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Fault-tolerant locating-dominating sets on the infinite tumbling block graph

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Abstract

A detection system, modeled in a graph, uses "detectors" placed on a subset of vertices to detect and uniquely identify the location of an "intruder" in the network. In this paper, we explore fault-tolerant variants for locating-dominating sets, a type of detection system in which a detector installed at a vertex can sense an intruder at said vertex or within its open neighborhood. In particular, we consider redundant locating-dominating sets, which permit a detector to be disabled or removed during normal operation, error-detecting locating-dominating sets, which permit a false negative from any sensor, and error-correcting locating-dominating sets, which correct any single sensor error (positive or negative). Specifically, we present bounds on the minimum densities of these fault-tolerant locating-dominating sets in the infinite tumbling block graph and show that these bounds are sharp.

1 Introduction

Let G be an (undirected) graph with vertices $V(G)$ and edges $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, denoted $N(v)$, is the set of vertices adjacent to v, that is, $N(v) = \{w \in V(G) : vw \in E(G)\}$. The *closed neighborhood* of a vertex $v \in V(G)$, denoted $N[v]$, is $N(v) \cup \{v\}$. If $S \subseteq V(G)$ and every vertex in $V(G)$ is within distance 1 of some $v \in S$ (i.e., $\bigcup_{v \in S} N[v] = V(G)$), then S is said to be

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a (closed) *dominating set*. Similarly, $S \subseteq V(G)$ is an *open dominating set* if every vertex is at distance 1 from some $v \in S$ (i.e., $\bigcup_{v \in S} N(v) = V(G)$). Given a graph G, a set $S \subseteq V(G)$ is called a *detection system* if each vertex in S is installed with a specific type of device for detecting an intruder such that the set of sensor data from all detectors in S can be used to uniquely determine the intruder's location in G.

Many types of detection systems with various properties have been explored, each with their own assumed detector capabilities. For example, an *Identifying Code (IC)* [1, 14] is a detection system where each detector at a vertex $v \in V(G)$ can sense an intruder within $N[v]$, but does not know the exact location within $N[v]$. A *Locating-Dominating (LD) set* is a detection system that extends the capabilities of an IC by allowing a detector at v to differentiate whether an intruder is at v itself or in $N(v)$ (does not know which vertex in $N(v)$) [1, 3, 21]. Still another system is called an *Open-Locating-Dominating (OLD) set*, where each detector at a vertex $v \in V(G)$ can sense an intruder only in $N(v)$, but not at v itself [15, 17]. Jean and Lobstein [7] have maintained a bibliography of currently over 500 articles published on various types of detection systems, and other related concepts including fault-tolerant variants of IC, LD and OLD sets.

Clearly, a detection system must cover every vertex, $v \in V(G)$, in order to locate an intruder at v; thus, IC and LD sets are closed-dominating sets and OLD sets are open-dominating sets. However, domination alone is not enough to determine the exact location of an intruder in the graph; for example, two vertices may be covered by the same set of detectors. In order to locate an intruder anywhere in the graph, we must be able to "distinguish" any two vertices based on their covering detectors. Given a detection system $S \subseteq V(G)$, two distinct vertices $u, v \in V(G)$ are said to be *distinguished* if we can eliminate u or v as the location of an intruder (if one is present). In an IC, S, vertices u and v are distinguished if $|N_S[u] \triangle N_S[v]| \geq 1$, where $\triangle$ denotes the symmetric difference and $N_S[u]$ is a shorthand notation for $N[u] \cap S$. In an LD set, S, vertex $x \in S$ is, by definition, distinguished from all other vertices, and $u, v \notin S$ are distinguished if $|N_S[u] \triangle N_S[v]| \geq 1$. In an OLD set, S, u and v are distinguished if $|N_S(u) \triangle N_S(v)| \geq 1$, where $N_S(u)$ denotes $N(u) \cap S$.

Figure 1 shows IC, LD, and OLD sets on the Petersen graph, G, where shaded vertices represent the detector vertices. In (a), we can verify $S_1 = \{v_1, v_3, v_9, v_{10}\}$ forms a dominating set, and all distinct $u, v \in V(G)$ pairs have $|N_{S_1}[u] \triangle N_{S_1}[v]| \geq 1$. Similarly, in (b), $S_2 = \{v_1, v_2, v_4, v_{10}\}$ forms a dominating set and all distinct pairs $u, v \notin S$ have $|N_{S_2}[u] \triangle N_{S_2}[v]| \geq 1$. In (c), we see that $S_3 = \{v_1, v_2, v_3, v_4, v_5\}$ forms an open-dominating set and gives that all distinct $u, v \in V(G)$ have $|N_{S_3}(u) \triangle N_{S_3}(v)| \geq 1$. Thus, we have confirmed the domination and distinguishing requirements for each type of set, and S_1, S_2, and S_3 are IC, LD, and OLD sets, respectively, on G.

Any superset of a detection system is clearly also a detection system, so naturally we are interested in the smallest sets with the given properties. Our goal of finding the minimum detection systems is especially important in real-world applications, as each detector represents a piece of physical hardware, making the smallest detection system the most cost-effective. Let $\mathrm{IC}(G)$, $\mathrm{LD}(G)$, and $\mathrm{OLD}(G)$ denote the minimum cardinalities of LD, IC, and OLD sets, respectively. Then, $\mathrm{IC}(G) = 4$, LD(G) = 4, and OLD(G) = 5 for G in Figure 1 because no smaller sets satisfy the required properties. The problems of determining IC(G), LD(G), and OLD(G) for an arbitrary graph G are known to be NP-complete [1, 2, 3, 17]. For more information about NP-completeness, refer to Garey and Johnson [4].

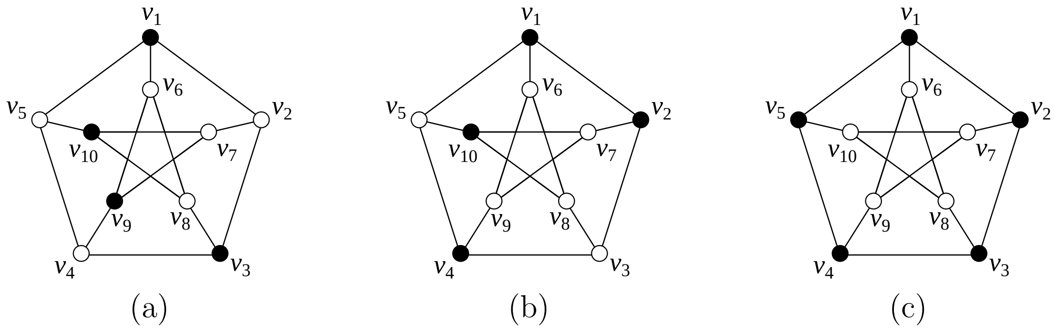


Figure 1: Minimum IC (a), LD (b), and OLD (c) sets on the Petersen graph.

Although the detection systems we have illustrated are relatively simple and typically require a small number of detectors to cover the network, they are arguably insufficient for real-world applications. For instance, these parameters assume that detectors never have any downtime when they are operational, and that they always correctly report the presence or absence of an intruder (i.e., they do not allow any sensing errors). Thus, we often impose additional constraints on these parameters to provide the various types of fault tolerance we might need for use in real-world applications. In particular, three varieties of fault tolerance we will explore are *redundant detection systems* which allow a sensor to go offline or be removed while the system is still functioning [10, 13, 19], *error-detecting detection systems* that permit a false negative from a sensor [11, 12, 19, 22], and *error-correcting detection systems* which correct any single error reported from a sensor (i.e., any false positive or false negative) [8, 9]. In this paper we focus on three fault-tolerant versions of LD sets: redundant locating-dominating (RED:LD) sets, error-detecting locating-dominating (DET:LD) sets, and error-correcting locating-dominating (ERR:LD) sets, specifically for the infinite tumbling block graph.

In Section 2, we introduce the infinite tumbling block graph (TMB) [18] and present previous results regarding the detection systems on TMB. In Section 3 we describe the purposes and characteristics of the three fault-tolerant LD sets and explain how to verify a given detection system meets their requirements. We conclude the section by proving the tight bounds on the minimum RED:LD, DET:LD, and ERR:LD sets for TMB.

2 Detection Systems on TMB

The *infinite tumbling block graph* (TMB) is an infinite bipartite graph with one part being degree 3 and the other being degree 6; example segments of this graph are shown in Figure 2. Seo and Slater [18] described various finite and infinite tumbling block graphs and determined the values of domination-related and detection system parameters for them. In this section, we present some of the previous results [18] as well as the values that we have improved for TMB.

In Section 1, we showed LD, IC, and OLD sets on the Petersen Graph with minimum cardinalities denoted as IC(G), LD(G), and OLD(G), respectively. Because TMB is an infinite graph, we cannot meaningfully measure the cardinality of the detector set. Instead, we measure the *density* of the detector set, which represents the ratio of detector vertices to total vertices. The density-based minimum notations for these parameters are LD%(G), IC%(G), and OLD%(G), respectively.

Let $B_r(v) = \{u \in V(G) : d(u,v) \leq r\}$ denote the ball of radius r around $v \in V(G)$. Formally, the *density* of $S \subseteq V(G)$ where G is locally-finite (i.e., $B_r(v)$ is finite for any r and v) is defined to be $\limsup_{r\to\infty} |B_r(v) \cap S|/|B_r(v)|$ for some choice of center point $v \in V(G)$. Notably, this limit always exists for any choice of v due to the fact that the sequence is bounded (i.e., $|B_r(v) \cap S|/|B_r(v)| \in [0,1]$). However, for some infinite graphs, the density value may be a function of the center point $v \in V(G)$. Recently, it has been proven that any graph satisfying the so-called "slow-growth" property, that is, $\lim_{r\to\infty} |B_{r+1}(v)|/|B_r(v)| = 1$, has the convenience that the density is invariant of v [16]. Fortunately, TMB has the slow-growth property, allowing us to determine LD%(TMB), IC%(TMB), and OLD%(TMB) by inspecting only a single center point or the local density within a periodic tiling.

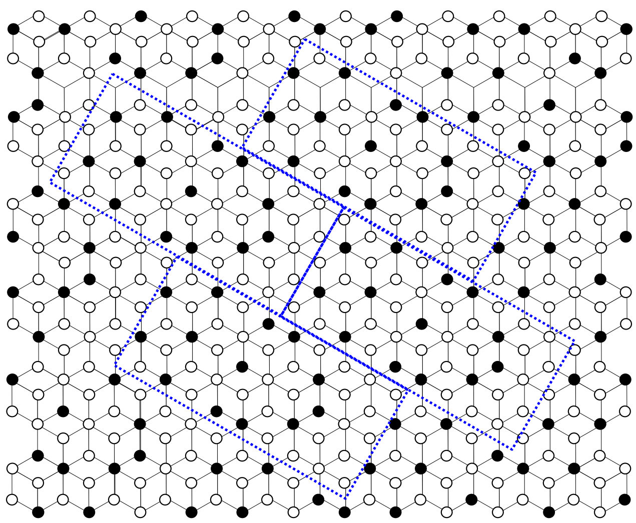


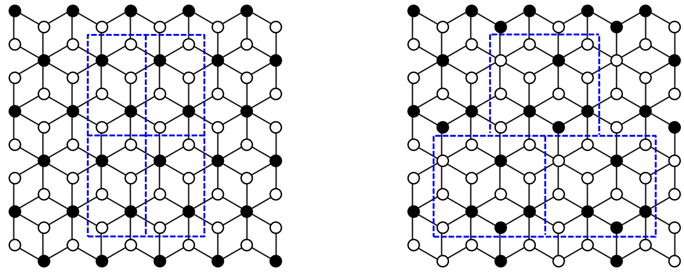
Figure 2: LD%(TMB) $\leq \frac{13}{45}$. The shaded vertices represent detectors.

Upper and lower bounds for LD%(TMB)

Seo and Slater [18] initially showed $\frac{1}{4}$ < LD%(TMB) $\leq \frac{8}{27} \approx 0.296$; we have narrowed the gap by improving the upper bound to LD%(TMB) $\leq \frac{13}{45} \approx 0.289$. Figure 2 shows a solution that achieves the upper bound with a tile of 45 vertices.

Upper and lower bounds on IC%(TMB)

Seo and Slater [18] also found two solutions with density $\frac{1}{3}$ for an IC on TMB as shown in Figure 3 and showed $\frac{3}{11} \leq$ IC%(TMB) $\leq \frac{1}{3}$.



(a) (b)

Figure 3: Two solutions showing IC%(TMB) $\leq \frac{1}{3}$.

Tight bound on OLD%(TMB)

Seo and Slater [18] proved the tight bound of OLD%(TMB) to be $\frac{7}{18}$ with an optimal solution with a tile of 18 vertices as shown in Figure 4.

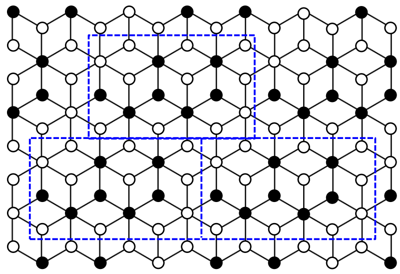


Figure 4: OLD%(TMB) $\leq \frac{7}{18}$.

3 Fault-Tolerant LD sets on TMB

In real-world applications, it is often not enough to be able to locate an intruder in the graph: we also require some level of fault-tolerance in the system to allow for maintenance or unpredictable sensor malfunctions. In this section we will discuss three fault-tolerant variants of LD sets, which include detector redundancies and the system's ability to handle false negatives and false positives. To discuss these concepts further, we will define an LD detector at vertex v to transmit one of three possible values: 0 to represent no intruder in $N[v]$, 1 to represent an intruder in $N(v)$, and 2 to represent an intruder at v. The same scheme of transmission values described above is used in other literature covering fault-tolerant LD sets [10, 21, 22].

The first fault-tolerant variant is a *redundant locating-dominating* (RED:LD) set [10], which is an LD set that can tolerate at most one detector being removed or going offline. Note that a removed/offline detector transmits no value, which differs from an online detector transmitting a false negative; i.e., in a RED:LD, any transmitted value can be assumed to be correct. The redundant property of a RED:LD set is useful if the detectors are known to give correct output when working, but may need to be cycled off (individually) for maintenance while the system is still running. This type of fault is discussed in Hernando et al. [5] for locating sets and in Honkala et al. [6] for identifying codes, and in Seo and Slater [19] for open-locating-dominating sets. More general types of fault-tolerant detection systems have also been studied by Seo and Slater [20].

Definition 3.1 ([23])**.** *A* redundant LD *(RED:LD) set is an LD set $S \subseteq V(G)$ such that for any detector $v \in S$, $S - \{v\}$ is also an LD set.*

As an example, in Figure 5 (a), the shaded vertex set $S_4 = \{v_2, v_3, v_4, v_5, v_8, v_9\}$ makes up a RED:LD set because for any detector vertex $w \in S_4$, $S_4 - \{w\}$ remains an LD set.

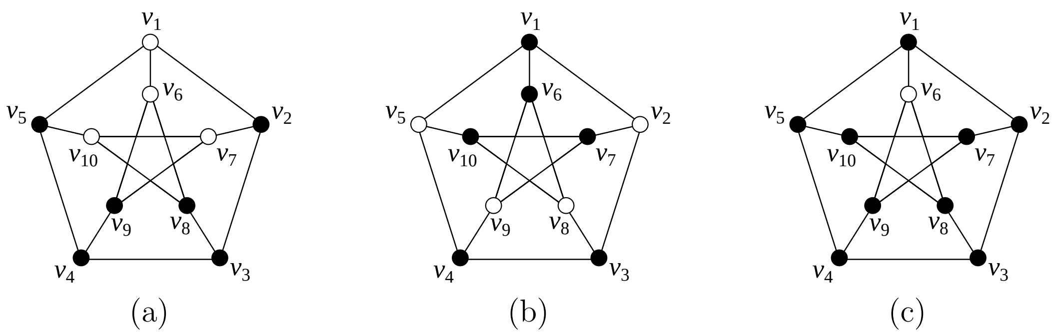


Figure 5: Minimum RED:LD (a), DET:LD (b), and ERR:LD (c) sets on the Petersen graph.

Next, we consider another fault-tolerant variant known as *error-detecting locating-dominating* (DET:LD) sets, which were introduced as *fault-tolerant LD* sets by Slater [22] and fully characterized by Jean and Seo [12]. DET:LD sets can tolerate at most one false negative, which in the context of LD detectors means transmitting 0 instead of 1 or 2.

Definition 3.2 ([12, 22])**.** *An* error-detecting LD *(DET:LD) set is an LD set which can tolerate at most one false negative.*

As an example, refer to Figure 5 (b); the shaded vertices make up a DET:LD set. Because DET:LD only allows false negatives, we may assume that any non-zero value is correct. Suppose the detector at v_{10}, which is responsible for sensing an intruder at v_5, v_7, v_8, and v_{10}, incorrectly transmits 0 instead of 1 or 2. If the intruder is at v_{10}, because we allow at most one error, we can assume v_7 transmits 1, meaning the system narrows the possible intruder locations to v_2, v_9, or v_{10}. The intruder cannot be at v_2 because both v_1 and v_3 transmitted 0; similarly, v_9 can be eliminated because both v_4 and v_6 transmitted 0. Thus, the system correctly determines the intruder is at v_{10}. If the intruder is at v_5, then v_1 and v_4 both transmit 1; we see that $N(v_1) \cap N(v_4) = \{v_5\}$, so the intruder must be at v_5. Similarly, we can correctly determine if the intruder is at v_7 or v_8, or indeed at any vertex. It can also be exhaustively shown that any other fault location (e.g., other than v_{10}) can similarly locate any arbitrary intruder location despite the presence of said fault.

The third variant is an *error-correcting locating-dominating* (ERR:LD) set, which can tolerate one false negative or false positive (including transmitting 1 instead of 2 or vice versa); thus, it has the property of correcting any single transmission error. The concept of a general error-correcting detection system was introduced by Seo and Slater [19, 20, 23] and an ERR:LD set was fully characterized by Jean and Seo [9].

Definition 3.3. *An* error-correcting LD *(ERR:LD) set is an LD set which can tolerate any single transmission error.*

To see how ERR:LD sets work, refer to Figure 5 (c) where the shaded vertices make up an ERR:LD set. Consider the scenario when the detector at v_1, which is responsible for sensing an intruder at v_1, v_2, v_5, and v_6, transmits an incorrect value. If the intruder is at v_1, the system knows that the sensor at v_1 is in error because both v_2 and v_5 detectors would transmit the correct value 1. Similarly, if the intruder is at v_2, then the detectors at v_2 would transmit 2 and both detectors at v_3 and v_7 would transmit 1, so we know the intruder is at v_2 even if the detector at v_1 transmits an incorrect value, 0 or 1. If the intruder is at v_6, then both detectors in $\{v_8, v_9\}$ would transmit 1, so we know the intruder is at v_6 even if the detector at v_1 erroneously transmits 0 or 2. Indeed, we find that all possible scenarios of intruder location and fault location result in the system still correctly locating the intruder.

Similar to the LD(G) notation, we use RED:LD(G), DET:LD(G), and ERR:LD(G) to denote the cardinality of the smallest RED:LD, DET:LD, and ERR:LD sets in G, respectively. We can verify the RED:LD, DET:LD, and ERR:LD sets in Figure 5 are of the minimum cardinality on the Petersen graph, G, hence we have RED:LD(G) = 6, DET:LD(G) = 6, and ERR:LD(G) = 9. The problems of determining RED:LD(G), DET:LD(G), and ERR:LD(G) for an arbitrary graph G are known to be NP-complete [9, 10, 12].

As stated in Section 1, when working with detection systems, we often make use of characterizations in terms of how many vertices must dominate a given vertex, or how many vertices must distinguish a pair of vertices. These definitions vary depending on the type of detector being used, e.g., LD, IC, and OLD set detectors have different detection regions and thus different definitions for which vertices dominate or distinguish others. However, the fault tolerant variants of the detection systems we will discuss can all be expressed in terms of different "k-dominated," "k-distinguished," and "$k^{\#}$-distinguished" requirements. We will now state these definitions for LD detectors.

Definition 3.4. *Let S be an LD set. A vertex $v \in V(G)$ is* k-dominated *if* $|(N(v) \cup \{v\}) \cap S| = k$.

Definition 3.5. *Let S be an LD set. Distinct vertices $u, v \in V(G)$ are* k-distinguished *if* $|((N(u) \triangle N(v)) \cup \{u, v\}) \cap S| \geq k$.

Definition 3.6. *Let S be an LD set. Distinct vertices $u, v \in V(G)$ are* $k^{\#}$-distinguished *if* $|((N(u) - N(v)) \cup \{u\}) \cap S| \geq k$ *or* $|((N(v) - N(u)) \cup \{v\}) \cap S| \geq k$.

With these definitions established, we can refer to Table 1 for the characterization of various fault-tolerant LD sets. The original characterizations for fault-tolerant LD sets were proven by Jean and Seo [9, 10, 12], and here we have presented alternative definitions of k-distinguished and $k^{\#}$-distinguished such that it is compatible with other fault-tolerant detection systems. Using the characterizations for the three fault-tolerant LD sets given in Table 1, we can confirm the sets of shaded vertices in Figure 5 constitute RED:LD, DET:LD, and ERR:LD sets for the Petersen graph. Specifically, in (a) each vertex is 2-dominated and every pair of vertices are 2-distinguished by the shaded vertex set $S_4 = \{v_2, v_3, v_4, v_5, v_8, v_9\}$, in (b) each vertex is 2-dominated and every pair of vertices are $2^{\#}$-distinguished by the shaded vertex set $S_5 = \{v_1, v_3, v_4, v_6, v_7, v_{10}\}$, and in (c) each vertex is 3-dominated and every pair of vertices are 3-distinguished by $S_6 = \{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}\}$. Note that the shaded vertex set S_4 in Figure 5 (a) is not a DET:LD set because some vertex pairs such as (v_1v_7), (v_1v_{10}), (v_6v_7), and (v_6v_{10}) are only 2-distinguished, rather than the required $2^{\#}$-distinguished.

Fault-tolerant LD Set	Min. Domination Requirement	Min. Distinguishing Requirement
LD Set	1-dominated	1-distinguished
Redundant LD Set	2-dominated	2-distinguished
Error-Detecting LD Set	2-dominated	$2^{\#}$-distinguished
Error-Correcting LD Set	3-dominated	3-distinguished

Table 1: Characterizations of fault-tolerant LD sets. Domination requirements apply to all vertices while distinguishing requirements apply to all distinct pairs of vertices.

Next, we explore three fault-tolerant variants of locating dominating (LD) sets on the infinite tumbling block graph (TMB).

3.1 Redundant LD sets on TMB

Notation 3.1. *Given an LD set $S \subseteq V(G)$ and a vertex $v \in V(G)$, let* $\mathrm{dom}(v) = |N_S[v]|$ *denote the domination count of v.*

Notation 3.2. *Let D_k denote the set of vertices which are exactly k-dominated, $\{v \in V(G) : \mathrm{dom}(v) = k\}$, and D_{k+} denote the set of vertices which are at least k-dominated, $\bigcup_{j \geq k} D_j$.*

Notation 3.3. *Let V_k denote the set of vertices of degree k, $\{v \in V : \deg(v) = k\}$.*

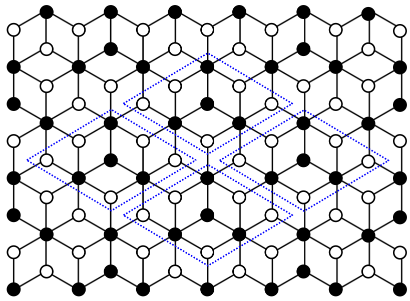


Figure 6: RED:LD%(TMB) $\leq \frac{4}{9}$.

Consider the set S of shaded vertices in Figure 6. It can be verified that every vertex is at least 2-dominated and every distinct pair is 2-distinguished (as per Definitions 3.4 and 3.5, respectively), so S is a RED:LD set of G with density $\frac{4}{9}$. Thus, we have an upper bound for the optimal density: RED:LD%(TMB) $\leq \frac{4}{9}$. Next, we will show $\frac{4}{9}$ is also the lower bound density for a RED:LD set for TMB, that is RED:LD%(TMB) $\geq \frac{4}{9}$.

To construct a lower bound for RED:LD%(TMB), we will use a technique known as a *share argument* [22]. In a share argument, instead of computing a lower bound for RED:LD directly, we prove an upper bound for the amount of "sharing" of the domination of the vertices in the graph. Specifically, for any LD variant, the share of a detector vertex $v \in S$ is $\sum_{w \in N[v]} \frac{1}{\text{dom}(w)}$. Each k-dominated vertex contributes $\frac{1}{k}$ to the share of each of its k dominators, for a total share value of 1 per dominated vertex. Because all LD variants are dominating sets, the sum of shares of all detectors is equal to $|V(G)|$, implying that the inverse of the average share over all detectors is equal to $\frac{|S|}{|V(G)|}$, i.e., the density of S in $V(G)$. Thus, we will find an upper bound for the average share of a detector vertex in S, and take the inverse to give a lower bound for the density of S. For the proof that follows, we will use the following shorthand notations introduced by Jean and Seo [10].

Notation 3.4. *For a sequence of* 1*-digit natural numbers, abc..., let* $\sigma_{abc\ldots} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots$.

Notation 3.5. *For a set of vertices,* $A = \{u, v, w, \ldots\}$*, let* $\text{sh}[A] = \text{sh}[uvw\ldots] = \frac{1}{\text{dom}(u)} + \frac{1}{\text{dom}(v)} + \frac{1}{\text{dom}(w)} + \cdots$ *be the* partial share *of* A.

Notation 3.6. *Let* $\text{sh}(v) = \text{sh}[N[v]]$ *be the (total)* share *of* v.

Lemma 3.1. *If* $x \in V_3 \cap S$ *for a RED:LD set* S *on the TMB, then* $\text{sh}(x) \leq \frac{7}{4}$.

Proof. As illustrated in Figure 7 (where vertex k is denoted by v_k), consider three non-isomorphic configurations for $N(x)$ of vertex x with $\deg(x) = 3$.

Case 1: We require $v_5 \in S$, as otherwise it cannot be 2-dominated. Thus, vertices v_2 and v_3 have two common neighbors; in order to 2-distinguish them we require $\text{sh}[v_2v_3] \leq \max\{\sigma_{33}, \sigma_{24}\} = \sigma_{24}$. We now have an upper bound for the partial

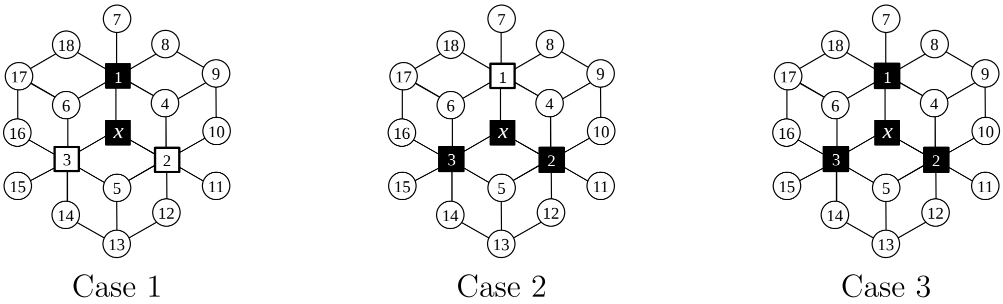


Figure 7: Three non-isomorphic cases for $N(x)$ when $\deg(x) = 3$. The shapes ■, □, and ○ represent detectors, non-detectors, and unknown vertices, respectively.

share of v_2 and v_3; the other vertices in $N[x]$ are at least 2-dominated, so we have $\mathrm{sh}(x) \le \sigma_{2224} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$ and we are done.

Case 2: If $\{v_4, v_{10}\} \cap S = \varnothing$, then v_4 and v_{10} cannot be 2-distinguished, a contradiction; therefore, $\{v_4, v_{10}\} \cap S \ne \varnothing$, so $\mathrm{dom}(v_2) \ge 3$ and by symmetry $\mathrm{dom}(v_3) \ge 3$. Thus, $\mathrm{sh}(x) \le \sigma_{3233} = \frac{3}{2} < \frac{7}{4}$ and we are done.

Case 3: Immediately, we have that $\mathrm{sh}(x) \le \sigma_{4222} = \frac{7}{4}$, completing the proof. □

Next, we will show that the average share of all detectors is at most $\frac{9}{4}$. Lemma 3.1 demonstrated that the degree-3 vertices in TMB have share at most $\frac{7}{4}$. However, as we will see, a bound of $\frac{9}{4}$ cannot be proven for the degree-6 vertices directly as they may have share exceeding this target. To handle the situation, we will use what is known as a *discharging argument* [19] whereby we allow some of the excess share (i.e., share exceeding $\frac{9}{4}$) to be transferred to its degree-3 detector neighbors. Each degree-3 detector neighbor can accept at most $\frac{9}{4} - \frac{7}{4}$ additional share value from its neighboring degree-6 vertices. However, at most three degree-6 detectors may choose to discharge into any given degree-3 detector. Thus, to be safe, we will only allow a degree-3 detector vertex to accept at most $\frac{1}{3}(\frac{9}{4} - \frac{7}{4})$ share from each of its detector numbers. By accounting for the total amount of discharging allowed to neighboring vertices, we can construct a larger *adjusted target* for the share of a degree-6 detector vertex. It should be noted that discharging simply moves around total share values, and so does not affect the sum of all shares or the average share we will prove.

Theorem 3.1. *The average share of all detectors in a RED:LD set S on the TMB graph is at most $\frac{9}{4}$.*

Proof. As illustrated in Figure 8 (where vertex k is denoted by v_k), consider twelve non-isomorphic configurations for $N(x)$ of vertex x with $\deg(x) = 6$.

Case 1: Because there is only a single detector vertex, namely v_6, adjacent to x (with maximum share $\frac{7}{4}$), we can use the adjusted target $\frac{9}{4} + 1 \times \frac{1}{3}(\frac{9}{4} - \frac{7}{4}) = \frac{29}{12}$. We see that $\{v_{12}, v_8\} \subseteq S$ to 2-distinguish v_1 and v_2. Similarly, $\{v_7, v_9\} \subseteq S$ to 2-distinguish v_2 and v_3, and by symmetry $\{v_{10}, v_{11}\} \subseteq S$. Thus, $\mathrm{sh}(x) \le \sigma_{2333334} = \frac{1}{2} + \frac{1}{4} + 5 \times \frac{1}{3} = \frac{29}{12}$

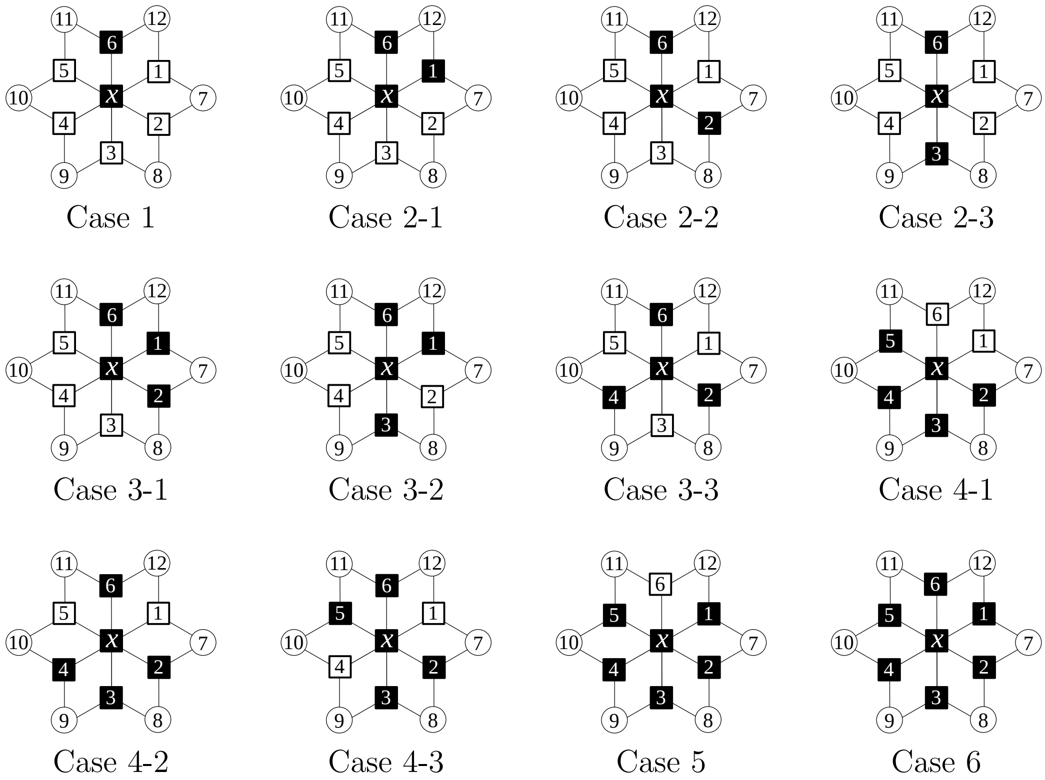


Figure 8: Twelve non-isomorphic cases for $N(x)$ when $\deg(x) = 6$. The shapes ■, □, and ○ represent detectors, non-detectors, and unknown vertices, respectively.

and we are done.

For the next three cases, we can use the adjusted target $\frac{9}{4} + 2 \times \frac{1}{3}(\frac{9}{4} - \frac{7}{4}) = \frac{31}{12}$ since there are two detector vertices in $N(x)$.

Case 2-1: To 2-distinguish v_3 and v_4 we require $\{v_8, v_{10}\} \subseteq S$, and similarly to 2-distinguish (v_2, v_3) and (v_4, v_5) we require $\{v_7, v_9, v_{11}\} \subseteq S$. Thus, every vertex in $N[x]$ is at least 3-dominated, so $\mathrm{sh}(x) \leq \frac{7}{3} < \frac{31}{12}$ and we are done.

Case 2-2: To 2-distinguish v_3 and v_4 we require $\{v_8, v_{10}\} \subseteq S$; similarly $\{v_9, v_{11}\} \subseteq S$ to 2-distinguish v_4 and v_5. Thus, $\mathrm{sh}(x) \leq \sigma_{3233333} < \frac{31}{12}$ and we are done.

Case 2-3: To 2-distinguish v_1 and v_2 we require $\{v_8, v_{12}\} \subseteq S$, and by symmetry $\{v_9, v_{11}\} \subseteq S$. If $v_7 \in S$ then $\mathrm{sh}(x) \leq \sigma_{3334224} < \frac{31}{12}$ and we will be done. Now we can assume $v_7 \notin S$ and by symmetry $v_{10} \notin S$, and we will establish even stronger adjusted target. Refer to Figure 9. If $\{\alpha, \beta\} \cap S = \varnothing$ then α and β cannot be 2-distinguished, a contradiction; therefore, $\{\alpha, \beta\} \cap S \neq \varnothing$, inducing $v_{12} \in D_{3+}$ and by symmetry $v_{11} \in D_{3+}$. Thus, $\mathrm{sh}(v_6) \leq \sigma_{4333} = \frac{5}{4}$, and by symmetry $\mathrm{sh}(v_3) \leq \frac{5}{4}$. Therefore, we can form a stronger adjusted target of $\frac{9}{4} + 2 \times \frac{1}{3}(\frac{9}{4} - \frac{5}{4}) = \frac{35}{12}$. We see that $\mathrm{sh}(x) \leq \sigma_{3224224} < \frac{35}{12}$ and we are done.

For the next three cases, we can use the adjusted target $\frac{9}{4} + 3 \times \frac{1}{3}(\frac{9}{4} - \frac{7}{4}) = \frac{11}{4}$

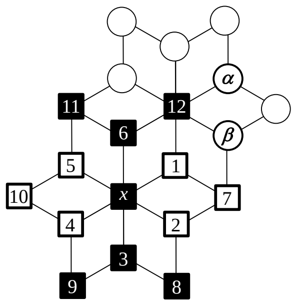


Figure 9: Case 2-3.

since there are three detector vertices in $N(x)$.

Case 3-1: Similar to the previous cases, we require $\{v_8, v_9, v_{10}, v_{11}\} \subseteq S$ to 2-distinguish non-detector pairs in $N(x)$. Thus, $\mathrm{sh}(x) \leq \sigma_{4233333} < \frac{11}{4}$ and we are done.

Case 3-2: To 2-distinguish v_4 and v_5, we need $\{v_9, v_{11}\} \subseteq S$. If $v_7 \in S$ then $\mathrm{sh}(x) \leq \sigma_{4323223} = \frac{11}{4}$ and we are done; otherwise we can assume $v_7 \notin S$, so we require $v_8 \in S$ to 2-dominate v_2. If $v_{12} \in S$, then $\mathrm{sh}(x) \leq \sigma_{4324224} < \frac{11}{4}$ and we are done; otherwise we assume $v_{12} \notin S$. By Case 2 of Lemma 3.1 we see that $\mathrm{sh}(v_6) \leq \frac{3}{2}$; thus, we can form a stronger adjusted target of $\frac{9}{4} + 2 \times \frac{1}{3}(\frac{9}{4} - \frac{7}{4}) + 1 \times \frac{1}{3}(\frac{9}{4} - \frac{3}{2}) = \frac{17}{6}$. We see that $\mathrm{sh}(x) \leq \sigma_{4224223} = \frac{17}{6}$ and we are done.

Case 3-3: If $\{v_2, v_4, v_6\} \cap D_3 = \varnothing$, then to 2-dominate v_1 we require $v_7 \in S$ or $v_{12} \in S$; without loss of generality let $v_7 \in S$, and $v_8 \in S$ because $v_2 \notin D_3$. To 2-dominate v_5 we require $v_{10} \in S$ or $v_{11} \in S$; without loss of generality let $v_{10} \in S$, and $v_9 \in S$ because $v_4 \notin D_3$. Then $\mathrm{sh}(x) \leq \sigma_{4243422} < \frac{11}{4}$ and we are done. Now we can assume $\{v_2, v_4, v_6\} \cap D_3 \neq \varnothing$, then without loss of generality let $v_2 \in D_3$ with $v_7 \in S$ and $v_8 \notin S$; then by Case 2 of Lemma 3.1 $\mathrm{sh}(v_2) \leq \frac{3}{2}$ and we can use a stronger adjusted target of $\frac{17}{6}$, as shown in Case 3-3. To 2-dominate v_3 we require $v_9 \in S$. To 2-dominate v_5 we require $v_{10} \in S$ or $v_{11} \in S$; then $\mathrm{sh}[v_4v_5v_6] \leq \max\{\sigma_{422}, \sigma_{333}\} = \sigma_{422}$, so $\mathrm{sh}(x) \leq \sigma_{4232422} = \frac{17}{6}$ and we are done.

For the next three cases, we know that $x \in D_{5+}$, so for all $v \in N(x)$, $\mathrm{sh}(v) \leq \sigma_{5222}$. Thus, we can use the adjusted target $\frac{9}{4} + 4 \times \frac{1}{3}(\frac{9}{4} - \sigma_{5222}) = \frac{179}{60}$.

Case 4-1: To 2-distinguish v_1 and v_6 we require $\{v_7, v_{11}\} \subseteq S$; thus, $\mathrm{sh}(x) \leq \sigma_{5232232} < \frac{179}{60}$ and we are done.

Case 4-2: To 2-dominate v_1 and v_5, we see that

$$\mathrm{sh}[v_1v_2v_4v_5v_6] \leq \max\{\sigma_{33222}, \sigma_{42222}\} = \sigma_{42222}.$$

Thus, $\mathrm{sh}(x) \leq \sigma_{5222224} < \frac{179}{60}$ and we are done.

Case 4-3: To 2-dominate v_1 we find that $\mathrm{sh}[v_1v_2v_6] \leq \sigma_{322}$, and by symmetry $\mathrm{sh}[v_3v_4v_5] \leq \sigma_{322}$ as well. Thus, $\mathrm{sh}(x) \leq \sigma_{5322322} < \frac{179}{60}$ and we are done.

Cases 5 and 6: Because $x \in D_{6+}$, we can strengthen our adjusted target $\frac{9}{4} + 5 \times \frac{1}{3}(\frac{9}{4} - \sigma_{6222}) = \frac{29}{9}$. We see that $\mathrm{sh}(x) \leq \sigma_{6222222} < \frac{29}{9}$ and we are done. □

From Theorem 3.1 we have a lower bound of $\frac{4}{9}$ for RED:LD on TMB. We see that the solution given in Figure 6 achieves this minimum value, yielding the following tight bound.

Corollary 3.1. RED:LD%(TMB) $= \frac{4}{9}$.

3.2 Error-Detecting LD sets on TMB

Consider the set S of shaded vertices in Figure 10. It can be verified that every vertex is at least 2-dominated and every distinct pair is $2^{\#}$-distinguished (as per Definitions 3.4 and 3.6, respectively), so from Table 1 we see that S is a DET:LD set of G with density $\frac{2}{3}$. Thus, we have an upper bound for the optimal density: DET:LD%(TMB) $\leq \frac{2}{3}$.

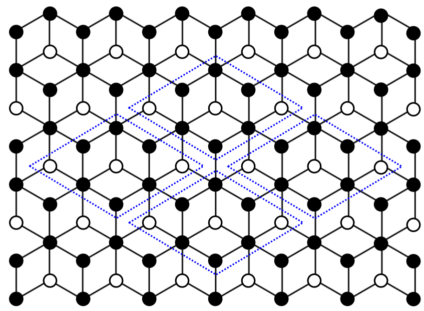


Figure 10: DET:LD%$(TMB) \leq \frac{2}{3}$.

Next, we will show that $\frac{2}{3}$ is a lower bound of DET:LD%(TMB).

Observation 1. *Let S be a DET:LD set for TMB and $u, v \in V_3$ with $u \neq v$ and $|N(u) \cap N(v)| = 2$. Then, $\{u, v\} \cap S \neq \varnothing$.*

Proof. Suppose to the contrary that $\{u, v\} \cap S = \varnothing$. Vertices u and v can be at most 3-dominated, but by hypothesis $|N(u) \cap N(v)| = 2$. Thus, $|(N(v) \cap S) - (N(u) \cap S)| \leq 1$, and by symmetry $|(N(u) \cap S) - (N(v) \cap S)| \leq 1$, contradicting that S is a DET:LD set. □

By partitioning the six vertices in $N(w)$ for some $w \in V_6$ into three disjoint (u, v) pairs such that $|N(v) \cap N(u)| = 2$ and applying Observation 1, we see that $|N(w) \cap S| \geq 3$, which yields the following corollary.

Corollary 3.2. *Let S be a DET:LD set for TMB. If $w \in V_6$, then $w \in D_{4+}$ for $w \in S$, and $w \in D_{3+}$ for $w \notin S$.*

Lemma 3.2. *Let S be a DET:LD set for TMB. If $x \in V_3 \cap S$ then* $\mathrm{sh}(x) \leq \frac{16}{15}$.

Proof. As illustrated in Figure 7 (where vertex k is denoted by v_k), consider the three non-isomorphic configurations for $N(x)$ of vertex x with $\deg(x) = 3$.

Case 1: $\mathrm{dom}(x) = 2$; without loss of generality let $N(x) \cap S = \{v_1\}$. We require $v_5 \in S$, as otherwise v_5 cannot be 2-dominated. If $v_4 \notin S$, we see that x and v_4 can be at most 2-distinguished, but not the required $2^{\#}$-distinguished; therefore, we must have $v_4 \in S$. Similarly, we require $v_8 \in S$ and $v_{12} \in S$ to $2^{\#}$-distinguish (v_4, v_8) and (v_5, v_{12}), respectively. By symmetry we need $\{v_6, v_{18}, v_{14}\} \subseteq S$ as well. Furthermore, by Observation 1, we require $\{v_{10}, v_{11}\} \cap S \neq \varnothing$ and $\{v_{15}, v_{16}\} \cap S \neq \varnothing$. Therefore $\mathrm{sh}(x) = \sigma_{2655} = \frac{16}{15}$.

Case 2: $\mathrm{dom}(x) = 3$; without loss of generality let $N(x) \cap S = \{v_2, v_3\}$. We require $v_5 \in S$ to $2^{\#}$-distinguish x and v_5. By Observation 1, $\{v_4, v_{10}\} \cap S \neq \varnothing$ and $\{v_{11}, v_{12}\} \cap S \neq \varnothing$; therefore, $\mathrm{dom}(v_2) \geq 5$ and by symmetry $\mathrm{dom}(v_3) \geq 5$. And by Corollary 3.2, $\mathrm{dom}(v_1) \geq 3$. Thus, $\mathrm{sh}(x) \leq \sigma_{3355} = \frac{16}{15}$.

Case 3: $\mathrm{dom}(x) = 4$, implying $N(x) \subseteq S$. By Corollary 3.2, $N(x) \subseteq D_{4+}$. Therefore, $\mathrm{sh}(x) \leq \sigma_{4444} < \frac{16}{15}$, completing the proof. □

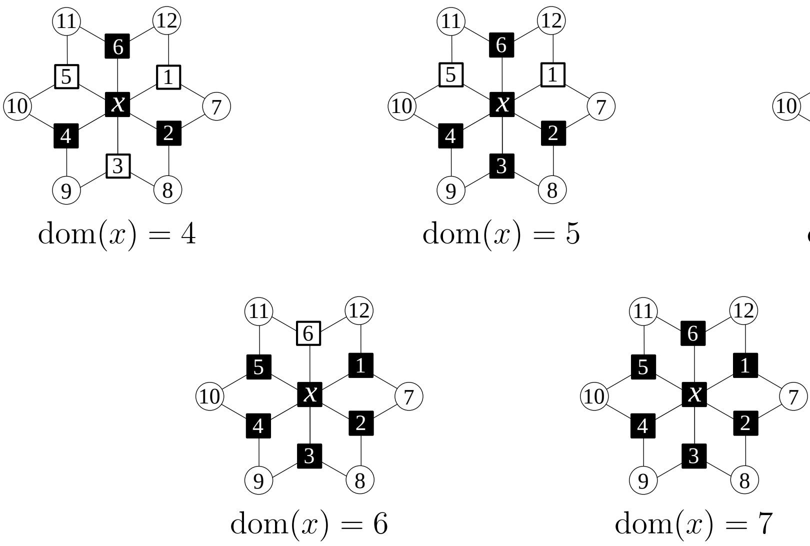


Figure 11: Five non-isomorphic cases for $N(x)$ where $\deg(x) = 6$. The shapes ■, □, and ○ represent detectors, non-detectors, and unknown vertices, respectively.

Theorem 3.2. *Let S be a DET:LD set for TMB. The average share of all detectors is no more than $\frac{3}{2}$.*

Proof. Let $x \in S$. If $x \in V_3$, then Lemma 3.2 yields that $\mathrm{sh}(x) \leq \frac{16}{15} < \frac{3}{2}$ and we are done; thus, we assume $x \in V_6$. Unfortunately, the minimum value for $\mathrm{sh}(x)$ (when $B_2(x) \subseteq S$) is $\sigma_{7444444} > \frac{3}{2}$, so every possible sub-configuration will require the use of adjusted targets. We know that $N(x) \subseteq V_3$, and $\forall u \in N(x) \cap S$, $\mathrm{sh}(u) < \frac{3}{2}$. Thus, where $k = |N(x) \cap S|$ and $q = \max\{\mathrm{sh}(u) : u \in N(x) \cap S\}$, we can use the adjusted target $\frac{3}{2} + k \times \frac{1}{3}(\frac{3}{2} - q)$. Corollary 3.2 yields that $\mathrm{dom}(x) \geq 4$, so we will consider the five non-isomorphic sub-cases for $\mathrm{dom}(x) \in \{4, 5, 6, 7\}$, as shown in Figure 11.

First, consider the case when $\mathrm{dom}(x) = 4$; then by Observation 1 we can assume that $N(x) \cap S = \{v_2, v_4, v_6\}$. We require $v_7 \in S$ to $2^{\#}$-distinguish v_2 and v_3, and by symmetry $\{v_8, v_9, v_{10}, v_{11}, v_{12}\} \subseteq S$. By Corollary 3.2, $\mathrm{dom}(v_7) \geq 4$ and $\mathrm{dom}(v_8) \geq 4$, so $\mathrm{sh}(v_2) \leq \sigma_{4444} = 1$ and by symmetry $\mathrm{sh}(v_4) \leq 1$ and $\mathrm{sh}(v_6) \leq 1$. Therefore, we can use the adjusted target $\frac{3}{2} + 3 \times \frac{1}{3}(\frac{3}{2} - 1) = 2$. We see that $\mathrm{sh}(x) \leq \sigma_{4343434} = 2$ and we are done.

Next, consider the first case with $\mathrm{dom}(x) = 5$, where $N(x) \cap S = \{v_2, v_3, v_4, v_6\}$. We require $v_8 \in S$ to $2^{\#}$-distinguish v_1 and v_2, and by symmetry $v_9 \in S$ as well. Similarly, we need $v_{12} \in S$ to $2^{\#}$-distinguish v_5 and v_6, and by symmetry $v_{11} \in S$ as well. Furthermore, to $2^{\#}$-distinguish v_1 and v_5 we require $v_7 \in S$ or $v_{10} \in S$; without loss of generality let $v_7 \in S$. We see that $\forall w \in \{v_2, v_3, v_6\}$, $\mathrm{sh}(w) \leq \sigma_{4444} = 1$ and by Lemma 3.2 $\mathrm{sh}(v_4) \leq \frac{16}{15}$; thus, we can use the adjusted target $\frac{3}{2} + 3 \times \frac{1}{3}(\frac{3}{2} - 1) + 1 \times \frac{1}{3}(\frac{3}{2} - \frac{16}{15}) = \frac{193}{90}$. We see that $\mathrm{sh}(x) \leq \sigma_{5344324} < \frac{193}{90}$ and we are done.

Now consider the other case with $\mathrm{dom}(x) = 5$, which has $N(x) \cap S = \{v_2, v_3, v_5, v_6\}$. We require $v_8 \in S$ to $2^{\#}$-distinguish v_1 and v_2, and by symmetry $v_{11} \in S$ as well. To $2^{\#}$-distinguish v_1 and v_4 we require $\{v_7, v_{12}\} \subseteq S$ or $\{v_9, v_{10}\} \subseteq S$; without loss of generality let $\{v_7, v_{12}\} \subseteq S$. In order to 2-dominate v_4 we require $\{v_9, v_{10}\} \cap S \neq \varnothing$; without loss of generality let $v_9 \in S$. From here, we find the same adjusted target and bound for $\mathrm{sh}(x)$ as in the previous $\mathrm{dom}(x) = 5$ case, and we are done.

Next, consider the case where $\mathrm{dom}(x) = 6$ with $N(x) \cap S = \{v_1, v_2, v_3, v_4, v_5\}$. By applying Lemma 3.2, we can use the adjusted target $\frac{3}{2} + 5 \times \frac{1}{3}(\frac{3}{2} - \frac{16}{15}) = \frac{20}{9}$. We require $v_7 \in S$ to $2^{\#}$-distinguish v_1 and v_6, and by symmetry $v_{10} \in S$ as well. To 2-dominate v_6 we need $v_{11} \in S$ or $v_{12} \in S$; without loss of generality let $v_{11} \in S$, so $\mathrm{sh}[v_4 v_5] \leq \sigma_{34}$. We require $v_{12} \in S$ or $v_8 \in S$ to $2^{\#}$-distinguish v_1 and v_2; in either case we have $\mathrm{sh}[v_3 v_6] \leq \sigma_{23}$ and $\mathrm{sh}[v_1 v_2] \leq \sigma_{34}$. Therefore, $\mathrm{sh}(x) \leq \sigma_{6233344} < \frac{20}{9}$ and we are done.

Lastly, consider the case where $\mathrm{dom}(x) = 7$, then $N(x) \subseteq S$. We can use the adjusted target $\frac{3}{2} + 6 \times \frac{1}{3}(\frac{3}{2} - \frac{16}{15}) = \frac{71}{30}$. Let $\Delta_1 = \{v_7, v_9, v_{11}\}$ and $\Delta_2 = \{v_8, v_{10}, v_{12}\}$. We see that $|\Delta_1 \cap S| \geq 2$, to $2^{\#}$-distinguish vertices in $N(x)$, and by symmetry $|\Delta_2 \cap S| \geq 2$. If $N(x) \subseteq D_{3+}$, then $\mathrm{sh}(x) \leq \sigma_{7333333} < \frac{71}{30}$ and we would be done; otherwise, without loss of generality let $v_1 \in D_2$, implying $(\Delta_1 \cup \Delta_2) \cap S = \{v_8, v_9, v_{10}, v_{11}\}$. We see that $\mathrm{sh}(x) \leq \sigma_{7234443} < \frac{71}{30}$, completing the proof. □

From Theorem 3.2, we have a lower bound density of $\frac{2}{3}$ for DET:LD on TMB. We see that the upper bound solution given in Figure 10 achieves this minimum value, so we have the following tight bound.

Corollary 3.3. DET:LD%(TMB) $= \frac{2}{3}$.

3.3 Error-Correcting LD sets on TMB

From Table 1, we see that every ERR:LD set is also a DET:LD set, implying ERR:LD%(TMB) $\geq$ DET:LD%(TMB) $= \frac{2}{3}$. Consider the set S of shaded vertices in Figure 10. It can be verified that every vertex is at least 3-dominated and every distinct pair is 3-distinguished (as per Definitions 3.4 and 3.5, respectively), so from

Table 1 we see that S is an ERR:LD set for G with density $\frac{2}{3}$. Thus, S achieves the minimum density on TMB, yielding the following tight bound.

Corollary 3.4. ERR:LD%(TMB) $= \frac{2}{3}$.

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