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Assorted musings on dimension-critical graphs

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Abstract

Let G be a finite simple graph. We say G is of dimension n, writing $\dim(G) = n$, if n is the smallest integer such that G can be represented as a unit-distance graph in $\mathbb{R}^n$. Define G to be dimension-critical if every proper subgraph of G has dimension less than G. In this article, we determine exactly which complete multipartite graphs are dimension-critical. It is then shown that for each $n \geq 2$, there is an arbitrarily large dimension-critical graph G with $\dim(G) = n$. We then pose and expound upon a number of questions related to this subject matter.

1 Introduction

Define a finite simple graph G to be *representable* (or alternately *embeddable*) in $\mathbb{R}^n$ if G can be drawn with its vertices being points of $\mathbb{R}^n$ where any two adjacent vertices are necessarily placed at a unit-distance apart. Say G is of dimension n, and denote $\dim(G) = n$, if G is representable in $\mathbb{R}^n$ but not in $\mathbb{R}^{n-1}$. For a non-empty graph G, define G to be *dimension-critical* if for every proper subgraph H of G, $\dim(H) < \dim(G)$.

This notion of graph dimension was initially put forth in a 1965 note by Erdős, Harary, and Tutte [8]. There the authors establish the dimension of a few common families of graphs and, as typical of a paper authored or co-authored by Erdős, conclude by stirring the pot with a number of questions for future investigation. Indeed, one of these questions serves as an impetus for our present work. Erdős, Harary, and Tutte ask the reader to "...characterize the critical n-dimensional graphs, at least for $n = 3$ (this is trivial for $n = 2$)." Indeed, it takes only a moment's thought to conclude that if G is a dimension-critical graph with $\dim(G) = 2$, then G is either a cycle or the star $K_{1,3}$. For higher dimensions, the situation is murkier, and for an arbitrary graph G, an efficiently-computed condition that is both necessary and sufficient for G to be dimension-critical seems unlikely to exist. We can, however, claim success in characterizing the criticality of certain families of graphs.

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In Section 2, we give a full description of which complete multipartite graphs are dimension-critical. To more succinctly phrase our result, we will implement notation similar to that seen in [10]. For non-negative integers α, β, and γ, define $G(\alpha, \beta, \gamma)$ to be the complete multipartite graph with $\alpha+\beta+\gamma$ parts, α of which are of size 1, β of which are of size 2, and γ of which are of size 3. We first observe that any complete multipartite graph having a part of size 4 or larger is in fact not dimension-critical, and then determine exactly which assignments of α, β, and γ result in $G(\alpha, \beta, \gamma)$ being dimension-critical.

In Section 3, for any $n \geq 2$ and positive integer c, we show through an explicit construction the existence of a dimension-critical graph G with $\dim(G) = n$ and $|E(G)| > c$. This generalizes a result of Boza and Revuelta [2] where they show it is possible for $n = 3$.

In Section 4, we conclude with a number of observations and questions that will hopefully re-stir the pot and prompt future research. In particular, we make a beginning foray into the problem of determining for which n, k there exists an arbitrarily large dimension-critical graph G having $\dim(G) = n$ and chromatic number $\chi(G) = k$.

2 Dimension-critical Complete Multipartite Graphs

In [10], Maehara determines the Euclidean dimension of all complete multipartite graphs. We ourselves will not be concerned with this particular graph parameter, however for those interested readers, we remark that the Euclidean dimension of a graph G is defined similarly to the dimension of G with the added stipulation that in any representation in $\mathbb{R}^n$, non-adjacent vertices of G are forbidden to be placed a unit-distance apart. Regardless, the following theorem is an easily established corollary of the work done in [10].

Theorem 2.1 *Let G be a complete $(\alpha + \beta + \gamma)$-partite graph having exactly α parts of size one, exactly β parts of size two, and exactly γ parts of size greater than or equal to three. If $\beta + \gamma \leq 1$, then* $\dim(G) = \alpha + \beta + 2\gamma - 1$*. If $\beta + \gamma \geq 2$, then* $\dim(G) = \alpha + \beta + 2\gamma$*.*

Theorem 2.1 will figure prominently in this section, and indeed it has an immediate and relevant bearing. Letting G be a complete multipartite graph containing a part of size four or larger, and letting G' be the graph formed by deleting from G a vertex of that part, we have that $\dim(G) = \dim(G')$. This gives us the corollary below.

Corollary 2.2 *Let G be a complete multipartite graph having at least one part of size four or larger. Then G is not dimension-critical.*

Now let G be equal to some $G(\alpha, \beta, \gamma)$, and let $e \in E(G)$. In deciding whether or not G is dimension-critical, we will often consider $G - e$ as a subgraph of some other complete multipartite graph. As an example, consider $G = G(1, 0, 2)$. Label the partite sets of G as $\{a\}, \{b_1, c_1, d_1\}, \{b_2, c_2, d_2\}$, and let $e = b_1b_2$. Then $G - e$ is a subgraph of $G(1, 3, 0)$ whose partite sets are $\{a\}, \{b_1, b_2\}, \{c_1, d_1\}, \{c_2, d_2\}$.

We list those dimension-critical complete multipartite graphs in the theorem below.

Theorem 2.3 *Each of the following complete multipartite graphs is dimension-critical.*

(i) K_α *for* $\alpha \geq 3$*;*
(ii) C_4*;*
(iii) $K_{1,3}$*;*
(iv) $K_{2,3}$*;*
(v) $G(\alpha, 0, \gamma)$ *for* $\alpha \geq 0$ *and* $\gamma \geq 2$*.*

Proof In [8], it is observed that $\dim(K_\alpha) = \alpha - 1$ and that $\dim(K_\alpha - e) = \alpha - 2$ for any $e \in E(K_\alpha)$, so we have that K_α is dimension-critical. It is obvious that the cycle C_4 and star $K_{1,3}$ are dimension-critical. It is also fairly easy to see that $\dim(K_{2,3}) = 3$ and $\dim(K_{2,3} - e) = 2$, although it is noted as well in [6] that $K_{2,3}$ is a dimension 3 graph with minimum edge-set, which in turn implies that $K_{2,3}$ is dimension-critical.

Now let $G = G(\alpha, 0, \gamma)$ for $\alpha \geq 0$ and $\gamma \geq 2$, and note that Theorem 2.1 gives $\dim(G) = \alpha + 2\gamma$. Label the partite sets of G as $\{a_1\}, \dots, \{a_\alpha\}$, $\{b_1, c_1, d_1\}, \dots,$ $\{b_\gamma, c_\gamma, d_\gamma\}$. Let $e_1 = a_1a_2$, $e_2 = b_1b_2$, and $e_3 = a_1b_1$. For any $e \in E(G)$, there is an automorphism of G mapping e to one of e_1, e_2, or e_3, so to show that G is dimension-critical, we just need to show that for $i \in \{1, 2, 3\}$, $\dim(G) > \dim(G - e_i)$. First note that $G - e_1$ is a subgraph of $G(\alpha - 2, 1, \gamma)$ which by Theorem 2.1 is of dimension $\alpha + 2\gamma - 1$. Secondly, note that $G - e_2$ is a subgraph of $G(\alpha, 3, \gamma - 2)$ which again by Theorem 2.1 is of dimension $\alpha + 2\gamma - 1$. Finally, we have that $G - e_3$ is a subgraph of $G(\alpha - 1, 2, \gamma - 1)$ which is of dimension $\alpha + 2\gamma - 1$ as well. Since for arbitrary graphs H and K, H being a subgraph of K implies that $\dim(H) \leq \dim(K)$, we have now shown that for any $e \in E(G)$, $\dim(G - e) \leq \alpha + 2\gamma - 1 < \dim(G)$. This completes the proof that G is dimension-critical. □

Theorem 2.4 *Let* $G = G(\alpha, \beta, \gamma)$ *where* $\alpha \geq 0$*,* $\beta \geq 1$*, and* $\beta + \gamma \geq 3$*. Then* G *is not dimension-critical.*

Proof Let $v \in V(G)$ where v is contained in a part of size 2. Then $G \setminus \{v\} = G(\alpha + 1, \beta - 1, \gamma)$ and by Theorem 2.1, $\dim(G \setminus \{v\}) = \alpha + \beta + 2\gamma$. Since $\dim(G) = \alpha + \beta + 2\gamma$ as well, we have that G is not dimension-critical. □

In light of Theorems 2.3 and 2.4, the only remaining complete multipartite graphs that we must investigate are K_2, $G(\alpha, 1, 0)$ for $\alpha \geq 1$, $G(\alpha, 1, 1)$ for $\alpha \geq 1$, $G(\alpha, 2, 0)$ for $\alpha \geq 1$, and $G(\alpha, 0, 1)$ for $\alpha \geq 2$. We show that each of these graphs is not dimension-critical in the theorem below.

Theorem 2.5 *Each of the following complete multipartite graphs is not dimension-critical.*

(i) K_2;
(ii) $G(\alpha, 1, 0)$ *for* $\alpha \geq 1$;
(iii) $G(\alpha, 1, 1)$ *for* $\alpha \geq 1$;
(iv) $G(\alpha, 2, 0)$ *for* $\alpha \geq 1$;
(v) $G(\alpha, 0, 1)$ *for* $\alpha \geq 2$.

Proof We consider each of these cases individually and apply Theorem 2.1 throughout.

(i) Quite obviously $\dim(K_2) = 1$, however deletion of the only edge of K_2 results in a graph just consisting of two isolated vertices which cannot be embedded in $\mathbb{R}^0$ (which by convention consists of a single point). So K_2 is not dimension-critical.

(ii) Let $G = G(\alpha, 1, 0)$ for $\alpha \geq 1$. Then $\dim(G) = \alpha$. Letting $v \in V(G)$ where v is contained in the part of size 2, $G \setminus \{v\}$ is equal to $K_{\alpha+1}$. Since $\dim(K_{\alpha+1}) = \alpha$, we have that G is not dimension-critical.

(iii) Let $G = G(\alpha, 1, 1)$ for $\alpha \geq 1$, and note that $\dim(G) = \alpha + 3$. Label the partite sets of G as $\{a_1\}, \ldots, \{a_\alpha\}, \{b_1, c_1\}, \{b_2, c_2, d_2\}$. Form a new graph G' by deleting from G the edges a_1b_1 and a_1c_1. Observe that $G' = G(\alpha - 1, 0, 2)$ and $\dim(G') = \alpha + 3$. Again, we have that G is not dimension-critical.

(iv) Let $G = G(\alpha, 2, 0)$ for $\alpha \geq 1$, and note that $\dim(G) = \alpha + 2$. Just as in the last case, let $G' = G \setminus \{a_1b_1, a_1c_1\}$ where $\{a_1\}$ and $\{b_1, c_1\}$ are parts of sizes one and two, respectively, and note that $G' = G(\alpha - 1, 1, 1)$. We have that $\dim(G') = \alpha + 2$ which implies that G is not dimension-critical.

(v) Finally, let $G = G(\alpha, 0, 1)$ for $\alpha \geq 2$, which gives $\dim(G) = \alpha + 1$. Let $\{a_1\}$ and $\{a_2\}$ both be partite sets of size 1, and let G' be formed by deleting edge a_1a_2 from G. Then $G' = G(\alpha - 2, 1, 1)$ and $\dim(G') = \alpha + 1$. We conclude that G is not dimension-critical. □

Theorem 2.5 shows that the graphs shown to be dimension-critical by Theorem 2.3 are in fact the only dimension-critical complete multipartite graphs.

3 Arbitrarily Large Dimension-critical Graphs

In this section, we show that for any $n \geq 2$, there exists an arbitrarily large dimension-critical graph G with $\dim(G) = n$. This is immediate for $n = 2$ as the cycle C_k is of dimension 2 for any $k \geq 3$, and deletion of any edge of C_m results in a path which has a unit-distance representation on the real number line $\mathbb{R}$. In [2], Boza and Revuelta construct an arbitrarily large dimension-critical graph of dimension 3. However, the authors of [2] do not comment on the existence of such graphs in higher dimensions, and it does not appear that their construction has a clear generalization.

We will obtain our result by considering the graph $G = K_n + C_m$. That is, G is formed by starting with the cycle C_m for some $m \geq 3$, then placing n vertices adjacent to each other and to each of the vertices of the copy of C_m. Along the way, we will employ a number of lemmas and theorems of a geometric sort, and we will make frequent reference to the *n-dimensional simplex*, which is formally defined as the convex hull of points $P_0, \ldots, P_n$ having the property that the n vectors $\overrightarrow{P_0P_1}, \ldots, \overrightarrow{P_0P_n}$ are linearly independent. A simplex is deemed to be *regular* if all its edge lengths are the same, or, in other words, if the distance $|P_i - P_j| = d$ for some $d > 0$ and all distinct $i, j \in \{0, \ldots, n\}$. A *unit simplex* is regular with $d = 1$. Lemma 3.1 is observed in the previously mentioned [8].

Lemma 3.1 *For any $n \geq 1$, $\dim(K_n) = n - 1$.*

Although proof of Lemma 3.1 is not given in [8], it follows from the fact that if points $p_1, \ldots, p_n$ are such that $|p_i - p_j| = 1$ for distinct i, j, the vectors $\overrightarrow{p_1p_2}, \ldots, \overrightarrow{p_1p_n}$ are linearly independent, and if p_1 is assumed to be the origin, then their span is a subspace isomorphic to $\mathbb{R}^{n-1}$. This observation is certainly known to the mathematical community, but a direct citation has been tough to find, so in the interest of completion, we include proof in Lemma 3.2. This lemma will then be used with Lemmas 3.3 through 3.6 to characterize sets of points in some $\mathbb{R}^d$ that are at distance 1 from each vertex of a unit simplex. This characterization will play a central role in Theorems 3.9 and 3.10, which together make up the main result of this section.

Lemma 3.2 *Let $v_1, \ldots, v_n \in \mathbb{R}^d$ for some $d \geq n$ where $|v_i| = 1$ for all $i \in \{1, \ldots n\}$ and $|v_i - v_j| = 1$ for all $i \neq j$. Then the vectors $v_1, \ldots, v_n$ are linearly independent.*

Proof First, note that any distinct v_i, v_j form two edges of an equilateral triangle and so the angle θ between them is equal to $60°$. The commonly used formula $\cos\theta = \frac{v_i \cdot v_j}{|v_i||v_j|}$ then gives $v_i \cdot v_j = \frac{1}{2}$.

Now let $\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbf{0}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Select v_i for any $i \in \{1, \ldots, n\}$, and then consider the dot product $\alpha_1 v_1 \cdot v_i + \cdots + \alpha_n v_n \cdot v_i = \mathbf{0}$. We may then rewrite this expression to obtain

$$\sum_{j \neq i} \tfrac{1}{2}\, \alpha_j = -\alpha_i \implies \sum_{j \neq i} \alpha_j = -2\alpha_i \implies \sum_{j=1}^{n} \alpha_j = -\alpha_i.$$

Since the selection of v_i was arbitrary, we have that $\alpha_1 = \cdots = \alpha_n$. It then follows that

$$\sum_{j=1}^{n} \alpha_j = -\alpha_i \implies n \sum_{j=1}^{n} \alpha_j = -n\alpha_i \implies n(\alpha_1 + \cdots + \alpha_n) = -(\alpha_1 + \cdots + \alpha_n)$$

which means $\alpha_1 + \cdots + \alpha_n = 0$. Thus $\alpha_i = 0$ for all $i \in \{1, \ldots, n\}$ and the vectors $v_1, \ldots, v_n$ are linearly independent. □

Note that in the proof of the above lemma, the size of d (and specifically, whether it was equal to or larger than n) did not come into play. We conclude that the span of vectors $v_1, \ldots, v_n$ as described by Lemma 3.2 is either a linear or affine subspace S of $\mathbb{R}^d$ with S being isomorphic to $\mathbb{R}^n$.

Lemma 3.3 *Let $p_0, \ldots, p_n \in \mathbb{R}^n$ with $|p_i - p_j| = 1$ for distinct i, j. There is exactly one point $c \in \mathbb{R}^n$ that is simultaneously equidistant to each of $p_0, \ldots, p_n$.*

Proof Without loss of generality, assume that p_0 is the origin. Write $p_i = (a_{i,1}, \ldots, a_{i,n})$ for each $i \in \{1, \ldots, n\}$. Note that for each $i \in \{1, \ldots, n\}$, the set of all points equidistant to p_0 and p_i consists of the hyperplane $a_{i,1}x_1 + \cdots + a_{i,n}x_n = \frac{1}{2}$ which we designate as H_i. Note also that $\bigcap_{i=1}^n H_i$ consists of all points in $\mathbb{R}^n$ that are simultaneously equidistant to each of $p_0, \ldots, p_n$. Let M be the $n \times n$ matrix given below.

$$M = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{bmatrix}.$$

By Lemma 3.2, the vectors $p_1, \ldots, p_n$ are linearly independent, and so the system of equations formed by augmenting M with the vector $(\frac{1}{2}, \ldots, \frac{1}{2})$ has a solution set which consists of a single point c. □

The point c is of course the circumcenter of the hypersphere circumscribed about the unit simplex with vertices $p_0, \ldots, p_n$. The distance from c to any of $p_0, \ldots, p_n$ can be found as an extension of Lemma 3.4, which can be found, for example, in [3].

Lemma 3.4 *Let S be a regular n-dimensional simplex embedded on a unit sphere in $\mathbb{R}^n$. Then for any vertices P_1, P_2 of S, $|P_1 - P_2| = \sqrt{2 + \frac{2}{n}}$.*

Since, in any representation of a graph in $\mathbb{R}^n$, we require edges to be of unit length, a quick calculation allows Lemma 3.4 to be restated as the following corollary.

Corollary 3.5 *Let K_n have a unit-distance representation in $\mathbb{R}^{n-1}$ on a sphere S in $\mathbb{R}^{n-1}$ having radius r. Then $r = \sqrt{\frac{n-1}{2n}}$.*

Define an *isometry* as any transformation of $\mathbb{R}^d$ (or a subset of $\mathbb{R}^d$) which preserves distance. Lemma 3.6 is given with proof as Theorem 11.4 in [13].

Lemma 3.6 *Let $S \subset \mathbb{R}^d$. Let $f : S \to \mathbb{R}^d$ be an isometry of S. There exists an isometry $g : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $s \in S$, $g(s) = f(s)$.*

The above lemma will be used in upcoming arguments as follows. Let G be a graph which has K_n as a subgraph, and suppose that we desire to show that $\dim(G) > d$ for some value d. We assume to the contrary that G does have a unit-distance representation in $\mathbb{R}^d$ with its vertices being placed at points $v_1, \ldots, v_m \in \mathbb{R}^d$. If f is any isometry of $\mathbb{R}^d$, then their images $f(v_1), \ldots, f(v_m)$ also give rise to a unit-distance representation of G in $\mathbb{R}^d$. Lemma 3.6 guarantees that, if it is useful in eventually establishing a contradiction, we may without loss of generality assume this isometry maps those vertices of K_n to the vertices of a regular simplex lying in $\mathbb{R}^{n-1}$.

Theorem 3.7 is found in [12] and will be implemented in the proof of Lemma 3.8 below.

Theorem 3.7 *Let $r \in \mathbb{Q}$ with $0 \leq r \leq 1$. The number $\frac{1}{\pi}\arcsin(\sqrt{r})$ is rational if and only if r is equal to 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, or 1.*

Lemma 3.8 *Let S be a circle of radius $r = \sqrt{\frac{n+1}{2n}}$ for some integer $n \geq 2$. Then no cycle of edge-length 1 is embeddable on S.*

Proof Consider a cycle C_m, and assume to the contrary that C_m is embeddable on S. We then must have that, for some integer $z \in \mathbb{Z}^+$, the angle θ given in Figure 1 satisfies $m\theta = z(2\pi)$.

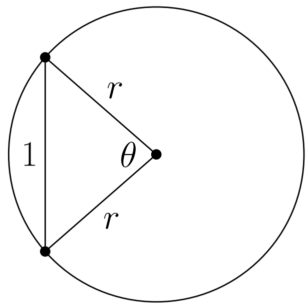

Figure 1

Solving for θ, we have $\sin(\frac{\theta}{2}) = \frac{1}{2r}$. Combining this with the equality given above, we have that $\frac{1}{\pi}\arcsin\left(\sqrt{\frac{n}{2(n+1)}}\right) = \frac{z}{m}$, or in other words, $\frac{1}{\pi}\arcsin\left(\sqrt{\frac{n}{2(n+1)}}\right)$ is rational. By Theorem 3.7, we see $\frac{n}{2(n+1)} \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. However, letting $f(x) = \frac{x}{2(x+1)}$, we have $f'(x) = \frac{1}{2(x+1)^2}$ which implies that $f(x)$ is strictly increasing. Since $f(2) = \frac{1}{3}$ and $\lim_{n\to\infty} f(n) = \frac{1}{2}$, we have a contradiction. □

We now determine the dimension of $G = K_n + C_m$.

Theorem 3.9 *Let $G = K_n + C_m$ for $m \geq 3$, $n \geq 2$. Then $\dim(G) = n + 2$.*

Proof Label the vertices of K_n as $a_1, \ldots, a_n$ and those of C_m as $w_1, \ldots, w_m$. Our first goal is to find an embedding of G in $\mathbb{R}^{n+2}$. We do this by representing $a_1, \ldots, a_n$ as the vertices of a regular $(n-1)$-dimensional simplex of edge-length 1 centered at the origin, which means that, for each a_i, the last three coordinates of that point are each zero. Let r_1 be the radius of this simplex, and by Corollary 3.5, we have $r_1 = \sqrt{\frac{n-1}{2n}}$. Each of the vertices $w_1, \ldots, w_m$ will then be represented as points in $\mathbb{R}^{n+2}$ of the form $(0, \ldots, 0, x_i, y_i, z_i)$ where $x_i^2 + y_i^2 + z_i^2 = 1 - r_1^2$ for $i \in \{1, \ldots, m\}$. Let $r_2 = \sqrt{1 - r_1^2}$, and note that $r_2 = \sqrt{\frac{1}{2} + \frac{1}{2n}}$. To complete our embedding of G in $\mathbb{R}^{n+2}$, it now suffices to show that the cycle C_m is representable in $\mathbb{R}^3$ with each of its vertices lying on a sphere of radius r_2.

Designate by S a sphere of radius r_2. First, we claim that for any points P_1, P_2 lying on S with $|P_1 - P_2| = 1$, there exists a point P_3 on S at distance 1 from each of P_1, P_2. To see this, we will show that there exists a point on S at distance less than 1 from each of P_1, P_2 and also a point on S at distance greater than 1 from each of P_1, P_2, whereby continuity guarantees the existence of the desired P_3. Consider the great circle of S containing both P_1 and P_2, and then label distances as in Figure 2 below.

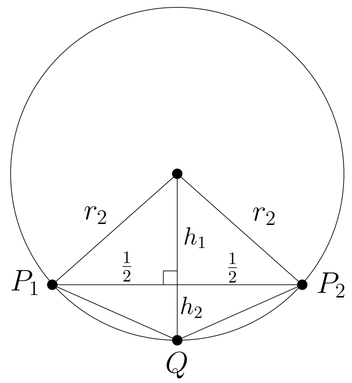

Figure 2

We have the relationships $h_1 + h_2 = r_2$, $h_1 = \sqrt{r_2^2 - \frac{1}{4}}$, and $|P_1 - Q|^2 = |P_2 - Q|^2 = h_2^2 + \frac{1}{4}$. We claim that $|P_1 - Q|^2 < 1$ which amounts to showing that $h_2 < \frac{\sqrt{3}}{2}$. To see this, combine the above equalities to write $h_2 = r_2 - \sqrt{r_2^2 - \frac{1}{4}}$. Letting $f(x) = x - \sqrt{x^2 - \frac{1}{4}}$, we have $f'(x) = 1 - \frac{x}{\sqrt{x^2 - \frac{1}{4}}} < 0$ which implies $f(x)$ is decreasing. Since $\frac{\sqrt{2}}{2} < r_2$, and $f(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}-1}{2} < \frac{\sqrt{3}}{2}$, we have established that there is a point on S at a distance less than 1 from each of P_1 and P_2. To see that there is a point on S at a distance greater than 1 from each of P_1 and P_2, just take an endpoint of the diameter of S that is orthogonal to the plane containing this great circle. The distance from this point to each of P_1 and P_2 is $r_2\sqrt{2}$, which is greater than 1 since $r_2 = \sqrt{\frac{1}{2} + \frac{1}{2n}} > \sqrt{\frac{1}{2}}$. This completes proof of our claim. We note also that a similar argument shows that for any two points on S that are a distance less than 1 apart, there is a point on S at distance 1 from each of them as well.

Now, to embed a cycle C_m on S, we perform the following procedure. If m is odd, place $w_1, w_3, w_5, \ldots, w_m$ on a great circle of S where w_1 and w_m are a Euclidean distance 1 apart, and $w_3, \ldots, w_{m-2}$ lie on the arc of the great circle between w_1 and w_m. For each pair of consecutive vertices in $\{w_1, w_3, \ldots, w_m\}$, there is a point on S at distance one from each of them. We may select these points to be the $w_2, w_4, \ldots, w_{m-1}$ and we have completed the embedding. If m is even, we may embed the cycle C_{m-1} in the fashion as just described, then delete the edge w_1w_2, and place new edges w_1P and Pw_2 where vertex P is a point on S at distance 1 from each of w_1 and w_2. This completes the proof that G is representable in $\mathbb{R}^{n+2}$.

Now suppose to the contrary that G is embeddable in $\mathbb{R}^{n+1}$. In light of Lemma 3.6 and its accompanying discussion, we may without loss of generality assume that $a_1, \ldots, a_n$ are represented as the vertices of a regular $(n-1)$-dimensional simplex of edge-length 1 which lies in $\mathbb{R}^{n-1}$ and is centered at the origin. Note that with this assumption, the last two coordinates entries of each a_i are both zero. Each of $w_1, \ldots, w_m$ is simultaneously at distance 1 from each of $a_1, \ldots, a_n$, and we claim that each w_i is of the form $(0, \ldots, 0, x_i, y_i)$, for the following reason. Suppose for a moment that we did in fact have some $w \in \{w_1, \ldots, w_m\}$ where w does not have each of its first $n-1$ coordinates equal to zero. Lemma 3.3 tells us that the origin is the *only* point in $\mathbb{R}^{n-1}$ that is equidistant from each of $a_1, \ldots, a_n$. So, for this hypothetical w, there would exist $a_i, a_j \in \{a_1, \ldots, a_n\}$ where $|a_i - w| \neq |a_j - w|$, a contradiction completing the justification of our claim. By Corollary 3.5, each a_i is distance $\sqrt{\frac{n-1}{2n}}$ from the origin, and it follows that $\frac{n-1}{2n} + x_i^2 + y_i^2 = 1$, which gives us $x_i^2 + y_i^2 = \frac{n+1}{2n}$. We conclude that the cycle C_m must have a unit-distance representation on a circle of radius $\sqrt{\frac{n+1}{2n}}$. This contradicts Lemma 3.8. □

We are now ready for the main result of this section.

Theorem 3.10 *Let $n \geq 2$, and $c \in \mathbb{Z}^+$. Then there exists a dimension-critical graph H satisfying* $\dim(H) = n+2$ *and* $|E(H)| > c$.

Proof Again, consider the graph $G = K_n + C_m$ where $m > c$, and label the vertices of G as in the proof of Theorem 3.9. For any edge of the form w_iw_j, there is an automorphism of G mapping that edge to $e = w_1w_m$. We aim to show then that e is critical to the dimension of G, or, in other words, that $\dim(G) > \dim(G-e)$. In light of Theorem 3.9, this amounts to showing that $G - e$ is representable in $\mathbb{R}^{n+1}$.

Just as in the proof of Theorem 3.9, we represent $a_1, \ldots, a_n$ as the vertices of a unit simplex which lies in $\mathbb{R}^{n-1}$, is centered at the origin, and has radius $r_1 = \sqrt{\frac{n-1}{2n}}$. We then represent each of the vertices $w_1, \ldots, w_m$ as points of the form $(0, \ldots, 0, x_i, y_i)$ where $x_i^2 + y_i^2 = \frac{n+1}{2n}$ for $i \in \{1, \ldots, m\}$. To see that this does indeed give a valid representation of $G - e$ in $\mathbb{R}^{n+1}$, we need only show that a path of arbitrary length has a unit-distance embedding on a circle, call it S, of radius $r_2 = \sqrt{\frac{n+1}{2n}}$. Since

$r_2 > \frac{1}{2}$, for any point p on S, there are two points on S at distance 1 from p. Since Lemma 3.8 guarantees that no cycle is embeddable on S, we have established that $G - e$ is embeddable in $\mathbb{R}^{n+1}$.

To then create a dimension-critical graph H with $\dim(H) = n + 2$, start with G and iteratively delete any edges that are not critical to the dimension of the graph. As observed above, all edges of the form w_iw_j are critical, so no matter how many edges of the form a_ia_j or a_iw_j are deleted, we have that $\dim(H) = n+2$ and $|E(H)| > c$.□

4 Further Work

In this section we collect a few observations and questions that have arisen during our investigations into the topic of dimension-critical graphs. To the best of our knowledge, each of these is open. We begin with a question in computational complexity. A full digression into the terminology, history, and methodology of this subject would take us far afield, so we will make do with assuming some familiarity of our readers, and point those uninitiated to the introductory texts [1] and [5].

Question 1 *For an arbitrary graph G, what is the complexity of determining whether G is dimension-critical?*

In [11], Schaefer proves that for a general graph G, it is NP-complete to determine whether or not G has a unit-distance representation in $\mathbb{R}^2$. An immediate extension is the fact that it is NP-hard to precisely determine $\dim(G)$. However, one can also use Schaefer's result to prove that for a given $e \in E(G)$, it is NP-hard to decide whether $\dim(G) > \dim(G - e)$. We do this below.

First, observe that a graph G has a unit-distance representation in $\mathbb{R}$ if and only if G is acyclic and contains no vertices of degree greater than 2 — in other words, if and only if every component of G is a path. There are linear-time algorithms for deciding if G has either of these two properties. Secondly, we note the impossibility of the existence of a graph H with $\dim(H) = 1$ where the creation of a graph H' by placing an edge between two non-adjacent vertices of H results in $\dim(H') > 2$. This is easy to see considering that H' would have at most one non-path component with that component being a tree, a cycle, or a cycle with one or two paths attached to single vertices of the cycle. In either case, that component, and by extension the entirety of H', is embeddable in $\mathbb{R}^2$.

Now suppose to the contrary that there does exist a polynomial-time algorithm to decide whether $e \in E(G)$ is critical to the dimension of G. Label the edges of G as $e_1, \ldots, e_m$ and, starting with $i = 1$, implement this algorithm to decide whether e_i is critical. If it is not, delete e_i from G, and implement the algorithm again to decide whether e_{i+1} is critical to the dimension of $G \setminus \{e_1, \ldots, e_i\}$. Eventually we must reach some edge, call it e_j, that is critical to the dimension of graph $G' = G \setminus \{e_1, \ldots, e_{j-1}\}$. We may now run polynomial-time algorithms to decide whether G' is representable in $\mathbb{R}$. If $\dim(G') = 1$, we have that $\dim(G) = 2$, and if $\dim(G') \neq 1$, we have that

$\dim(G) \neq 2$. The existence of this polynomial-time algorithm to determine whether or not G has a representation in $\mathbb{R}^2$ contradicts Schaefer's result.

From the above observations, the existence of a polynomial-time algorithm to determine if G is dimension-critical seems very unlikely. However, we (somewhat abashedly) remark that we see no way to completely resolve Question 1.

Question 2 *For an arbitrary graph G and $e \in E(G)$, is it true that* $\dim(G) - \dim(G - e) \leq 1$*?*

Of course, one can produce myriad examples of G and $e \in E(G)$ where the deletion of e either does not change the dimension of the graph or reduces the dimension of the graph by 1. However, we were unable to find a single instance where $\dim(G) - \dim(G-e) \geq 2$. Our guess is that such graphs do not exist, and we would be very interested to see a proof. Incidentally, if one instead considers the deletion of a vertex of G, there is a little more that can be said.

Question 3 *Does there exist an integer c such that for all graphs G and $v \in V(G)$, we are guaranteed to have* $\dim(G) - \dim(G \setminus \{v\}) \leq c$*? If so, can we let $c = 2$?*

Again, it is easy to construct examples of G and $v \in V(G)$ where $\dim(G) - \dim(G \setminus \{v\})$ is equal to 0 or 1. However, if we let G be the graph $K_2 + C_6$, we have by Theorem 3.9 that $\dim(G) = 4$. Designating by v one of the vertices of G of degree 7, we have that $G \setminus \{v\}$ is isomorphic to W_6. The wheel W_6 is embeddable in $\mathbb{R}^2$ with the usual representation of a regular hexagon of edge-length 1 along with a vertex placed at its center, so here, $\dim(G) - \dim(G \setminus \{v\}) = 2$. We were unable to construct an example where $\dim(G) - \dim(G \setminus \{v\}) \geq 3$.

Question 4 *For which n does there exist an arbitrarily large bipartite graph G which is dimension-critical with* $\dim(G) = n$*?*

The question above is the easiest non-trivial case of a very deep question that we will present at the end of this section, yet even it appears to be rather thorny. In [8], Erdős, Harary, and Tutte demonstrate that for any graph G, $\dim(G) \leq 2\chi(G)$ where $\chi(G)$ denotes the vertex-chromatic number of G. It follows that any bipartite graph G has $\dim(G) \in \{1, 2, 3, 4\}$. Note that $\dim(G) = 1$ if and only if every component of G is isomorphic to a path or an isolated vertex, so Question 4 is trivially answered in the negative when $n = 1$. Equally trivial is the case $n = 2$ where Question 4 is answered in the affirmative. Just take an arbitrarily large even cycle as the desired G. For $n = 3$, we will show that it has an affirmative answer in the theorem below.

Theorem 4.1 *There exist arbitrarily large bipartite graphs G which are dimension-critical with* $\dim(G) = 3$.

Proof For an integer $n \geq 2$, define the *Möbius Ladder* M_{2n} to be the graph of order $2n$ constructed by beginning with two copies of the path P_n, say with the standard vertex sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, respectively, and then placing the additional edges a_ib_i for $i \in \{1, \ldots, n\}$ along with a_1b_n and a_nb_1. As a reference, M_{10} is drawn in Figure 3.

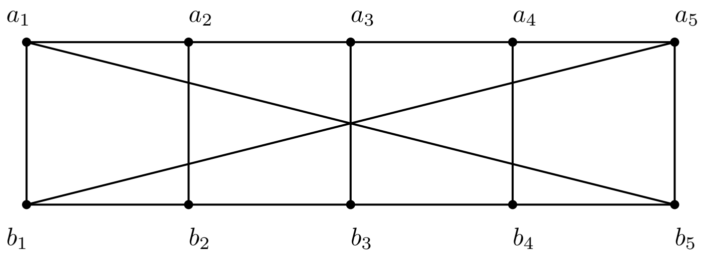

Figure 3

Note that M_{2n} is bipartite when when n is odd. We will now show that $\dim(M_{2n}) > 2$. Indeed, suppose to the contrary that M_{2n} has been drawn as a unit-distance graph in $\mathbb{R}^2$. In such a representation, for $i = 1, 2, \ldots, n-1$, the vertices $a_i, a_{i+1}, b_i, b_{i+1}$ form the vertices of a rhombus. Since opposite sides of a rhombus are parallel, the vector with initial point a_{i+1} and terminal point b_{i+1} must be a translate of the vector with initial point a_i and terminal point b_i. Without loss of generality, we may assume that in this supposed unit-distance drawing of M_{2n} in $\mathbb{R}^2$, we have a_1 placed at the origin and b_1 placed at $(1, 0)$. Now consider circles C_{a_1} and C_{b_1} drawn in Figure 4, each of radius 1 and centered at $(0, 0)$ and $(1, 0)$, respectively. Since $a_1b_n \in E(G)$, we must have b_n placed on C_{a_1}, and similarly, a_n placed on C_{b_1}. However, by the rationale we described above, the line segment connecting a_n and b_n must be horizontal with a_n to the left and b_n to the right. This is a contradiction as it would force a_n to be placed in the exact same position as b_1 (as well as b_n being placed in the same position as a_1).

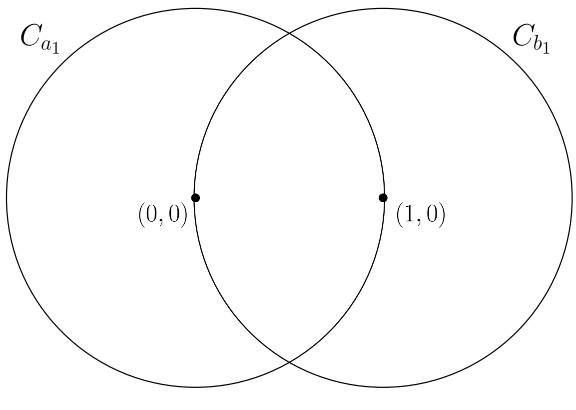

Figure 4

When $n = 3$, the graph M_{2n} is isomorphic to $K_{3,3}$ and it has already been seen that $\dim(K_{3,3}) = 4$. For all higher n, it is the case that $\dim(M_{2n}) = 3$ and furthermore, M_{2n} is dimension-critical. However, we do not need this fact to establish proof of the theorem. One need only observe that, should one start with M_{2n} and then delete vertices (if necessary) until a dimension-critical subgraph has been found, for each $i \in \{1, \dots, n\}$, the vertices a_i and b_i would not both be deleted. The theorem immediately follows. □

We have been unable to resolve Question 4 when $n = 4$. In fact, other than $K_{3,3}$, we have not been able to supply any concrete examples of dimension-critical bipartite graphs G with $\dim(G) = 4$. As it turns out, though, such graphs do exist, which can be seen by observing two major results in extremal combinatorics. In [4], Brown constructs a family of bipartite graphs of order n which do not have $K_{3,3}$ as a subgraph, and whose number of edges is asymptotically on the order of $n^{\frac{5}{3}}$. It is independently shown by Kaplan, Matoušek, Safernová, and Sharir in [9] and by Zahl in [14] that an upper bound for the number of edges in a graph G of order n and satisfying $\dim(G) = 3$ is asymptotically on the order of $n^{\frac{3}{2}}$. Thus for sufficiently large n, a graph G of order n produced via Brown's construction will automatically satisfy $\dim(G) = 4$. Unfortunately (at least, from our point of view), Brown's construction is entirely algebraic, and it seems quite difficult to determine what a dimension-critical subgraph of this G would actually be.

The general formulation of Question 4 is given below.

Question 5 *For which n, k does there exist an arbitrarily large dimension-critical graph G with $\chi(G) = k$ and $\dim(G) = n$?*

A full resolution of this question is far beyond our present reach. For example, a torrent of work has been produced in the past few years on coloring unit-distance graphs in $\mathbb{R}^2$, much of it stemming from de Grey's stunning construction [7] of a 5-chromatic unit-distance graph in the plane. Yet still, it is unknown whether there even exists G satisfying $\dim(G) = 2$ and $\chi(G) \in \{6, 7\}$, let alone an arbitrarily large dimension-critical G with those properties. However, if a successful approach could resolve Question 4, perhaps it could be applied to the more modest Question 6.

Question 6 *For which k does there exist an arbitrarily large dimension-critical graph G with $\chi(G) = k$ and $\dim(G) = 2k$?*

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