of order of summation and the last equality follows from 2.8(ii) and (g) (applied to $\Omega_{\leq v'}$ instead of $\Omega$). □

## 2.11

According to Table 1, Theorem 2.2(a) is an analog of Theorem 1.2(b), and Theorem 2.2(f) is an analog of (5) for $K$ in $\mathcal{N}$. Theorem 2.2(e) (respectively, (g), (h)) corresponds to Theorem 1.1(b) (respectively, (d), (e)). For the rest of this section, we discuss the remaining "missing" analogs of results from Sections 1 and 2. Parts (a)–(b) of the next result give analogs of Theorem 1.1(f)–(g).

**Corollary 2.3.** *(a) The constant term of the formal power series expansion of $\theta(F_\infty)$ in $R = \mathbb{Z}[[(x_v)_{v \in V}]]$ is $-\widetilde{\chi}(\Delta(\Omega)) = -\mu_{\widehat{\Omega}}(m, \infty)$.*

*(b) If there is an integer $N$ such that $\mu(v, \infty) = (-1)^N \epsilon_v$ for all $v$ in $\Omega'$, then $\theta(F_\infty) = (-1)^{N+1} F_\infty$.*

*Proof.* (a) Since $\theta \frac{x_{v_i}}{1-x_{v_i}} = \frac{1}{x_{v_i}-1}$, Theorem 2.2(a) implies

$$\theta(F_\infty) = \sum_{n \in \mathbb{N}} \sum_{m=v_0<\cdots<v_n} \frac{1}{(x_{v_1}-1)\cdots(x_{v_n}-1)}$$

where the right-hand side is in $\widehat{R}$ and, using 2.8(iv), has constant term

$$\sum_{n\in\mathbb{N}}\sum_{m=v_0<\cdots<v_n}(-1)^n=\sum_{\sigma\in\Delta(\Omega)}(-1)^{\dim(\sigma)+1}=-\widetilde{\chi}(\Delta(\Omega))=-\mu(m,\infty).$$

(b) This follows by applying $\theta$ to 2.2(g) and comparing with 2.2(h). □

## 2.12

Assume henceforward that $\Omega'$ is an order ideal of an Eulerian poset $\widetilde{\Omega}$. For $q$ in $\widetilde{\Omega}$, let $\Omega_{\leq q} := \{\, p\in\Omega \mid p\leq q\,\}$ and $\Omega_{<q} := \{\, p\in\Omega \mid p<q\,\}$, extending the notation introduced previously for $q$ in $\Omega'$. The rank function of $\widetilde{\Omega}$ extends the rank function $r$ of $\Omega'$ and will still be denoted as $r$. Let $\epsilon_q=(-1)^{r(q)}$ for $q$ in $\widetilde{\Omega}$ (which extends the definition of $\epsilon_p$ for $p$ in $\Omega'$). Since $\widetilde{\Omega}$ is Eulerian, one has $\mu_{\widetilde{\Omega}}(x,y)=\epsilon_x\epsilon_y$ for $x\leq y$ in $\widetilde{\Omega}$. Let $M := M_{\widetilde{\Omega}}$ denote the maximum element of $\widetilde{\Omega}$. For $q$ in $\widetilde{\Omega}$, define $F_q := F(\Delta(\Omega_{\leq q}))$ and $F^q := \frac{F_\infty}{F_q}$. Set $x_v := 0$ for all $v\in\widetilde{\Omega}\setminus\Omega'$. This explains the notations involving $\widetilde{\Omega}$ in Table 1. These additional entries in the table are consistent with the others where they overlap; in particular, the two entries corresponding to $\frac{1}{P_{W_J}}$ for $J\in\mathcal{N}$ are the same. Note also that $F_M=F_\infty$ since $\Omega_{\leq M}=\Omega$.

One obtains from above an analog ($F_qF^q=F_\infty$ for $q\in\widetilde{\Omega}$) of Theorem 1.1(a) just by definition of $F^q$; it is not clear if there is an independent description of $F^q$ which would make this equation a theorem instead. Analogs of Theorem 1.2(a) and Theorem 1.1(c) are given by parts (a) and (b) of the following result.

**Theorem 2.4.** *(a) One has*

$$F_\infty=\sum_{n\in\mathbb{N}}\sum_{m=v_0<\cdots<v_n=M}\frac{(-1)^n\epsilon_M}{(1-x_{v_1})\cdots(1-x_{v_n})}$$

*where the $v_i$ range over $\widetilde{\Omega}$ and $x_v := 0$ for $v\in\widetilde{\Omega}\setminus\Omega$.*

*(b) One has*

$$\sum_{q\in\widetilde{\Omega}}\epsilon_qF_q=\begin{cases}\epsilon_Mx_MF_\infty, & \text{if } M\in\Omega' \text{ i.e. } \Omega'=\widetilde{\Omega}\\ 0, & \text{otherwise.}\end{cases}$$

*Proof.* (a) Set $\Lambda := \widetilde{\Omega}\setminus\{m\}$ and $\Lambda' := \widetilde{\Omega}$. Applying Theorem 2.2(a),(f) to $\Lambda$ instead of $\Omega$ gives identities

$$F(\Delta(\Lambda))=\sum_{n\in\mathbb{N}}\sum_{m=v_0<\cdots<v_n}\frac{x_{v_1}\cdots x_{v_n}}{(1-x_{v_1})\cdots(1-x_{v_n})}$$

and

$$F(\Delta(\Lambda_{\leq M}))=\sum_{n\in\mathbb{N}}\sum_{m=v_0<\cdots<v_n=M}\frac{(-1)^n\epsilon_M}{(1-x_{v_1})\cdots(1-x_{v_n})}$$

where $(x_v)_{v\in\Lambda}$ are indeterminates and the $v_i$ range over $\Lambda'$. Since $\Lambda_{\leq M} = \Lambda$, one has $F(\Delta(\Lambda)) = F(\Delta(\Lambda_{\leq M}))$. Now substitute $x_v = 0$ for all $v \in \Lambda \setminus \Omega'$ in these identities. By Theorem 2.2(a) applied to $\Omega$, $F(\Delta(\Lambda))$ specializes to $F_\infty$ under this substitution, while $F(\Delta(\Lambda_{\leq M}))$ obviously specializes to the right-hand side of the identity in (a) which is to be proved. The result follows.

(b) By Theorem 2.2(g) (applied to the posets $\Omega_{\leq q}$ for $q \in \widetilde{\Omega}$ instead of $\Omega$) the left-hand side is

$$\begin{aligned} L &:= \sum_{q\in\widetilde{\Omega}} \epsilon_q F(\Delta(\Omega_{\leq q})) = \sum_{q\in\widetilde{\Omega}} \epsilon_q \sum_{\substack{v\in\Omega' \\ v\leq q}} \epsilon_v \theta(F(\Delta(\Omega_{\leq v}))) \\ &= \sum_{v\in\Omega'} \Bigg( \sum_{\substack{q\in\widetilde{\Omega} \\ M\geq q\geq v}} \mu_{\widetilde{\Omega}}(v,q) \Bigg) \theta(F(\Delta(\Omega_{\leq v}))). \end{aligned}$$

The inner sum over $q$ is zero unless $M = v \in \Omega'$, in which case it is 1. Hence if $M$ is not in $\Omega'$, then $L = 0$, and the "otherwise" case is proved. Assume now that $M$ is in $\Omega'$. This obviously is equivalent to $\Omega' = \widetilde{\Omega}$ and implies $\Omega = \Omega_{\leq M}$. In this case, one has $L = \theta(F(\Delta(\Omega_{\leq M}))) = \epsilon_M x_M F_\infty$ by Theorem 2.2(e), as required. □

## 2.13

All the results on Poincaré series of $P_W$ in Section 1 may now be deduced from Theorem 1.2(b) by specialization of results of this section. To be explicit, given the finite rank Coxeter system $(W, S)$, we take $\Omega' := \mathcal{N}$ (the nerve of $(W, S)$ as in Section 1) and $\widetilde{\Omega} = \mathcal{P}(S)$ (the power set of $S$, as Eulerian poset) with $\mathcal{N}$ as order ideal. One considers the specialization given by $x_J \mapsto (-1)^{|J|} X_{w(J)}$, where $X_{w(J)}$ is as in Section 1, for $J \in \mathcal{N}$. Table 2 then indicates how the rational functions $F_\infty$, $F_q$, $F^q$ etc. specialize to rational functions naturally associated to $(W, S)$. We observe that specialization of Theorem 2.1 leads to some other apparently new formulae for $P_W$, but shall not list them explicitly.

## 2.14

For completeness, we conclude this paper by discussing the translation of Theorem 2.2(c)–(d) back to the setting of Section 1. Theorem 2.2(c) suggests the entry in Table 1 according to which $\frac{1-x_J}{P_{W_J}}$ for $J \in \mathcal{N} \setminus \{\emptyset\}$ corresponds to $F(\Delta(\Omega_{<v}))$ for $v \in \Omega$. Theorem 2.2(d) then corresponds to the statement in the setting of Section 1 that for such $J$, one has $\frac{1-x_J^{-1}}{\tau(P_{W_J})} = -\epsilon_J \frac{1-x_J}{P_{W_J}}$, which is readily checked from Theorem 1.1(b).

Note that this last correspondence does not extend as stated to one for $J \in \mathcal{N}$ and $v \in \Omega'$, since $\frac{1-x_\emptyset}{P_{W_\emptyset}} = 0$, which is not the relevant specialization (namely, 1) of $F(\Delta(\Omega_{<m})) = F(\{\emptyset\}) = 1$. This may be fixed by setting $\Omega'_{<v} := \{\, x \in \Omega' \mid x < v \,\}$ for $v \in \Omega'$ and using $\mathrm{lk}_{\Delta(\Omega'_{<v})}(\{m\})$ instead of $\Delta(\Omega_{<v})$ from 2.2 on (note they differ

only for $v = m$); then Theorem 2.2(c)–(d) hold for all $v \in \Omega'$. Using also $\Delta(\Omega) = \mathrm{lk}_{\Delta(\Omega')}(\{m\})$ and $\Delta(\Omega_{\leq v}) = \mathrm{lk}_{\Delta(\Omega'_{\leq v})}(\{m\})$ for $v \in \Omega'$, where $\Omega'_{\leq v} := \{\, x \in \Omega' \mid x \leq v \,\}$, permits a more natural formulation of the results in terms of the lower Eulerian poset $\Omega'$ than in terms of its subposet $\Omega$, as we have given in this note.

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