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List proper connection of 2-edge-connected graphs

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Abstract

Given any colouring of the edges of a graph, we say that two vertices are properly connected if there exists a path between them which is properly coloured. The least number of colours in a colouring for which each pair of vertices is properly connected is called the proper connection number. Borozan et al. proved that each 2-edge-connected finite graph has proper connection number at most 3. We extend this result and prove that any finite or infinite 2-edge connected graph has a colouring from any set of lists of size 3 such that each pair of vertices is properly connected.

1 Introduction

Proper edge colourings is one of the most studied topics of graph theory. A fundamental theorem of Vizing states that every finite graph G has a proper colouring using at most $\Delta(G) + 1$ colours. Weakenings of proper colourings which may require fewer colours are also intensively studied. One such notion was introduced by Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza [2], where we require that any pair of vertices in a component of a given graph G are connected by a properly coloured path. We say that such vertices are *properly connected*. We call G properly connected if any two vertices of G are properly connected. The minimal number of colours in such a colouring is called the *proper connection number* of G, and it is denoted by $pc(G)$.

Clearly, $pc(G) \leq \chi'(G)$, and for some classes of graphs, the proper connection number and the chromatic index coincide. For example, Borozan et al. argued that for any finite tree T, we have $pc(T) = \Delta(T)$ because any colouring which makes every pair of vertices properly connected must be a proper colouring, and hence require $\Delta(T)$ colours by Kőnig's Theorem. However, the same authors proved that

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the bound obtained from Vizing's Theorem may be greatly reduced if we assume that the graph is 2-edge-connected.

Let x, y be two vertices of an edge-coloured graph G, and let P be a path from x to y. By $\mathrm{start}(P)$ we denote the colour on the first edge of P (i.e. the one incident to x), and by $\mathrm{end}(P)$, the colour on the last edge of P (i.e. the one incident to y).

Theorem 1 ([2]). *Let G be a finite 2-edge-connected graph. Then there exists a 3-edge colouring of G that makes it properly connected with the following strong property. For any pair of distinct vertices v, w there exist two paths P_1, P_2 between them, such that* $\mathrm{start}(P_1) \neq \mathrm{start}(P_2)$ *and* $\mathrm{end}(P_1) \neq \mathrm{end}(P_2)$.

For more results on proper connection, see [9, 4, 3, 8, 10, 5, 6]. None of these results, however, apply to the list variant. This may be a particularly interesting version of this problem, due to the famous List Colouring Conjecture by Bollobás and Harris [1] which states that any k-edge-colourable graph admits a proper colouring from any set of lists of size k. In this paper, we introduce the *list proper connection number* of a graph G, which is the minimum number k such that there exists an edge colouring from any set of lists of size k in which any two vertices are properly connected. We denote this parameter by $lpc(G)$. By analogy for the proper edge colourings, we propose the following conjecture.

Conjecture 2. *For any connected graph G, $lpc(G) = pc(G)$.*

Note that by the result of Galvin [7] confirming the List Colouring Conjecture for all bipartite graphs, the result of Borozan et al. about trees also holds in the case of list colourings. Therefore, Conjecture 2 holds for trees. We further support this conjecture by extending the claim of Theorem 1 to list colourings. Since the authors of [2] constructed a family of finite graphs such that $pc(G) = 3$, then our result is also tight, both for finite graphs and for infinite graphs of any infinite cardinality (where we can append a graph from that family as an induced subgraph connected to a given infinite graph by a single edge).

Before we proceed to the main result, we highlight one more novelty that we are introducing to this problem. Until now, the proper connection was studied only for finite graphs, whereas our result holds for both finite and infinite graphs. Note that both reasonings about trees stated above are true as well for infinite graphs.

In the proof of the main theorem, we use the ear decomposition of a graph. For any graph G, an *ear decomposition* of G is an edge-decomposition of G into a family of paths (open or closed) $P_0, \ldots, P_\alpha, \ldots$ (for $\alpha < \beta$) for some ordinal number β such that P_0 is a closed path (i.e. a cycle) and the intersection of any path P_i, $i > 0$ and $\bigcup_{j<i} P_j$ is precisely the two end-vertices of P_i. We call such paths *ears*. A well-known result of Robbins [11] states that a finite graph is 2-edge-connected graph if and only if it has an ear decomposition. We also generalize this result to infinite graphs.

Theorem 3. *An infinite graph is 2-edge-connected if and only if it admits an ear decomposition.*

2 Main result

We start with proving Theorem 3.

Proof. Let G be an 2-edge-connected infinite graph of order κ. Fix an ordering on $V(G)$, say $\{v_i : i < \kappa\}$. Note that this ordering induces a lexicographic ordering of the edge set. Let P_0 be a cycle containing v_0. We iteratively construct the remaining ears $(P_i :< \kappa)$, which shall form the ear decomposition of G.

Assume that we already defined the paths $(P_i : i < \alpha)$ for some $0 < \alpha < \kappa$. If there exists an edge that connects two vertices of $(P_i : i < \alpha)$ but it is not covered by any of these paths, then we consider the least (by the induced edge ordering) such edge e and define the path P_α as consisting of this single edge. Otherwise, if no such edge exists, then we consider the least vertex v_i which does not belong to $(P_i : i < \alpha)$ and has a neighbour w in $(P_i : i < \alpha)$. By 2-edge-connectivity, there exists a shortest path P from v_i to $(P_i : i < \alpha)$ in $G - v_i w$. We append the edge $v_i w$ to P and define the result as P_α. It is easy to see that P_α is a valid ear. □

Note that (similarly for finite graphs) if a graph is 2-edge-connected, we can further demand that the ear decomposition is *open*, which means that each path in an ear decomposition is open except the first one.

Now we proceed to prove the main theorem of this paper.

Theorem 4. *Let G be a 2-edge-connected graph. Then there exists an edge colouring of G from any set of lists for edges of size 3 that makes it proper connected with the following strong property. For any pair of distinct vertices v, w there exist two paths P_1, P_2 between them, such that* $\text{start}(P_1) \neq \text{start}(P_2)$ *and* $\text{end}(P_1) \neq \text{end}(P_2)$.

Proof. Denote by $\mathcal{L} = \{L(e) : e \in E(G)\}$ the set of lists for the edges of G, each of size 3. We consider a minimal 2-edge-connected spanning subgraph H of G. The proper list connectivity is monotone with respect to adding edges, so we can assume that $G = H$. Let $(P_i : i < \beta)$ be an ear decomposition of G for some $\beta \leq |G|$. The proof is by induction on the number of ears, i.e. the induction hypothesis is that the claim holds for the subgraph induced by $(P_i : i \leq \alpha)$, where $\alpha < \beta$.

If $\alpha = 0$, then we properly colour the cycle P_0 from its lists of size 3. Let $\alpha > 0$ and let G_α be a subgraph induced by the edges of $(P_i : i \leq \alpha)$. Moreover, let $G'_\alpha = G_\alpha - P_\alpha$. Notice that P_α must contain an internal vertex because G is assumed to be minimal. Denote $P_\alpha = uu_1u_2 \dots u_p v$, where $u, v \in G'_\alpha$. By induction, G'_α admits a colouring from the set of lists $\mathcal{L}$. We shall extend this colouring by choosing colours for the edges of P_α.

By the assumption, there exist two paths P_1, P_2 between the vertices u, v such that $\text{start}(P_1) \neq \text{start}(P_2)$ and $\text{end}(P_1) \neq \text{end}(P_2)$. The colouring of P_α depends on the configuration of colours on the pendant edges of these paths, and also on the length of P_α. Let $L'(uu_1) = L(uu_1) \setminus \{\text{start}(P_1), \text{start}(P_2)\}$ and $L'(u_p v) = L(u_p v) \setminus \{\text{end}(P_1), \text{end}(P_2)\}$.

Case 1. The path P_α contains at least two internal vertices, i.e. $p \geq 2$.

We choose a colour other than $\mathrm{start}(P_1)$ and $\mathrm{start}(P_2)$ for the edge uu_1 and a colour other than $\mathrm{end}(P_1)$ and $\mathrm{end}(P_2)$ for the edge u_pv. Then we choose the colours for the remaining edges of P_α so that it is properly coloured.

Case 2. The path P_α contains only one internal vertex u_1, and there are at least two different colours in $L'(uu_1) \cup L'(u_1v)$. We choose a colour for the edge uu_1 from the modified list $L'(uu_1)$ and another, different colour from $L'(u_1v)$ for the edge u_1v.

Case 3. The path P_α contains only one internal vertex u_1, and there is only one colour, say red, in $L'(uu_1) \cup L'(u_1v)$. We cannot choose this colour twice if we would like to have two paths from u_1 starting with different colours. Therefore, we choose red for the edge uu_1 and $\mathrm{end}(P_2)$ for u_1v (this colour must be in the list for u_1v).

We show that in all these cases the resulting colouring of G_α makes it proper connected with the strong property. Notice that for any two vertices of G'_α, the claim follows from the induction. Moreover, since $P_\alpha \cup P_1$ is a properly coloured cycle, any pair of distinct vertices on this cycle also has the desired paths with the strong property. Hence, we only need to check the pair u_i, y, where $y \in G'_\alpha - P_1$.

By induction, in G'_α there is a pair of paths P_{u_1} and P_{u_2} from u to y with the strong property, and similarly, a pair of paths P_{v_1} and P_{v_2} from v to y, also with the strong property. One of P_{u_1}, P_{u_2} (say, P_{u_1}) does not start with the colour $c(uu_1)$, and we append the path $u_iP_\alpha u$ and obtain a proper path $Q_1 = u_iP_\alpha uP_{u_1}y$. Similarly, one of P_{v_1}, P_{v_2} (say, P_{v_1}) does not start with the colour $c(u_pv)$, and we append the edge $u_iP_\alpha v$ and obtain a proper path $Q_2 = u_iP_\alpha vP_{v_1}y$. Then the starting colours of Q_1 and Q_2 are different, and if the ending colours are different, then we have the required pair of paths. Note that if we can choose P_{u_2} or P_{v_2} as a proper path, then we can replace one of these paths and obtain a pair with different terminal colours.

Now consider the case where $\mathrm{end}(P_{u_1}) = \mathrm{end}(P_{v_1})$. By the last remark in the previous paragraph, we can also assume that $\mathrm{start}(P_{u_2})$ is $c(uu_1)$ and $\mathrm{start}(P_{v_2}) = c(u_pv)$. The next candidate for the pair of paths satisfying the strong property is $R_1 = u_iP_\alpha uP_1vP_{v_2}y$ and $R_2 = u_iP_\alpha vP_{v_1}y$. If R_1 is indeed a path, then it is a proper path, since $\mathrm{end}(P_1) \neq c(u_pv) = \mathrm{start}(P_{v_2})$, and R_1 and R_2 do have the strong property. If R_1 is not a path, then pick the vertex z closest to y, which appears more than once on R_1. Then, $R'_1 = u_iP_\alpha uP_1zP_{v_2}y$ is a path, and if it is proper, then we can pair it with R_2 and obtain the claim.

Finally, consider the case where R'_1 is not proper, i.e. that $\mathrm{end}(uP_1z) = \mathrm{start}(zP_{v_2}y)$. Our last candidate is the pair $S_1 = u_iP_\alpha vP_1zP_{v_2}y$ and $S_2 = u_iP_\alpha uP_{u_1}y$. We already know by the choice of P_{u_1} that S_2 is a proper path. Then, S_1 is also a proper path, since P_1 is proper, and we approach z from the other side which has a different colour. To argue that S_1 and S_2 have the strong property, it is enough to recall that $\mathrm{end}(P_{u_1}) = \mathrm{end}(P_{v_1}) \neq \mathrm{end}(P_{v_2})$. □

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