Maximal Sets of Triangle-Factors

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Dedicated to the memory of Alan Rahilly, 1947 – 1992

ABSTRACT. A collection of edge-disjoint triangle-factors on K_{3n} is called maximal if it cannot be extended by a further triangle-factor. It is well-known that a maximal set must therefore contain at least $\frac{n}{2}$ triangle factors. We consider the following question: for which k with $\frac{n}{2} \leq k \leq \frac{(3n-1)}{2}$ is there a maximal set of k triangle-factors on K_{3n}?

1. Introduction

A *triangle-factor* on K_{3n} is a vertex-disjoint union of n triangles (K_3s). A collection C of edge-disjoint triangle-factors is called *maximal* if any (further) triangle-factor in K_{3n} shares an edge with some triangle-factors in C, i.e., C cannot be extended by a further triangle-factor. The following basic result is due to Corrádi and Hajnal, [CH].

Lemma 1.1. *Let G be a graph on $3n$ vertices with $\delta(G) \geq 2n$. Then G has a triangle-factor.*

Corollary 1.2. *If C is a maximal set of triangle-factors on $3n$ vertices, then $|C| \geq \frac{n}{2}$.*

Thus, a maximal set on $3n$ vertices must contain at least $\frac{n}{2}$ triangle-factors. At the other end of the spectrum, it is clear that a maximal set cannot contain more than $\frac{(3n-1)}{2}$ triangle-factors.

Theorem 1.3. *For every odd n there is a (maximal) set of $(3n-1)/2$ triangle-factors on $3n$ vertices. For every even $n \geq 6$ there is a (maximal) set of $(3n-2)/2$ triangle-factors on $3n$ vertices.*

Proof. These configurations are, respectively, Kirkman Triple Systems KTS $(3n)$ and Nearly Kirkman Triple Systems NKTS $(3n)$. □

Throughout this paper we will use the notation $F(3n)$ to represent the *spectrum for triangle-factors*, by which we mean $F(3n) = \{\frac{n}{2} \leq k \leq \frac{3n-1}{2}$: there exists a maximal set of k triangle-factors on $3n$ vertices$\}$. Our objective here is to study the behaviour of the function F.

Analogous problems that have been considered and solved recently include determining the spectrum for maximal sets of one-factors [RW1], and for maximal sets of two-factors and of Hamiltonian cycles [HRR]; see also [R6] for further problems of similar kind.

Before proceeding we will introduce some terminology and notation which we shall use throughout the paper. (For undefined design-theoretic terms, see, e.g. [DS].) A $TD(k,n)$ is a transversal design with k groups of size n. A restricted resolvable design $RRP(p,k)$ is a pairwise balanced design on p points, with block sizes two and three, whose block set can be partitioned into k parallel classes; we call the design uniform if it admits a partition so that each parallel class is either a one-factor or a triangle-factor. The spectrum for RRPs was given in a series of papers by Rees (see [R1], [R2], [R3], [R4]):

Theorem 1.4. *There exists an $RRP(p,k)$ if and only if $\lfloor p/2 \rfloor \leq k \leq p-1$ and $p(k-p+1) \equiv 0 \mod 3$, with the following exceptions:*

(i) $p \equiv 1 \mod 6$ *and* $k = (p-1)/2$, *or* p *is odd and* $k = p - 1$

(ii) $p \equiv 3 \mod 6, p \neq 3$ *and* $k = p - 2$

(iii) $p \equiv 3 \mod 6, p \neq 9$ *and* $k = p - 3$, *and*

(iv) $(p, k) = (6, 3)$ *or* $(12, 6)$

Moreover, when $p \equiv 0 \mod 6$ *the* $RRP(p, k)$ *may be taken to be uniform.*

By a $KTS(v) - KTS(w)$ we will mean a Kirkman Triple System of order v which is 'missing' a subsystem of order w, that being a triple (X, Y, B) where X is a set of v points, Y is a subset of X of size w (Y is called the 'hole') and B is a collection of triples on X so that (i) $(X, B \cup \{Y\})$ is a pairwise balanced design and (ii) B admits a partition into parallel classes and holey parallel classes (each holey parallel class being a partition of $X \backslash Y$). An $NKTS(v) - NKTS(w)$ is defined similarly. The spectrum for subsystems in Kirkman Triple Systems was determined by Rees and Stinson (see [RS]).

Theorem 1.5. *A* $KTS(v) - KTS(w)$ *exists if and only if* $v \equiv w \equiv 3$ *modulo 6 and* $v \geq 3w$.

As a useful application of Theorem 1.6 we have the following:

Corollary 1.6. *If* $v \equiv w \equiv 3$ *modulo 6,* $v \geq 3w$ *and* $k \in F(w)$ *then* $\frac{1}{2}(v - w) + k \in F(v)$.

Proof. From Theorem 1.6 we have a $KTS(v) - KTS(w)$. Now, in this design there are $\frac{1}{2}(v - w)$ parallel classes and $\frac{1}{2}(w - 1)$ holey parallel classes. Thus, if we build a maximal set of k triangle-factors on the hole (of size w) and throw away $\frac{1}{2}(w-1)-k$ of the holey parallel classes we are left with a maximal set of $\frac{1}{2}(v-w)+k$ triangle-factors on v points. □

We will begin in the next section by considering $\min F(3n)$ and $\max F(3n)$ for each n (i.e., the "extreme" values) and then in Section 3 we will consider the small values

$n = 1, 2, \ldots, 10$. For these values of n we will see that the only cases that we are presently unable to settle are whether or not $5 \in F(27)$ or $5 \in F(30)$. In Sections 4 and 5 we present some general results (see, e.g. Theorems 4.7 and 5.1), drawing on some of the constructions used in previous sections as well as bringing in some new ones. To this end, we will find the following result to be useful. If G is a graph we denote by $G \otimes I_w$ the graph obtained by taking w copies $x_1, x_2, \ldots, x_w$ of each vertex x in G , where x_i is adjacent to x'_j if and only if x is adjacent to x' in G.

Theorem 1.7. *(Rees, [R5]) If the graph G admits an edge-decomposition into an even number k of triangle-factors, then the graph $G \otimes I_2$ admits an edge-decomposition into $2k$ triangle-factors.*

Corollary 1.8. *If there is a maximal set of an even number k of triangle-factors on $3n$ vertices whose leave graph contains a component on m vertices, $m \not\equiv 0 \mod 3$, then there is a maximal set of $2k$ triangle-factors on $6n$ vertices.*

Proof. Apply Theorem 1.7. The set of $2k$ triangle-factors so produced will have a leave graph with a component on $2m \not\equiv 0 \mod 3$ vertices and so will form a maximal set. □

We end this section with an observation which we shall take advantage of quite frequently throughout the paper. If a graph G on $3n$ vertices has independence number $\alpha(G) > n$ then G cannot contain a triangle-factor; consequently, if C is a collection of triangle-factors whose leave graph contains a large independent set (i.e., on more than one-third the number of vertices) then C is maximal.

2. Extreme values of F

In this section we consider $\max F(3n)$ and $\min F(3n)$. We have in fact already determined $\max F(3n)$ in Theorem 1.3:

Theorem 2.1. *For any positive integer $n \neq 2$ or 4, we have*

$$\max F(3n) = \begin{cases} (3n-1)/2 & \text{if } n \text{ is odd} \\ (3n-2)/2 & \text{if } n \text{ is even} \end{cases}$$

Furthermore, $\max F(6) = 1$ *and* $\max F(12) = 4$.

Proof. See Theorem 1.3. Now NKTS(6) and NKTS(12) do not exist (see [KR]), whence $\max F(6) \leq 1$ and $\max F(12) \leq 4$. It is trivial to construct one triangle-factor on 6 vertices, while to get four (disjoint) triangle-factors on 12 vertices we consider the blocks in a resolvable $TD(3,4)$. □

We turn out attention now to $\min F(3n)$. From Corollary 1.2 we know that $\min F(3n) \geq \frac{n}{2}$.

Theorem 2.2. *If* $n \equiv 1, 2$ *or* $5 \mod 6$ *and* $n \neq 5, 11$ *then* $\min F(3n) = \lceil \frac{n}{2} \rceil$. *Furthermore,* $\min F(15) = 4$, *while* $\min F(33) = 6$ *or* 7.

Proof. If $n \equiv 2 \mod 6$ we take as our vertex set $A \cup B$ where $|A| = n+1$ and $|B| = 2n-1$. From Theorem 2.1 we can construct $\frac{n}{2}$ disjoint triangle-factors $T_1, T_2, \ldots, T_{\frac{n}{2}}$ on A and a further $\frac{n}{2}$ disjoint triangle-factors $T_1', T_2', \ldots, T_{\frac{n}{2}}'$ on B. Then the collection $\mathcal{C} = \{T_1 \cup T_1', T_2 \cup T_2', \ldots, T_{\frac{n}{2}} \cup T_{\frac{n}{2}}'\}$ is a maximal set on $A \cup B$ as all pairs from A are exhausted.

If $n \equiv 1 \mod 6$ take as our vertex set $A \cup B$, where $|A| = n+2$ and $|B| = 2n-2$. Now we proceed as before, appealing to Theorem 2.1 to construct $(n+1)/2$ disjoint triangle-factors on each of A and B. Again all pairs from A are exhausted and so a maximal set on $A \cup B$ is obtained.

For $n \equiv 5 \mod 6, n \geq 17$, we take as our vertex set $A \cup B$ where $|A| = n+1$ and $|B| = 2n-1$. Use Theorem 2.1 to construct $(n-1)/2$ disjoint triangle-factors on each of A and B. This time there remains on A a one-factor, call it P, of pairs that are not covered by any triangle. We will construct one further triangle-factor on $A \cup B$, exhausting these pairs, as follows. Theorem 2.1 assures us that we can construct $n-1$ disjoint triangle-factors on B; hence the $(n-1)/2$

triangle-factors on B previously referred to can be chosen so that there remains on B a further (disjoint) triangle-factor T. Let $P = \{a_0a_1, a_2a_3, \ldots, a_{n-1}a_n\}$ and let $T = \{b_0b_1b_2, b_3b_4b_5, \ldots b_{2n-4}b_{2n-3}b_{2n-2}\}$; then our extra triangle-factor on $A \cup B$ is $\{b_0a_0a_1, b_1a_2a_3, \ldots, b_{(n-1)/2}a_{n-1}a_n, b_{(n+1)/2}b_{(n+3)/2}b_{(n+5)/2}, \ldots, b_{2n-4}b_{2n-3}b_{2n-2}\}$. The result is a maximal set of $(n+1)/2$ triangle-factors on $A \cup B$.

There remain the values $n = 5, 11$ to be considered. It has been shown in [FMR] that $F(15) = \{4, 5, 6, 7\}$; in particular any set of three disjoint triangle-factors on 15 vertices can be extended to include a fourth triangle-factor. There are, in fact, exactly 1409 nonisomorphic maximal sets of 4 disjoint triangle-factors on 15 vertices.

For $n = 11$, we do not yet know whether $6 \in F(33)$. We can show that $7 \in F(33)$, as follows.

Points $\{1, 2, \ldots 12\} \cup (\mathbb{Z}_7 \times \{1, 2, 3\})$

Triangle-Factors: Construct a uniform $RRP(12, 7)$ on $A = \{1, 2, \ldots, 12\}$ with triangle-factors T_1, T_2, T_3, T_4 and one-factors P_1, P_2, P_3. On the set $B = \mathbb{Z}_7 \times \{1, 2, 3\}$ construct the following $KTS(21)$:

$$\begin{aligned} T_i' &= \begin{array}{lll} i_1(i+1)_1(i+3)_1 & i_2(i+2)_2(i+6)_2 & i_3(i+2)_3(i+3)_3 \\ (i+2)_1(i+4)_2(i+6)_3 & (i+4)_1(i+1)_2(i+5)_3 & (i+5)_1(i+3)_2(i+1)_3 \\ (i+6)_1(i+5)_2(i+4)_3, & i \in \mathbb{Z}_7 & \end{array} \\ T_1'' &= j_1j_2j_3,\ j \in \mathbb{Z}_7 \\ T_2'' &= j_1(j+1)_2(j+2)_3,\ j \in \mathbb{Z}_7 \\ T_3'' &= j_1(j+3)_2(j+6)_3,\ j \in \mathbb{Z}_7 \end{aligned}$$

We get four triangle-factors on $A \cup B$ by taking $T_i \cup T_i'$ for $i = 1, 2, 3, 4$.

Now let the edges in P_i be $e_{i1}, e_{i2}, \ldots, e_{i6}$; the remaining triangle-factors on $A \cup B$ are

$$\{e_{11}0_1, e_{12}0_2, e_{13}0_3, e_{14}1_1, e_{15}1_2, e_{16}1_3\} \cup T_1'' \backslash \{0_10_20_3, 1_11_21_3\},$$

and

$$\{e_{21}2_1, e_{22}3_2, e_{23}4_3, e_{24}4_1, e_{25}5_2, e_{26}6_3\} \cup T_2'' \backslash \{2_13_24_3, 4_15_26_3\},$$

and

$$\{e_{31}3_1, e_{32}6_2, e_{33}2_3, e_{34}6_1, e_{35}2_2, e_{36}5_3\} \cup T_3''\backslash\{3_16_22_3, 6_12_25_3\}.$$

As all pairs in A are exhausted, we indeed have a maximal set of triangle-factors on $A \cup B$.

This completes the proof of Theorem 2.2. □

Theorem 2.3. *If* $n \equiv 3, 4$ *or* $0 \mod 6$ *and* $n \neq 3$ *then there is a maximal set of* $\lceil \frac{n}{2} \rceil + 1$ *triangle-factors on* $3n$ *vertices. Also,* $\min F(9) = 4$.

Proof. If $n \equiv 0 \mod 6$ take the vertex set $A \cup B$ where $|A| = n+3$ and $|B| = 2n-3$. From Theorem 2.1 we can construct $\frac{1}{2}(n+2)$ triangle-factors on each of A and B; in this way we obtain a collection of $\frac{1}{2}(n+2)$ triangle-factors on $A \cup B$ which forms a maximal set, as all pairs from A are exhausted.

If $n \equiv 3 \mod 6$ and $n \geq 15$ take the vertex set $A \cup B$ where $|A| = n+3$ and $|B| = 2n-3$. By Theorem 2.1 we can construct $\frac{1}{2}(n+1)$ triangle-factors on A and $\frac{1}{2}(n+1)+1$ triangle-factors on B, from which $\frac{1}{2}(n+1)$ triangle-factors on $A \cup B$ can be constructed. The pairs remaining on A form a one-factor; these together with the extra triangle-factor on B can be used to create a further triangle-factor on $A \cup B$, whereupon all pairs from A are exhausted (see the $n \equiv 5 \mod 6$ case in Theorem 2.2).

Now for $n = 3$ it is not difficult to see that a maximal set of triangle-factors on 9 vertices actually forms a $KTS(9)$. For $n = 9$, we have the following construction for a maximal set of 6 triangle-factors on 27 vertices:

Points $\{a, b, c, d, e, f, g, h, i, j\} \cup \{1, 2, \ldots, 17\}$

Triangle-Factors

a b 2	a c h	a g j	a d f	b d j	b f g
c d 16	b e i	c f i	e h j	c e g	d h i
e f 17	d g 2	b h 3	b c 4	a i 5	a e 1
g h 7	f j 11	d e 12	g i 13	f h 14	c j 15
i j 12	1 6 17	1 6 28	1 6 39	16 4 10	16 5 6
1 6 11	17 6 12	17 7 13	17 8 14	17 9 15	17 10 11
3 8 13	3 14 15	4 15 11	5 11 12	1 12 13	2 13 14
4 9 14	4 5 8	5 1 9	1 2 10	2 3 6	3 4 7
5 10 15	9 10 13	10 6 14	6 7 15	7 8 11	8 9 12

Note that the triples induce an $RRP(10,6)$ on the set $A = \{a, b, \ldots, j\}$ and so exhaust the pairs on A.

Finally, we consider the case $n \equiv 4 \mod 6$. If $n \geq 16$ take the vertex set $A \cup B$ where $|A| = n + 2$ and $|B| = 2n - 2$. By Theorem 2.1 we can construct $\frac{n}{2}$ triangle-factors on A and $\frac{n}{2} + 1$ triangle-factors on B, leaving on A a one-factor of uncovered pairs; now continue as in the $n \equiv 3 \mod 6$ case to get, in all, $\frac{n}{2} + 1$ triangle-factors on $A \cup B$ which form a maximal set. For $n = 4$ we have the following set of three triangle-factors on 12 vertices which forms a maximal set:

1 5 9	1 6 11	1 7 10
2 6 10	2 7 12	2 8 11
3 7 11	3 8 9	3 5 12
4 8 12	4 5 10	4 6 9

Finally, for $n = 10$ we take as our point set $\{a, b, c, \ldots, j, x\} \cup \{1, 2, \ldots, 19\}$ and take the following triangle-factors:

a b 18	a c h	a d f	b d j	b f g	a g j
c d 6	b e i	e h j	c e g	d h i	c f i
e f 12	d g x	b c x	f h x	a e x	b h 5
g h 13	f j 14	g i 15	a i 16	c j 17	d e 18
i j x	1 7 13	2 8 14	3 9 15	4 10 16	x 11 17
1 8 15	2 9 16	3 10 17	4 11 18	5 12 13	6 7 14
4 9 14	5 10 15	6 11 16	1 12 17	2 7 18	3 8 13
2 10 11	3 11-12	4 1 27	5 7 8	6 8 9	1 9 10
3 5 16	4 6 17	5 7 18	6 8 13	1 9 14	2 10 15
19 7 17	19 8 18	19 9 13	19 10 14	19 11 15	19 12 16

The triples induce on $RRP(10,6)$ on $\{a, b, c, \ldots, j\}$, and the point x meets with every point in this set whence all pairs on $\{a, b, c, \ldots, j, x\}$ are exhausted.

This completes the proof of Theorem 2.3. □

We conclude this section with the remark that we do not know of any examples for $n \equiv 3, 4$ or $0 \mod 6$ where the Corrádi-Hajnal bound of $\lceil \frac{n}{2} \rceil$ is actually achieved. It is easily seen that $2 \notin F(9)$ and that $2 \notin F(12)$, and exhaustive computer search has shown that $3 \notin F(18)$. Thus, the first case that arises is the question of whether or not $5 \in F(27)$.

Basically, the algorithm employed to show that $3 \notin F(18)$ goes as follows:

Step 1. Compute all non-isomorphic ways to put two triangle-factors together;

Step 2. For each of the configurations in Step 1, compute all compatible third factors (employing isomorph rejection);

Step 3. For each of the configurations in Step 2 search for a compatible fourth factor.

Note that if for some configuration C in Step 2 there is no compatible fourth factor, then C is maximal and we would have 3 in $F(18)$. What actually happended, however, was that every configuration from Step 2 was able to be extended by Step 3 with a fourth factor, whence no collection of 3 disjoint triangle-factors on 18 vertices is maximal, i.e., $3 \notin F(18)$.

3. Small Values of n

In this section we consider the small cases $n = 1, 2, \ldots, 10$; we will determine $F(3n)$ completely for each $n \neq 9, 10$ (we still do not know whether $5 \in F(27)$ or $5 \in F(30)$). So far we have $F(3) = \{1\}, F(6) = \{1\}, F(9) = \{4\}$ and $F(12) = \{3, 4\}$. We now consider $F(15)$.

Lemma 3.1. $F(15) = \{4, 5, 6, 7\}$

Proof. See [FMR]. □

Next we consider $F(18)$. From Theorem 2.1 we have $\max F(18) = 8$; on the other hand from Theorem 2.3 and the remark following it, we have $\min F(18) = 4$.

Lemma 3.2. $F(18) = \{4, 5, 6, 7, 8\}$

Proof. From the foregoing we must show that 5,6 and 7 are in $F(18)$. We start with 5 triangle-factors.

Points $\{a, b, c, d, e, f, g, h\} \cup \{1, 2, \dots, 10\}$

Triangle-Factors

d e f	d h c	a d g	b e h	c f 2
a b c	g b f	e c 7	a f 1	a h 3
g h 10	a e 10	h f 5	g c 8	d b 7
1 2 3	1 4 7	b 3 4	d 4 9	g e 6
4 5 6	2 5 8	1 6 8	2 6 7	1 5 9
7 8 9	3 6 9	2 9 10	3 5 10	4 8 10

These triangle-factors do in fact form a maximal set, as all pairs from $\{a, b, c, d, e, f, g, h\}$ are exhausted (in fact, the design induced on these points can be obtained by deleting a point from an $RRP(9, 5)$).

For 6 triangle-factors take the point set $\mathbb{Z}_6 \times \{1, 2, 3\}$ and develop each of the following sets of base blocks modulo six: $\{0_1 0_2 0_3\}$, $\{0_1 1_2 2_3\}$, $\{0_1 2_2 4_3\}$, $\{0_1 3_2 1_3\}$, $\{0_1 4_2 3_3\}$ and $\{0_1 2_1 4_1, 0_2 2_2 4_2, 0_3 2_3 4_3\}$.

Finally, for 7 triangle-factors we simply put a triangle-factor on the 'missing' subdesign in an $NKTS(18) - NKTS(6)$ (this design is due to Brouwer [B]). □

We next determine $F(21)$ and $F(24)$.

Lemma 3.3. $F(21) = \{4, 5, 6, 7, 8, 9, 10\}$

Proof. From Theorem 2.1 and Theorem 2.2 we have $\max F(21) = 10$ and $\min F(21) = 4$. Thus, we must show that $\{5, 6, 7, 8, 9\} \subseteq F(21)$.

For 5 triangle-factors we have the following solution on the point set $\{a, b, c, d, e, f, g, h, i\} \cup \{1, 2, \dots, 12\}$:

a b c	a e i	a d g	b e h	c f i
d e f	d h c	b i 6	a f 4	a h 5
g h i	g b f	e c 9	d i 7	d b 8
1 2 3	1 7 11	h f 12	g c 10	g e 11
4 7 10	2 5 9	1 4 5	1 8 9	1 10 12
5 8 11	3 4 12	2 7 8	2 11 12	2 4 6
6 9 12	10 8 6	3 10 11	3 5 6	3 7 9

The triples induce an $RRP(9,5)$ on $\{a,b,c,\dots,i\}$.

For 6 triangle-factors we again use as our point set $\{a,b,c,\dots,i\} \cup \{1,2,\dots,12\}$:

a b c	d e f	g h i	a d g	b e h	c f i
d h 3	c h 4	c d 5	f h 6	d i 1	a h 2
e i 5	a i 6	a e 1	c e 2	a f 3	b d 4
f g 11	b g 12	b f 7	b i 8	c g 9	e g 10
1 7 8	2 8 9	3 9 10	4 10 11	5 11 12	6 12 7
2 10 12	3 11 7	4 12 8	5 7 9	6 8 10	1 9 11
4 6 9	5 1 10	6 2 11	1 3 12	2 4 7	3 5 8.

The triples include an $RRP(9,6)$ on $\{a,b,c,\dots,i\}$.

To get $7 \in F(21)$ we take as our triangles the blocks of a resolvable $TD(3,7)$, while to get $8 \in F(21)$ we take as our point set $(\mathbb{Z}_8 \times \{1,2\}) \cup \{a,b,c,d,e\}$ and develop the following triangle-factor modulo 8:

$$\begin{array}{cc} 0_1 1_1 3_1 & 0_2 1_2 3_2 \\ a 4_1 7_2 & b 4_2 7_1 \\ c 5_1 6_2 & d 5_2 6_1 \\ e 2_1 2_2 & \end{array}$$

To see that we do indeed get a maximal set note that the leave graph contains a K_5 (on the vertices a,b,c,d,e) as a component.

Finally, for 9 triangle-factors we take the point set $(\mathbb{Z}_9 \times \{1,2\}) \cup \{a,b,c\}$ and develop the following triangle-factor modulo 9:

$$\begin{array}{cc} 0_1 1_1 3_1 & 0_2 1_2 3_2 \\ 4_1 8_1 5_2 & 4_2 8_2 5_1 \\ 2_1 6_2 a & 2_2 6_1 b \\ 7_1 7_2 c & \end{array}$$

In this case the leave graph consists of a triangle and an 18-cycle. □

Lemma 3.4. $F(24) = \{4,5,6,7,8,9,10,11\}$

Proof. By Theorems 2.1 and 2.2 we have $\max F(24) = 11$ and $\min F(24) = 4$, and so we must show that $\{5,6,7,8,9,10\} \subseteq F(24)$.

We start with five triangle-factors. Take as our point set $\{a,b,c,d,e,f,g,h,i\} \cup \{1,2,\dots 15\}$ and consider the following factors:

a b c	a e i	a d g	b e h	c f i
d e f	d h c	b i 2	a f 3	a h 4
g h i	g b f	e c 10	d i 6	d b 7
1 6 11	1 7 13	h f 14	g c 15	g e 11
2 7 12	2 8 14	13 1 2	2 4 13	3 5 14
3 8 13	3 9 15	4 6 7	5 7 8	1 8 9
4 9 14	4 10 11	5 9 13	1 10 14	2 6 15
5 10 15	5 6 12	8 11 15	9 12 11	10 13 12

Note that the triples induce an $RRP(9,5)$ on $\{a, b, c, \dots, i\}$.

We now construct a maximal set of 6 triangle-factors, again taking the point set $\{a, b, c, \dots, i\} \cup \{1, 2, \dots, 15\}$:

a b c	d e f	g h i	a d g	b e h	c f i
d h 3	c h 4	c d 5	f h 6	d i 1	a h 2
e i 5	a i 6	a e 1	c e 2	a f 3	b d 4
f g 8	b g 9	b f 10	b i 11	c g 12	e g 7
13 1 7	13 2 8	13 3 9	13 4 10	13 5 11	13 6 12
14 2 9	14 3 10	14 4 11	14 5 12	14 6 7	14 1 8
15 4 12	15 5 7	15 6 8	15 1 9	15 2 10	15 3 11
6 10 11	11 1 12	2 12 7	3 7 8	4 8 9	5 9 10

Here the triples induce an $RRP(9,6)$ on $\{a, b, c, \dots, i\}$.

For 7 triangle-factors we take as our point set $\mathbb{Z}_8 \times \{1, 2, 3\}$ and develop each of the base triples $0_1 0_2 0_3, 0_1 1_2 2_3, 0_1 2_2 4_3, 0_1 3_2 6_3, 0_1 4_2 1_3, 0_1 5_2 3_3$ and $0_1 6_2 5_3$ modulo 8, while for 8 triangle-factors we take as our triangles the blocks of a resolvable $TD(3,8)$. To get a maximal set of 9 triangle-factors we take the point set $(\mathbb{Z}_9 \times \{1,2\}) \cup \{a, b, c, d, e, f\}$ and develop the triangle-factor

$a 1_1 1_2$	$e 7_1 5_2$
$b 3_1 7_2$	$f 8_1 0_2$
$c 4_1 6_2$	$0_1 2_1 5_1$
$d 6_1 3_2$	$2_2 4_2 8_2$

modulo 9 (the only triangles in the leave are contained entirely within the vertex set $\{a, b, c, d, e, f\}$).

Finally, for 10 triangle-factors we put a triangle factor on the 'missing' subdesign in an $NKTS(24) - NKTS(6)$ (see [R2]).

This completes the proof of Lemma 3.4. □

We complete this section by considering $F(27)$ and $F(30)$; in each case there remains one value of k which we are presently unable to include in or exclude from F.

Lemma 3.5. $F(27) \supseteq \{6, 7, 8, 9, 10, 11, 12, 13\}$.

Proof. By Theorem 2.1 we have $\max F(27) = 13$, while we have $6 \in F(27)$ by Theorem 2.3 (we do not yet know whether $5 \in F(27)$), and so we must show that $\{7, 8, 9, 10, 11, 12\} \subseteq F(27)$. We begin with seven triangle-factors. Our ingredients will be a uniform $RRP(12,7)$ (on the point set $A = \{a, b, c, \ldots, l\}$) and a $KTS(15)$ (on the point set $B = \{1, 2, \ldots, 14\} \cup \{\infty\}$):

				a b	a c	a d
a e i	a f j	a g k	a h l	c d	b d	b c
b h k	b g l	b f i	b e j	e f	e g	e h
c f l	c e k	c h j	c g i	g h	f h	f g
d g j	d h i	d e l	d f k	i j	i k	i l
				k l	j l	j k

∞ 1 8	∞ 2 9	∞ 3 10	∞ 4 11	∞ 5 12	∞ 6 13	∞ 7 14
2 11 12	3 12 13	4 13 14	5 14 8	6 8 9	7 9 10	1 10 11
3 9 14	4 10 8	5 11 9	6 12 10	7 13 11	1 14 12	2 8 13
4 6 7	5 7 1	6 1 2	7 2 3	1 3 4	2 4 5	3 5 6
5 10 13	6 11 14	7 12 8	1 13 9	2 14 10	3 8 11	4 9 12

We pair off the first four triangle-factors in each design to yield four-triangle-factors on $A \cup B$. The remaining three triangle-factors are obtained by dismantling two triangles in each of the last three triangle-factors on B and assigning to each set of six points so produced one of the one-factors on A:

∞ a b	∞ e g	∞ i l
5 c d	6 a c	7 a d
12 e f	13 b d	14 b c
6 g h	2 i k	1 e h
8 i j	4 j l	10 f g
9 k l	5 f h	11 j k

In this way a maximal set of 7 triangle-factors on $A \cup B$ is obtained. The constructions for 8 and 9 triangle-factors are similar to the foregoing. For 8 triangle-factors we take a uniform $RRP(12, 8)$ which can be obtained from the foregoing $RRP(12, 7)$ by arranging the pairs covered by the first triangle-factor and the first one-factor into three one-factors:

a b	e f	i j
c d	g h	k l
e i	a i	a e
h k	b k	b h
f l	c l	c f
g j	d j	d g

Now take the $KTS(15)$ given above and pair off the three triangle-factors in the $RRP(12,8)$ with the first three triangle-factors in the $KTS(15)$. Of the triangles that remain on the $KTS(15)$ we can pull out five disjoint subsets, each made up of three disjoint triangles:

∞ 4 11	7 13 11	7 9 10	2 8 13	7 2 3
5 14 8	1 3 4	1 14 12	3 5 6	1 13 9
6 12 10	2 14 10	3 8 11	4 9 12	∞ 5 12

In each case we extend the three disjoint triangles to a triangle-factor on $A \cup B$ by assigning to each point not covered by the three triangles an edge of one of the one-factors on A:

7 a c	∞ a b	∞ e f	∞ i j	14 a d
2 b d	5 e i	6 a i	7 b h	4 e h
3 e g	12 f l	13 g h	14 c f	6 b c
1 f h	6 hk	2 c l	1 k l	8 i l
13 i k	8 g j	4 b k	10 a e	10 f g
9 j l	9 c d	5 d j	11 d g	11 j k

To get 9 triangle-factors we start with a uniform $RRP(12,9)$ which we obtain from the $RRP(12,8)$ by arranging the pairs covered by the triangle-factor a f j, b g l, c e k, d h i and the one-factor a c, b d, e g, f h, i k, j l into three one-factors:

a c	e g	i k
b d	f h	j l
f j	a j	a f
g l	b l	b g
e k	c k	c e
h i	d i	d h

Now take the $KTS(15)$ and pair off the first and third triangle-factors with the two triangle-factors in the $RRP(12,9)$; this gives two triangle-factors on $A \cup B$. From the remaining triangles on the $KTS(15)$ we extract seven disjoint subsets each made up of three disjoint triangles:

∞ 4 11	7 13 11	7 9 10	2 8 13	7 2 3	∞ 6 13	∞ 7 14
5 14 8	1 3 4	1 14 12	3 5 6	1 13 9	2 4 5	6 8 9
6 12 10	2 14 10	3 8 11	4 9 12	∞ 5 12	1 10 11	3 12 13

As before we extend each subset to a triangle-factor on $A \cup B$ by assigning to each point not covered by the three triangles an edge from a one-factor on A.

7 a c	∞ a b	∞ e f	∞ i j	14 a d	3 b l	1 c e
2 b d	5 e i	6 a i	7b h	4 e h	7 d i	10 d h
3 h i	12 f l	13 g h	14 c f	6 b c	8 f h	11 a f
1 f j	6 h k	2 c l	1 k l	8 i l	9 a j	2 i k
13 e k	8g j	4 b k	10 a e	10 f g	12 c k	4 j l
9 g l	9 c d	5 d j	11 d g	11 j k	14 e g	5 b g

We move now to 10 triangle-factors. Take as our point set $(\mathbb{Z}_{10} \times \{1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_7\}$ and develop the base triangle-factor $\infty_1 1_1 0_2$, $\infty_2 2_1 7_2$, $\infty_3 0_1 6_2$, $\infty_4 5_1 3_2$, $\infty_5 7_1 8_2$, $\infty_6 8_1 5_2$, $\infty_7 9_1 9_2$, $3_1 4_1 6_1$, $1_2 2_2 4_2$ modulo 10. We get a maximal set, as the leave contains a K_7(on $\infty_1, \ldots, \infty_7$) as a component. The solution for 11 triangle-factors is similar, taking $(\mathbb{Z}_{11} \times \{1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_5\}$ as our point set and developing the base triangle-factor $\infty_1 6_1 2_2$, $\infty_2 7_1 4_2$, $\infty_3 8_1 6_2$, $\infty_4 9_1 8_2$, $\infty_5 10_1 10_2$, $0_1 3_1 5_1$, $1_1 2_1 3_2$, $0_2 1_2 5_2$, $4_1 7_2 9_2$ modulo 11; the leave contains a K_5 (on $\infty_1, \infty_2, \ldots, \infty_5$) as a component. Finally, to get a maximal set of 12 triangle-factors we take as our point set $(\mathbb{Z}_{12} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3\}$ and develop the triangle-factor $\infty_1 2_1 3_2$, $\infty_2 10_1 5_2$, $\infty_3 1_1 7_2$, $0_1 5_1 2_1$, $3_1 7_1 6_2$, $0_2 8_2 9_1$, $6_1 8_1 9_1$, $4_2 9_2 11_1$, $1_2 10_2 11_2$ modulo 12. Note that the leave graph here consists of a disjoint union of one triangle and six four-cycles. This completes the proof of Lemma 3.5. □

Lemma 3.6. $F(30) \supseteq \{6,7,8,9,10,11,12,13,14\}$.

Proof. From Theorem 2.1 we have $\max F(30) = 14$, while $6 \in F(30)$ by Theorem 2.3 (we do not know yet whether $5 \in F(30)$). Hence we must show that $\{7,8,9,10,11,12,13\} \subseteq F(30)$.

For a maximal set of 7 triangle-factors we simply take two (disjoint) copies of a $KTS(15)$. From here we can easily get 9 triangle-factors, as follows. Let one of the triangle-factors (of the set of 7) be

1 2 3	4 5 6	7 8 9	10 11 12	13 14 15
1' 2' 3'	4' 5' 6'	7' 8' 9'	10' 11' 12'	13' 14' 15'

We dismantle this factor and create three new ones, viz:

1 2 6'	4 5 9'	7 8 12'	10 11 15'	13 14 3'
1' 2' 6	4' 5' 9	7' 8' 12	10' 11' 15	13' 14' 3

1 3 5'	4 6 8'	7 9 11'	10 12 14'	13 15 2'
1' 3' 5	4' 6' 8	7' 9' 11	10' 12' 14	13' 15' 2

2 3 4'	5 6 7'	8 9 10'	11 12 13'	14 15 1'
2' 3' 4	5' 6' 7	8' 9' 10	11' 12' 13	14' 15' 1

For 8 triangle-factors we start with a uniform $RRP(12,8)$ and the following collection of three triangle-factors and five partial triangle-factors on $(\mathbb{Z}_5 \times \{1,2,3\}) \cup \{\infty_1, \infty_2, \infty_3\}$:

$\infty_1\infty_2\infty_3$	$\infty_1 0_1 1_1$	$\infty_1 3_1 4_1$		
$0_1 0_2 0_3$	$\infty_2 1_2 2_2$	$\infty_2 0_2 3_2$	$0_1 3_2 1_3$	
$1_1 1_2 1_3$	$\infty_3 2_3 3_3$	$\infty_3 0_3 4_3$	; $\infty_1 1_2 0_3$	mod 5
$2_1 2_2 2_3$	$2_1 3_2 4_3$	$0_1 1_2 2_3$	$\infty_2 4_1 2_3$	
$3_1 3_2 3_3$	$3_1 4_2 0_3$	$1_1 2_2 3_3$	$\infty_3 1_1 0_2$	
$4_1 4_2 4_3$	$4_1 0_2 1_3$	$2_1 4_2 1_3$		

(For the $RRP(12,8)$ we take the design constructed in Lemma 3.5.) We pair off the three triangle-factors above with the three triangle-factors in the $RRP(12,8)$. Now extend each partial triangle-factor above by assigning to each point not in the factor an edge from a one-factor of the $RRP(12,8)$:

$2_1\ a\ c$	$3_1\ e\ i$	$4_1\ e\ f$	$0_1\ i\ j$	$1_1\ b\ c$
$3_1\ b\ d$	$4_1\ a\ b$	$0_1\ c\ l$	$1_1\ a\ e$	$2_1\ e\ h$
$2_2\ e\ g$	$3_2\ f\ l$	$4_2\ a\ i$	$0_2\ b\ h$	$1_2\ a\ d$
$4_2\ f\ h$	$0_2\ g\ j$	$1_2\ b\ k$	$2_2\ c\ f$	$3_2\ j\ k$
$3_3\ i\ k$	$4_3\ h\ k$	$0_3\ g\ h$	$1_3\ k\ l$	$2_3\ i\ l$
$4_3\ j\ l$	$0_3\ c\ d$	$1_3\ d\ j$	$2_3\ d\ g$	$3_3\ f\ g$

As all blocks from the $RRP(12,8)$ are utilized, we get a maximal set of 8 triangle-factors (on $\{a,b,c,\ldots,l\} \cup (\mathbb{Z}_5 \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$).

We proceed now to 10 triangle-factors. A maximal set is obtained by taking as triangles the blocks of a resolvable $TD(3,10)$. For 11 triangle-factors we take as our point set $(\mathbb{Z}_{11} \times \{1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_8\}$ and we develop the base triangle-factor $\infty_1 0_1 9_2$, $\infty_2 0_2 9_1$, $\infty_3 1_1 8_2$, $\infty_4 1_2 8_1$, $\infty_5 2_1 7_2$, $\infty_6 2_2 7_1$, $\infty_7 3_1 4_2$, $\infty_8 3_2 4_1$, $5_1 6_1 10_1$, $5_2 6_2 10_2$ modulo 11; the leave contains a K_8 (on $\infty_1, \infty_2, \ldots, \infty_8$) as a component and so can not contain a triangle-factor.

For 12 triangle-factors, we proceed as follows. We start with the maximal set of six triangle-factors on 15 points given in Lemma 3.1. As the leave contains a component on 4 vertices (i.e., a 4-cycle) we can apply Corollary 1.9 to get a maximal set of twelve triangle-factors on 30 points, as desired. Similarly, if we start with a maximal set of seven triangle-factors on 15 points (i.e. a $KTS(15)$) and apply Theorem 1.8 to six of these triangle-factors we get a resolvable group-divisible design, with blocks of size three, having 5 groups of size 6. Building a triangle-factor on each group then yields a maximal set of 13 triangle-factors on 30 points.

This completes the proof of Lemma 3.6. □

We summarize the results of the foregoing lemmas in the following:

Theorem 3.7. *Let* $5 \leq n \leq 10$. *Then* $F(3n) = \{k : \frac{n}{2} \leq k \leq \frac{3n-1}{2}\}$, *with the exception of* $(k,n) = (3,5)$ *and* $(3,6)$ *and the possible exceptions of* $(k,n) = (5,9)$ *and* $(5,10)$.

Note that both possible exceptions in Theorem 3.7 are from the class $\{(\lceil \frac{n}{2} \rceil, n) : n \equiv 0, 3 \text{ or } 4 \text{ modulo } 6\}$ (see Theorem 2.3 and the remark following it).

Many of the constructions in this and the previous section may be generalized; this we will do in the next sections.

4. Constructing Maximal Sets From Restricted Resolvable Designs

By far, the most common construction used in Sections 2 and 3 is where we partition our $3n$ points (on which the maximal set is to be constructed) into two subsets A and B, where $|A| > n$, and then build the maximal set so that all pairs from A are exhausted. Usually this will occur by constructing the triples so as to induce an $RRP(p,k)$ on A, k being the number of triangle-factors in the maximal set.

Lemma 4.1. *Let* C *be a set of triangle-factors on* $A \cup B$ *where* $|A| = p$ *and* $|B| = q$ *and suppose that* C *induces an* $RRP(p,k)$ *on* A *where* $k = |C|$. *Then either* C *is a Kirkman Triple System, or* $q \geq p$.

Proof. Suppose that in the $RRP(p,k)$ there are k_i classes each with e_i pairs, $i = 1,\ldots,j$. Then

$$\sum_{i=1}^{j} k_i = k$$

and furthermore, since there are in all $\frac{1}{2}p(2k-p+1)$ pairs (i.e. blocks of size two) in an $RRP(p,k)$, we have

$$\sum_{i=1}^{j} k_i e_i = \frac{1}{2}p(2k-p+1) .$$

Now a parallel class in the RRP containing e_i pairs is induced by a triangle-factor containing $\frac{1}{3}(q-e_i)$ triples from B; hence

$$\binom{q}{2} \geq \sum_{i=1}^{j} k_i(q-e_i) .$$

The three above equations now yield

$$\binom{q}{2} \geq kq - \frac{1}{2}p(2k-p+1) ,$$

from which we get the inequality

$$(1) \qquad (q-p)(q-(2k+1-p)) \geq 0 .$$

Now $2k+1-p < p$ (as $k \leq p-1$ in an $RRP(p,k)$) and so from inequality (1) either $q \geq p$ or $q \leq 2k+1-p$; but in this latter case we get $k \geq \frac{1}{2}(q+p-1)$, which in fact means $k = \frac{1}{2}(q+p-1)$ and so C is a Kirkman Triple System $KTS(q+p)$ (having a Steiner Subsystem $STS(q)$ on B).

This completes the proof of Lemma 4.1 □

From Lemma 4.1 then we see that in order to take advantage of this construction we must take $p \leq q < 2p$ where $p > k$. While it seems certain that such a construction should apply whenever the numerical constraints are met we currently know of no way to prove this. As a result, many of our maximal sets from Sections 2 and 3 which are constructed using this technique are done on a case by case basis. Nonetheless, we will be able to use this construction to determine the bottom quarter of the spectrum for $F(v)$ (Theorem 4.7). We will first need the following result, which is a direct consequence of Hall's Theorem.

Lemma 4.2. *Let G be a subgraph of the complete bipartite graph $K_{n,n}$ with bipartition (v_1, v_2) and suppose that $\delta_1 + \delta_2 \geq n$, where δ_i is the minimum degree of the vertices in part v_i. Then G has a one-factor.*

We will use Lemma 4.2 as follows. If M is a matching in a graph H and S is a set of $|M|$ vertices of H none of which is covered by M, then we define $G(M,S)$ to be the graph whose vertex set is $M \cup S$ and whose edge set is $\{((x,y),z)\colon (x,y) \in M, z \in S$ and x, y, z is a triangle in $H\}$. Note that $G(M,S)$ is a subgraph of $K_{m,m}$ where $m = |M|$. We denote by δ_M and δ_S the minimum degrees, in $G(M,S)$, of the vertices in M and S respectively. Moreover, a one-factor in $G(M,S)$ corresponds to a disjoint set of triangles in H which cover the edges in M and the vertices in S. To facilitate the use of this idea we will denote by $\mathcal{L}(\mathcal{T})$ the leave graph of the collection $\mathcal{T}$ of triangle-factors, that is, the subgraph (of K_v) spanned by those edges which are *not* covered by any triangle-factor in $\mathcal{T}$. We begin with the following lemma, which will be central to the proof of Theorem 4.7.

Lemma 4.3. *Let $p \equiv 0$ modulo 6, $p \geq 18$ and let $v = p + q$ where $p < q < 2p$. Then* $\{\frac{1}{2}p, \frac{1}{2}p + 1, \frac{1}{2}p + 2\} \subseteq F(v)$.

Proof. We start with $k = \frac{1}{2}p$. Construct an $NKTS(p)$ on the p-set and either a $KTS(q)$ or an $NKTS(q)$ on the q-set (depending on whether $q \equiv 3$ or 0 modulo 6). Let $\{x_1x_2, x_3x_4, \ldots, x_{p-1}x_p\}$ be the one-factor on the p-set and let $\{y_1y_2y_3,\ y_4y_5y_6, \ldots, y_{q-2}y_{q-1}y_q\}$ be a triangle-factor on the q-set. We take as one triangle-factor the triples $x_1x_2y_1,\ x_3x_4y_2, \ldots,\ x_{p-1}x_py_{p/2},\ y_{p/2+1}\ y_{p/2+2}\ y_{p/2+3}, \ldots,\ y_{q-2}y_{q-1}y_q$. Then we pair off the $\frac{1}{2}p - 1$ triangle-factors on the p-set with the same number of triangle-factors on the q-set.

For $k = \frac{1}{2}p + 1$ we construct a uniform $RRP(p, \frac{1}{2}p + 1)$ on the p-set and either a $KTS(q)$ or an $NKTS(q)$ on the q-set. There are 3 one-factors and $\frac{1}{2}p - 2$ triangle-factors in the RRP-pair off the triangle-factors with the same number of triangle-factors on the q-set to obtain a collection $\mathcal{T}$ of triangle-factors on v points. Let M_1, M_2 and M_3 be the one-factors on the p-set; since $q > p$ there are (at least) three triangle-factors T_1, T_2, T_3 left on the q-set. We get three more triangle-factors (on v points) as follows.

Factor I: Let $M_1 = x_1^1x_2^1,\ x_3^1x_4^1, \cdots, x_{p-1}^1x_p^1$ and $T_1 = y_1^1y_2^1y_3^1,\ y_4^1y_5^1y_6^1, \cdots, y_{q-2}^1y_{q-1}^1y_q^1$; take the triangle-factor $T^1 = x_1^1x_2^1,\ y_1^1\ x_3^1,\ x_4^1y_2^1, \cdots, x_{p-1}^1x_p^1,\ y_{p/2}^1,\ y_{p/2+1}^1, y_{p/2+2}^1y_{p/2+3}^1, \cdots, y_{q-2}^1y_{q-1}^1y_q^1$.

Factor II: Let $M_2 = x_1^2x_2^2,\ x_3^2x_4^2, \cdots, x_{p-1}^2x_p^2$, $T_2 = y_1^2y_2^2y_3^2, y_4^2y_5^2y_6^2, \cdots, y_{q-2}^2y_{q-1}^2y_q^2$ and let $S = \{y_1^2, y_2^2, \cdots, y_{p/2}^2\}$. Then with respect to the graph $\mathcal{L}(\mathcal{T} \cup \{T^1\})$, $G(M_2, S)$ is a subgraph of $K_{p/2,p/2}$ with minimum degree $\delta(G) \geq p/2 - 2$. From Lemma 4.2 G has a one-factor so that by relabelling if necessary, we get our second triangle-factor $T^2 = x_1^2x_2^2y_1^2, x_3^2x_4^2y_2^2, \cdots, x_{p-1}^2x_p^2y_{p/2}^2, y_{p/2+1}^2y_{p/2+2}^2y_{p/2+3}^2, \cdots, y_{q-2}^2y_{q-1}^2y_q^2$.

Factor III: Let $M_3 = x_1^3x_2^3, x_3^3x_4^3, \cdots, x_{p-1}^3x_p^3$ and $T_3 = y_1^3y_2^3y_3^3,\ y_4^3y_5^3y_6^3, \cdots, y_{q-2}^3y_{q-1}^3y_q^3$, and let $S = \{y_1^3, y_2^3, \cdots, y_{p/2}^3\}$. Then, with respect to $\mathcal{L}(\mathcal{T} \cup \{T^1, T^2\})$, the graph $G(M_3, S)$ is a subgraph of $K_{p/2,p/2}$ having minimum degree $\delta(G) \geq p/2 - 4$; since $p \geq 18$ we can apply Lemma 4.2 to construct a one-factor on G, from which we get our last triangle-factor (again by relabelling if necessary) $T^3 = x_1^3x_2^3y_1^3,\ x_3^3x_4^3y_2^3, \cdots, x_{p-1}^3x_p^3y_{p/2}^3, y_{p/2+1}^3y_{p/2+2}^3y_{p/2+3}^3, \cdots, y_{q-2}^3y_{q-1}^3y_q^3$.

In all then we have a maximal set of $(\frac{1}{2}p - 2) + 3 = \frac{1}{2}p + 1$ triangle factors, as desired.

Finally we consider $k = \frac{1}{2}p + 2$. We proceed as before, starting with a uniform $RRP(p, \frac{1}{2}p + 2)$ on the p-set and either a $KTS(q)$ or an $NKTS(q)$ on the q-set. Pair off the $\frac{1}{2}p - 3$ triangle-factors in the RRP with the same number of triangle-factors on the q-set to get a collection $\mathcal{T}$ of triangle-factors on v points. There remain on the

p-set five one-factors $M_1, M_2, \cdots, M_5$ and on the q-set at least four triangle-factors T_1, T_2, T_3, T_4, with there being a fifth triangle-factor T_5 if $q \geq p+6$. We want in all five more triangle-factors on our v points. We consider two cases.

(i) $q = p + 3$

We begin by constructing five partial triangle-factors, each with $\frac{1}{6}(q+3)$ triangles, from T_1, T_2, T_3, T_4 (actually we will not need T_4). Let B be a fixed triangle of T_2 and let B'_1, B'_2 be fixed triangles of T_3, both of which intersect B. Now let T'_1 be any $\frac{1}{6}(q+3)$ triangles of T_1 which cover the vertices of B and let T'_2 be any $\frac{1}{6}(q-9)$ triangles of T_2, none of which intersects B'_1 or B'_2. Let T'_3 be any $\frac{1}{6}(q+3)$ triangles from $T_3 \backslash \{B'_1 . B'_2\}$. Our five partial triangle-factors are then $\pi_1 = T'_1$, $\pi_2 = (T_1 \backslash T'_1) \cup \{B\}$, $\pi_3 = T_2 \backslash (T'_2 \cup \{B\})$, $\pi_4 = T'_2 \cup \{B'_1, B'_2\}$ and $\pi_5 = T'_3$. Note that the triangle-factors T_1 and T_2 have been exhausted and so each point y in the q-set is covered by at least two of these partial triangle-factors. For each i let S_i denote those points of the q-set that are *not* covered by any triangle of π_i, where $i = 1, \cdots, 5$. Then each point y is covered by at most three of the S_is.

We can now construct our last five triangle-factors. We assume that the first $i-1$ factors $T^1, \cdots, T^{i-1}$ have been constructed; the i^{th} factor goes as follows. Let $M_i = x^i_1 x^i_2 \ x^i_3 x^i_4 \cdots x^i_{p-1} x^i_p$, $S_i = \{y^i_1, y^i_2 \cdots, y^i_{p/2}\}$ and consider the graph $G(M_i, S_i)$ (with respect to $\mathcal{L}(\mathcal{T} \cup \{T^1, \cdots, T^{i-1}\})$). From the foregoing each point $y \in S_i$ is contained in at most two of the S_js, $j = 1, \cdots, i-1$, whence G has parameters $\delta_S \geq \frac{1}{2}p - 4$ and $\delta_M \geq \frac{1}{2}p - 8$. By Lemma 4.2 G has a one-factor provided that $\delta_S + \delta_M \geq \frac{1}{2}p$, i.e. $p \geq 24$. By relabelling if necessary we obtain the triangle-factor $T^i = \pi_i \cup \{x^i_1 x^i_2 y^i_1, \ x^i_3 x^i_4 y^i_2, \ \cdots, x^i_{p-1} x^i_p y^i_{p/2}\}$, as desired.

There remains $p = 18$ to be dealt with. This corresponds to a maximal set of 11 triangle-factors on 39 points. We will achieve this by writing $39 = 15 + 24$ and utilizing an $RRP(15, 11)$ and a Kirkman frame of type 6^4 (see Stinson [S]). Now the $RRP(15, 11)$ has ten parallel classes each with 6 pairs and a triple, and a further parallel class with 5 triples (see Theorem 3.5 in [RW2]). One triangle factor on 39 points is obtained by constructing a triangle-factor on the holes of the frame and pairing this off with the triangle-factor on the RRP. The remaining ten triangle-factors are obtained as follows. First of all we give the parallel classes of the $RRP(15,11)$, written on the point set $\mathbb{Z}_5 \times \{1,2,3\}$:

$$\begin{array}{ccccccc} 0_11_22_3 & & 0_12_24_3 & & & & \\ 1_10_2 & & 1_13_1 & & 0_10_20_3 & & \\ 2_10_3 & & 1_23_2 & & 1_11_21_3 & & \\ 2_21_3 & ; & 1_33_3 & ; & 2_12_22_3 & & \text{mod } 5 \\ 3_14_1 & & 2_10_2 & & 3_13_23_3 & & \\ 3_24_2 & & 4_10_3 & & 4_14_24_3 & & \\ 3_34_3 & & 4_22_3 & & & & \end{array}$$

Now let $\{a,b,c,d,e,f\}$ be a hole in the frame, and let π_1, π_2 and π_3 be the holey parallel classes corresponding to this hole. Three triangle-factors on our 39 points are $\pi_1 \cup \{0_11_22_3,\ a1_10_2,\ b2_10_3,\ c2_21_3,\ d3_14_1,\ e3_24_2,\ f3_34_3\}$, $\pi_2 \cup \{1_12_23_3,\ a2_11_2,\ b3_11_3,$ $c3_22_3,\ d4_20_2,\ e4_30_3,\ f4_10_1\}$ and $\pi_3 \cup \{2_13_24_3,\ a3_12_2,\ b4_12_3,\ c4_23_3,\ d0_31_3,\ e0_11_1,\ f0_21_2\}$. The remaining triangle-factors are constructed analogously, using the remaining holes in the frame. A maximal set of 11 triangle-factors on 39 points results. This completes the consideration of case (i).

(ii) $q \geq p+6$

Our last five triangle-factors are constructed as follows. We will assume that the first $i-1$ factors $T^1, \cdots, T^{i-1}$ have been constructed. Then let $M_i = x_1^i x_2^i$ $x_3^i x_4^i \cdots x_{p-1}^i x_p^i$ and let $S_i = \{y_1^i, y_2^i, \cdots, y_{p/2}^i\}$, where $T_i = y_1^i y_2^i y_3^i,\ y_4^i y_5^i y_6^i, \cdots, y_{q-2}^i y_{q-1}^i y_q^i$. Now in the graph $G(M_i, S_i)$ we have $\delta(G) \geq p/2 - 2(i-1)$ whence by Lemma 4.2 G will have a one-factor if $p - 4i + 4 \geq p/2$; this occurs as long as $p \geq 36$, or when $p = 24, 30$ and $i \neq 5$, or when $p = 18$ and $i \neq 4$ or 5. By relabelling the ys if necessary our i^{th} triangle-factor becomes $T^i = x_1^i x_2^i y_1^i, x_3^i x_4^i y_2^i, \cdots, x_{p-1}^i x_p^i y_{p/2}^i,\ y_{p/2+1}^i y_{p/2+2}^i y_{p/2+3}^i,$ $\cdots, y_{q-2}^i y_{q-1}^i y_q^i$.

The foregoing settles things for $p \geq 36$. When $p = 24$ or 30 we do not get the last triangle-factor. In order to do these orders we will have to be more discriminating in how we choose the $S_i s$. We start with $p = 30$, constructing from T_1, T_2, T_3, T_4, T_5 five partial triangle-factors on the q-set, each with $\frac{1}{3}(q-15)$ triangles, so that each point y in the q-set is covered by at least one triangle. Take any $\frac{1}{3}(q-15)$ triangles from T_1 as the first partial triangle-factor π_1; let $t_1, \cdots, t_5$ be the remaining triangles of T_1. Now choose $\frac{1}{3}(q-21)$ triangles from T_2, none of which intersect with t_1 or t_2 and form from these triangles (together with t_1 and t_2) a second partial triangle-factor π_2. Similarly we choose from each of T_3, T_4 and T_5 $\frac{1}{3}(q-18)$ triangles none of which intersect with, respectively, t_3, t_4 and t_5 and so form three more partial factors π_3, π_4, π_5. Now take S_i to be the set of points on the q-set which are not covered by any triangle in π_i, $i = 1, \cdots, 5$. Then each point y is covered by at most four of the $S_i s$; this will insure that the last (fifth) triangle-factor can be constructed when we repeat the foregoing construction using these new $S_i s$ (in $G(M_5, S_5)$ we will have $\delta_S(G) \geq 9$ and $\delta_M(G) \geq 7$).

Regarding $p = 24$, we will construct on the q-set a set of nine triangle-factors together with five further partial triangle-factors, each with $\frac{1}{3}(q-12)$ triangles, so that each point y in the q-set is covered by at least two of the partial triangle-factors. When $q \equiv 3 \mod 6$ (i.e. $q = 33$, 39 or 45) we accomplish this by means of a $KTS(q) - KTS(9)$ (see Theorem 1.6). The partial triangle-factors are constructed as follows. Let T_1 and T_2 be two holey triangle-factors in the incomplete KTS. (We will take the hole to be $\{y_1, y_2, \cdots y_9\}$.) Let $t_1 \in T_1$; then our first partial triangle-factor is $\pi_1 = T_1 \backslash \{t_1\}$. Now let T_2' be any $\frac{1}{3}(q-21)$-subset of T_2 which covers the vertices of t_1, and take $\pi_2 = T_2' \cup \{y_1y_2y_3,\ y_4y_5y_6,\ y_7y_8y_9\}$. Note that between them π_1 and π_2 cover all points in the q-set. Thus if we repeat the foregoing, starting with the remaining two holey triangle-factors T_3 and T_4 we produce two further partial triangle-factors π_3 and π_4 which between them cover all of the points in the q-set. (Note that in constructing π_4 we must of course take a second triangle-factor on $y_1, y_2, \cdots, y_9$ which is edge-disjoint from that chosen for π_2.) Thus each point y in the q-set is covered by at least two of the partial triangle-factors $\pi_1, \pi_2, \pi_3, \pi_4$. As our fifth partial triangle-factor π_5 we can therefore take any $\frac{1}{3}(q\text{-}12)$-subset of one of the triangle-factors in the incomplete KTS. Note that there will remain at least eleven triangle-factors in the incomplete KTS, nine of which will be used to pair off with the nine triangle-factors in the uniform RRP (24, 14). Now, letting S_i be the set of points of the q-set which are not covered by any triangle in π_i, $i = 1, \cdots, 5$, we see that a given point y will be contained in at most three of the $S_i s$. This then will insure that the last (fifth) triangle-factor on our v points can be constructed, for in $G(M_5, S_5)$ we will have $\delta_S(G) \geq 8$ and $\delta_M(G) \geq 4$.

For $q \equiv 0 \mod 6$ (i.e. $q = 30$, 36, 42) proceed as follows. We start with $q = 30$, constructing on the q-set an $NKTS(30) - NKTS(6)$ (see [R2]) with $\{y_1, y_2, \cdots, y_6\}$ as the hole. Let T_1 and T_2 be the two holey triangle-factors in the incomplete $NKTS$, and let $t_1, t_2 \in T_1$. Take as the first partial triangle-factor $\pi_1 = T_1 \backslash \{t_1, t_2\}$. Now let t_3 and t_4 be triangles in T_2 each of which is disjoint from t_1 and t_2, and take as the second partial triangle-factor $\pi_2 = T_2 \backslash \{t_3, t_4\}$. Let T_3 be a triangle-factor on the incomplete $NKTS$, i.e. $T_3 = \{B_1, \cdots, B_6, B_7, B_8, B_9, B_{10}\}$ where B_i intersects the hole in the point y_i, $i = 1, \cdots, 6$. We take as our third and fourth partial triangle-factors $\pi_3 = \{B_1, B_2, B_3, B_7, B_8\} \cup \{\{y_4, y_5, y_6\}\}$ and $\pi_4 = \{B_4, B_5, B_6, B_9, B_{10}\} \cup \{\{y_1, y_2, y_3\}\}$. Note that each point y in the q-set is covered by at least two of the partial triangle-factors $\pi_1, \pi_2, \pi_3, \pi_4$ and so we take as π_5 any 6 triangles from a second triangle-factor T_4 in the incomplete $NKTS$. There remain ten triangle-factors in the incomplete $NKTS$, nine of which will be used to pair off with the nine triangle-factors in the uniform $RRP(24, 14)$. For $q = 36$ we construct on the q-set a resolvable $TD(3, 12)$. On each group G_j, $j = 1, 2, 3$, construct a (maximal) set of four triangle-factors $\pi_1^j, \pi_2^j, \pi_3^j, \pi_4^j$. Then our five partial triangle-factors are $\pi_1 = \pi_1^1 \cup \pi_2^2$, $\pi_2 = \pi_1^2 \cup \pi_2^3$, $\pi_3 = \pi_1^3 \cup \pi_2^1$, $\pi_4 = \pi_3^1 \cup \pi_4^2$, $\pi_5 = \pi_3^2 \cup \pi_4^3$. As desired each point y in the q-set is contained in at least two (in fact, at least three) of $\pi_1, \pi_2, \cdots, \pi_5$. Nine of the parallel classes on the TD will be paired off with the nine triangle-factors in the uniform $RRP(24,14)$. Finally, for $q = 42$ we start with a Kirkman Triple System $KTS(21)$ in which there are four triples abc, def, adg, beh where the first two triples are in the same parallel class T_1' and the remaining two triples are in the same parallel class T_2'. By Theorem 1.8 we can apply weight 2 to the $KTS(21)$ to yield an $NKTS(42)$; furthermore (see Rees [R5]) this can be done so as to produce from the above configuration of triples *two* such configurations, involving eight triples and four parallel classes T_1, T_2, T_3, T_4 on the $NKTS(42)$. We then obtain our five partial parallel classes of triples on the q-set as follows. Consider the parallel classes T_1, T_2 and the triples $\{y_1y_2y_3,\ y_4y_5y_6\} \subseteq T_1$ and $\{y_1y_4y_7,\ y_2y_5y_8\} \subseteq T_2$. As our first partial triangle-factor π_1 we take 10 triangles from T_1, each of which is disjoint from $y_1y_4y_7$ and $y_2y_5y_8$. For our second partial factor π_2 we take $T_1\backslash\pi_1$ together with six triangles from T_2, each of which is disjoint from each triangle in $T_1\backslash\pi_1$. Note that each point y in the q-set is covered by at least one of π_1 or π_2. Thus if we repeat the foregoing construction with the parallel classes T_3, T_4 and then take as π_5 any subset of ten triangles from a fifth parallel class T_5 of triangles on the $NKTS(42)$ we obtain five partial triangle-factors on the q-set which between them cover each point at least twice. There remain fifteen parallel classes of triples on the $NKTS(42)$, nine of which will be used to pair off with the nine triangle-factors on the uniform $RRP(24,14)$.

In each of the above cases where $q \equiv 0 \mod 6$ we define, for each $i = 1, \cdots, 5$, S_i to be the set of those points in the q-set which are not covered by any triangle in π_i, so that each point y in the q-set is covered by at most three of the S_is. Thus, as in the $q \equiv 3 \mod 6$ cases we will have $\delta_S(G(M_5, S_5)) \geq 8$ and $\delta_M(G(M_5, S_5)) \geq 4$ whence the last (fifth) triangle-factor on v points can be assembled. This settles $p = 24$.

There remains $p = 18$, corresponding to maximal sets of 11 triangle-factors on 42, 45, 48 and 51 points. We can construct these by writing $v = 21 + q'$, $q' = 21, 24, 27, 30$, constructing an $RRP(21,11)$ on the 21-set and then constructing either a $KTS(21)$,

$NKTS(24)$, $KTS(27)$ or resolvable $TD(3,10)$ - having an orthogonal parallel class O - on the q'-set. Now in the $RRP(21,11)$ we have seven parallel classes, each with 3 pairs and 5 triples, and four more parallel classes each with 7 triples (see Table I in the Appendix). Thus if $q' = 21$ we can get our maximal set of 11 triangle-factors by first pairing off the triangle-factors in the $RRP(21,11)$ with the triangle-factor O and the three triangle-factors on the $KTS(21)$ that are disjoint from O, and then constructing one-factors on each of the graphs $G(M_i, S_i)$ where M_i is the set of three pairs in the i^{th} parallel class on the RRP and S_i is the set of points covered by the i^{th} triangle in O, $i = 1, 2, \cdots, 7$. (The i^{th} triangle-factor so produced consists of the 5 triples in the i^{th} parallel class of the RRP together with the 6 triples in $P_i \backslash O$ (P_i being that parallel class in the KTS which contains the i^{th} triangle in O) and the 3 triples arising out of the one-factor on $G(M_i, S_i)$.) The constructions for $q' = 24, 27$ are virtually identical to the foregoing. For $q' = 30$ the four triangle-factors on the q'-set to be paired off with those on the RRP are as follows: take P_8, P_9 and P_{10} (P_i being that parallel class in the TD which contains the i^{th} triangle in O) and, finally, extend the first seven triangles in O by three new triples, each triple being contained in some group of the TD. The remaining seven triangle-factors on $v = 51$ points are constructed as in the $q' = 21$ case.

This completes the proof of Lemma 4.3. □

We need a few more results before proceeding to the main theorem of this section. The first uses what is, strictly speaking, a variation on the RRP construction since we do not quite exhaust all of the pairs on the A-set. Note that this Lemma corresponds to the case where $q = p$ in Lemma 4.3.

Lemma 4.4. *If* $v \equiv 0 \mod 12$ *then* $\{\frac{1}{4}v, \frac{1}{4}v+1, \frac{1}{4}v+2\} \subseteq F(v)$, *except when* $v = 12$ *and* $k = 5$, *and possibly when* $v = 48$ *and* $k = 14$ *or* $v = 60$ *and* $k = 17$.

Proof. We begin with $k = \frac{1}{4}v$. That $3 \in F(12)$ and $6 \in F(24)$ was determined in sections 2 and 3. Let $v \geq 36$ and write $v = 2p$ where $p \equiv 0 \mod 6$ and $p \geq 18$. Construct an $NKTS(p)$ on each of two disjoint sets of p points each. Pair off $\frac{1}{2}p - 2$ triangle-factors on one $NKTS$ with the same number of triangle-factors on the other $NKTS$ to yield a set $\mathcal{T}$ of $\frac{1}{4}v - 2$ triangle-factors on v points. There remains on each p-set a one-factor and a triangle-factor, which we call, respectively, $x_1^i x_2^i, x_3^i x_4^i, \cdots, x_{p-1}^i x_p^i$ and $y_1^i y_2^i y_3^i, y_4^i y_5^i y_6^i, \cdots, y_{p-2}^i y_{p-1}^i y_p^i$ for $i = 1, 2$. Our remaining two triangle-factors are constructed as follows. As our first triangle-factor we take $T = x_1^1 x_2^1 y_1^2, x_3^1 x_4^1 y_2^2, \cdots, x_{p-1}^1 x_p^1 y_{p/2}^2, y_{p/2+1}^2 y_{p/2+2}^2 y_{p/2+3}^2, \cdots, y_{p-2}^2 y_{p-1}^2 y_p^2$. Now let $M = \{x_1^2 x_2^2, x_3^2 x_4^2, \cdots, x_{p-1}^2 x_p^2\}$ and $S = \{y_1^1, y_2^1, \cdots, y_{p/2}^1\}$. Then with respect to the leave graph $\mathcal{L}(\mathcal{T} \cup \{T\})$, the graph $G(M, S)$ has minimum degree $\delta(G) \geq p/2 - 4$ and since $p \geq 18$ we can apply Lemma 4.2 to produce a one-factor in $G(M, S)$ and so in turn (by relabelling if necessary) our last triangle-factor $T' = x_1^2 x_2^2 y_1^1, x_3^2 x_4^2 y_2^1, \cdots, x_{p-1}^2 x_p^2 y_{p/2}^1, y_{p/2+1}^1 y_{p/2+2}^1 y_{p/2+3}^1, \cdots, y_{p-2}^1 y_{p-1}^1 y_p^1$. There remains on each of the p-sets $p/6$ vertex-disjoint triangles. It is easy to see, therefore, that it is impossible to form a further triangle-factor.

Consider now the case $k = \frac{1}{4}v + 1$. We know from sections 2 and 3 that $4 \epsilon F(12)$ and $7 \epsilon F(24)$, and we may therefore assume that $v \geq 36$. As above we write $v = 2p$ but this time we construct a uniform $RRP(p, \frac{1}{2}p+1)$ on one p-set and an $NKTS(p)$ on the other. Pair off $\frac{1}{2}p - 3$ triangle-factors from each of these two designs to yield a set $\mathcal{T}$ of $\frac{1}{4}v - 3$ triangle-factors on v points. Then construct the triangle-factors T and T' as above. There remain two one-factors on the RRP and a triangle-factor on the $NKTS$, from which we will form two more triangle-factors as follows. Let the triangle-factor on the $NKTS$ be $y_1 y_2 y_3, \; y_4 y_5 y_6, \cdots y_{p-2} y_{p-1} y_p$. Let $S_1 = \{y_1, y_2, \cdots, y_{p/2}\}$ and $S_2 = \{y_{p/2+1}, y_{p/2+2}, \cdots, y_p\}$ and let $M_1 = x_1 x_2, x_3 x_4, \cdots, x_{p-1} x_p$ and $M_2 = x'_1 x'_2, x'_3 x'_4, \cdots, x'_{p-1} x'_p$ be the two one-factors on the RRP. Then with respect to the leave graph $\mathcal{L}(\mathcal{T} \cup \{T, T'\})$ the graph $G(M_i, S_i)$ has minimum degrees $\delta_{S_i}(G) \geq \frac{1}{2}p - 3$ and $\delta_{M_i}(G) \geq \frac{1}{2}p - 6$; since $p \geq 18$ Lemma 4.2 applies and so we can construct the last two triangle-factors

$x_1x_2y_1, x_3x_4y_2, \cdots, x_{p-1}x_py_{p/2}, y_{p/2+1}y_{p/2+2}y_{p/2+3}, \cdots, y_{p-2}y_{p-1}y_p$ and

$x'_1x'_2y_{p/2+1}, x'_3x'_4y_{p/2+2} \cdots x'_{p-1}x'_py_p, y_1y_2y_3 \cdots y_{p/2-2}y_{p/2-1}y_{p/2}$.

Finally we let $k = \frac{1}{4}v + 2$. Here we will use a somewhat different idea. First of all we already know from sections 3 and 4 that $5 \notin F(12)$ and $8 \in F(24)$. The following construction shows $11 \in F(36)$. On each of two disjoint sets construct an $NKTS(18)$ which has the triangle-factor $T =$ 1 4 7 , 2 5 8 , 3 6 9 , 10 13 16 , 11 14 17 , 12 15 18 and the one-factor $F =$ 1 10 , 2 11 , 3 12 , 4 13 , 5 14 , 6 15 , 7 16 , 8 17 , 9 18 (see, e.g. Kotzig and Rosa [KR]). From T and its counterpart we construct three triangle-factors on 36 points:

1 4 8′	2 5 9′	3 6 7′	10 13 17′	11 14 18′	12 15 16′
1′4′8	2′5′9	3′6′7	10′13′17	11′14′18	12′15′16
1 7 5′	2 8 6′	3 9 4′	10 16 14′	11 17 15′	12 18 13′
1′7′5	2′8′6	3′9′4	10′16′14	11′17′15	12′18′13
4 7 2′	5 8 3′	6 9 1′	13 16 11′	14 17 12′	15 18 10′
4′7′2	5′8′3	6′9′1	13′16′11	14′17′12	15′18′10

A fourth triangle-factor is constructed from (some of) the pairs in F and its counterpart:

1 10 7′	2 11 8′	3 12 9′	4 13 16′	5 14 17′	6 15 18′
1′10′7	2′11′8	3′12′9	4′13′16	5′14′17	6′15′18

Now pair off the remaining seven triangle-factors on each of the two $NKTSs$ to obtain, in all, 11 triangle-factors on 36 points. There remain three disjoint pairs on each of the two $NKTSs$, and so we clearly have a maximal set.

Now let $v \geq 72$. Our design will be constructed using the case $v = 36$ as a model. Write $v = 2p$, $p \geq 36$, and construct an $NKTS(p)$ on each of two disjoint p-sets. Pair off $\frac{1}{2}p - 2$ triangle-factors from each of the two $NKTSs$ to yield a set $\mathcal{T}$ of $\frac{1}{4}v - 2$ triangle-factors on v points. There remains on each $NKTS$ a triangle-factor and a one-factor. Let the triangle-factors be

$$\begin{array}{lccc} & x_1x_2x_3 & x_4x_5x_6 & \cdots & x_{p-2}x_{p-1}x_p \\ \text{and} & y_1y_2y_3 & y_4y_5y_6 & \cdots & y_{p-2}y_{p-1}y_p \,. \end{array}$$

We get three triangle-factors as follows.

$$T_1 = \begin{matrix} x_1x_2y_6 & x_4x_5y_9 & \cdots & x_{3m+1}x_{3m+2}y_{3m+6} & \cdots & x_{p-2}x_{p-1}y_3 \\ y_1y_2x_6 & y_4y_5x_9 & \cdots & y_{3m+1}y_{3m+2}x_{3m+6} & \cdots & y_{p-2}y_{p-1}x_3 \end{matrix}$$

$$T_2 = \begin{matrix} x_1x_3y_5 & x_4x_6y_8 & \cdots & x_{3m+1}x_{3m+3}y_{3m+5} & \cdots & x_{p-2}x_py_2 \\ y_1y_3x_5 & y_4y_6x_8 & \cdots & y_{3m+1}y_{3m+3}x_{3m+5} & \cdots & y_{p-2}y_px_2 \end{matrix}$$

$$T_3 = \begin{matrix} x_2x_3y_4 & x_5x_6y_7 & \cdots & x_{3m+2}x_{3m+3}y_{3m+4} & \cdots & y_{p-1}x_py_1 \\ y_2y_3x_4 & y_5y_6x_7 & \cdots & y_{3m+2}y_{3m+3}x_{3m+4} & \cdots & y_{p-1}y_px_1 \end{matrix}$$

To get the last triangle-factor, let M be a $(p/3)$-subset of the edge set of the one-factor on one of the $NKTSs$ and let S be a $(p/3)$-subset of points from the other $NKTS$ which is exactly covered by $p/6$ edges from the one-factor on that set. Then with respect to the leave graph $\mathcal{L}(\mathcal{T} \cup \{T_1, T_2, T_3\})$ the graph $G(M, S)$ has parameters $\delta_M(G) \geq p/3 - 8$ and $\delta_S(G) \geq p/3 - 4$ whence by Lemma 4.2 G has a one-factor, since $p \geq 36$. From this we obtain $p/3$ vertex-disjoint triangles on our v points, covering $2p/3$ points from one of the $NKTSs$ and $p/3$ points from the other $NKTS$. Now repeat, choosing M' from the edge set of the second $NKTS$ (so that S does not intersect the point set covered by M') and choosing S' from the point set of the first $NKTS$ (so that S' does not intersect the point set covered by M). In all we obtain a triangle-factor on v points. What remains on each $NKTS$ is a set of $p/6$ mutually disjoint edges, and so a further triangle-factor cannot be constructed. Hence our $\frac{1}{4}v + 2$ triangle-factors so constructed form a maximal set.

This completes the proof of Lemma 4.4. □

The following result, which we state without proof, is a direct analogue to the $k = \frac{1}{4}v + 2$ case of Lemma 4.4.

Lemma 4.4A. If $v \equiv 6 \mod 12$ and $v \geq 18$ then $\frac{1}{4}(v + 6) \in F(v)$.

Lemma 4.5. *If* $v \equiv 3$ *or* $6 \mod 18$ *and* $v \geq 21$ *then* $\lceil \frac{v}{6} \rceil + 1 \in F(v)$.

Proof. If $v = 21$ or 24 apply Lemmas 3.3 and 3.4. Let $v \geq 39$ and write $v = p + q$ where $p = 2\lceil \frac{v}{6} \rceil + 1$; then $p \equiv 3 \mod 6$, $p \geq 15$ and $p < q < 2p$. Divide the v-set into a p-set and a q-set, and on the p-set construct an $RRP(p, \frac{1}{2}(p+1))$ (Theorem 1.5). On the q-set we construct either an $NKTS(q)$ or a $KTS(q)$. Now we will need to examine how the pairs (blocks of size two) occur in the RRP. The information in Table I (Appendix) can be obtained by analyzing the construction for these designs in [R3, Lemma 2.7]. By the *pair-type* of the RRP we will mean the expression $p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ where there are r_i parallel classes each with p_i pairs, $i = 1, \cdots, s$. We omit the term with $p_i = 0$.

Now let us suppose that the pair-type in the RRP is $p_1^{r_1}$ or $p_1^{r_1} p_2^{r_2}$. To a parallel class with p_i pairs we associate the p_i-set of points (in the q-set) covered by $p_i/3$ triangles of some triangle-factor on the q-set. As usual we will so generate a triangle-factor on our v-set provided that the graph $G(M, S)$ has a one-factor, where M is the p_i-set of pairs on the p-set and S is the p_i-set of points on the q-set (see, e.g. the case $p = 18$ in Lemma 4.3). Now since the pairs in the RRP form a 2-regular graph, the parameter $\delta_M(G)$ will always satisfy $\delta_M(G) \geq p_i - 2$, and so G will have a one-factor provided that $\delta_S(G) \geq 2$ (Lemma 4.2); this in turn will happen as long as $p_i \geq 2k$, where $k - 1$ is the maximum value, taken over all $y \in S$, of the number of graphs $G_j(M_j, S_j)$ considered *prior* to $G(M, S)$ with $y \in S_j$. Furthermore if $S' = S \cap (\cup_j S_j)$ and $|S'| = 0$ or 1 then $\delta_M(G) \geq p_i - |S'|$, so that we require only that $\delta_S(G) \geq |S'|$. This is automatic when $|S'| = 0$, while if $|S'| = 1$ we want $p_i \geq 2k - 1$. In particular when $p_i = 3$ this allows one element $y \in S$ to have occurred in one of the preceding S_js.

Referring to Table I we see that the pairs fall into any of three, five or seven parallel classes in the RRP. For those designs in which the pairs fall into three classes a graph $G(M, S)$ will, by the foregoing, have a one-factor provided that $p_i \geq 6$ or, when $p_i = 3$, that (at most) one element $y \in S$ has occurred in one of the preceding S_js. From Table I we see that (for $p \geq 15$) this can be easily arranged; for those designs with a 3 in the pair-type there is only one such 3, and we simply use these three pairs to set up the first graph $G(M, S)$. For those designs in which the pairs fall into five classes a graph $G(M, S)$ has a one-factor provided that $p_i \geq 12$ or, when $p_i = 3$, that at most one element $y \in S$ has occurred in one of the preceding $S_j s$. Now from Table I the pair-type in these designs is either 3^5 ($p = 15$) or $p_1^2 3^3$ where $p_1 \geq 21$. Thus we need consider only the graphs $G(M, S)$ corresponding to $p_i = 3$. When $p = 15$ we have $q = 24$ or $q = 27$; then given $\lfloor \frac{1}{2}(q-1) \rfloor$ triangle-factors on q points it is a simple matter to choose, in turn, five triangles, at most one from each triangle-factor, so that each triangle intersects at most one of its predecessors. This is precisely what we seek, since each of these triangles then gives rise to an S_j (i.e. S_j = point set of the j^{th} triangle, $j = 1, \cdots, 5$). Similarly, for $RRPs$ of pair-type $p_1^2 3^3$ we choose, in turn, three such triangles from the $(N)KTS(q)$ on the corresponding q-set and so form S_1, S_2 and S_3. We then choose any $p_1/3$ triangles from each of a fourth and fifth triangle-factor on the q-set and so form S_4 and S_5. Finally, for those designs in which the pairs fall into seven classes a graph $G(M, S)$ will have a one-factor as long as $p_i \geq 15$ or, when $p_i = 3$, as long as at most one element $y \in S$ has occurred in one of the preceding $S_j s$. From Table I the pair-type in these $RRPs$ is either $3^7 (p = 21)$, $p_1^2 3^5$ or $p_1^3 3^4$ where $p_1 \geq 21$, whence again it suffices to consider graphs $G(M, S)$ corresponding to $p_i = 3$; this can be done in similar fashion to the previous case (note that when $p = 21$ we have $q = 36$ or 39).

This completes the proof of Lemma 4.5. $\square$

The following result is proven similarly.

Lemma 4.6. *If* $v \equiv 0 \mod 18$ *then* $\frac{v}{6} + 2 \in F(v)$.

Proof. If $v = 18$ we apply Lemma 3.2. For $v \geq 36$ write $v = p + q$ where $p = \frac{v}{3} + 3$. Then we have $p \equiv 3 \mod 6$, $p \geq 15$ and $p < q < 2p$. Now proceed as in the proof of Lemma 4.5. The only significant changes occur when $p = 15$ or 21, when we have, respectively, $q = 21$ or 33. On these q-sets construct a $KTS(21)$ and a resolvable

$TD(3,11)$, each of which has an orthogonal parallel class of blocks. In this way we can extract the five (resp. seven) triangles on the q-set with the desired intersection pattern. $\square$

We now proceed to the main result of the section.

Theorem 4.7. *Let $v \equiv 0 \mod 3$, $v \geq 33$. Then $\{\lceil \frac{v}{6} \rceil, \lceil \frac{v}{6} \rceil + 1, \cdots, \lceil \frac{v-1}{4} \rceil\} \subseteq F(v)$, except possibly that $\lceil \frac{v}{6} \rceil \notin F(v)$ where either $v \equiv 0, 9, 12 \mod 18$ or $v = 33$.*

Proof. If $v \equiv 0 \mod 18$ and $k = \frac{v}{6} + 1$ or $\frac{v}{6} + 2$ apply, respectively, Theorem 2.3 or Lemma 4.6. If $v \equiv 3$ or $6 \mod 18$ and $k = \lceil \frac{v}{6} \rceil$ or $\lceil \frac{v}{6} \rceil + 1$ apply, respectively, Theorem 2.2 or Lemma 4.5. For $v = 33$ we need only construct a maximal set of 8 triangle-factors (the case $k = 7$ is dealt with in Theorem 2.2). Take the point set $\{1, 2, \cdots, 12\} \cup (\mathbb{Z}_7 \times \{1, 2, 3\})$, and construct a uniform $RRP(12, 8)$ on $\{1, 2, \cdots, 12\}$ with triangle-factors T_1, T_2, T_3 and one-factors $P_1, \cdots, P_5$. On $\mathbb{Z}_7 \times \{1, 2, 3\}$ construct the $KTS(21)$ given in the case $v = 33$ of Theorem 2.2. Our first three triangle-factors are $T_i \cup T_i'$, $i = 1, 2, 3$ and we get three more triangle-factors from P_1, P_2, P_3 and T_1'', T_2'', T_3'' in the same manner as that of Theorem 2.2. The last two triangle-factors are obtained as follows. Let $M = P_4$ and $S = \{2_1, 4_2, 6_3, 6_1, 5_2, 4_3\}$; with respect to the leave graph of the first six triangle-factors the graph $G(M, S)$ has minimum degree $\delta(G) \geq 3$ and so by Lemma 4.2 has a one-factor. By relabelling if necessary we get our seventh triangle-factor

$$\{e_1 2_1,\ e_2 4_2,\ e_3 6_3,\ e_4 6_1,\ e_5 5_2,\ e_6 4_3\} \cup (T_0' \backslash \{2_1 4_2 6_3,\ 6_1 5_2 4_3\}) .$$

The eight triangle-factor is constructed analogously, using $M = P_5$ and $S = \{1_1, 3_2, 5_3, 5_1, 4_2, 3_3\}(\{1_1 3_2 5_3,\ 5_1 4_2 3_3\} \subseteq T_6')$.

Otherwise, let $\lceil \frac{v}{6} \rceil \leq k \leq \lceil \frac{v-1}{4} \rceil$, $p = 2k - (2k \mod 6)$ and $q = v - p$. Then $v = p + q$, where $p \equiv 0 \mod 6$ and $p \leq q < 2p$ (with $p = q$ occurring precisely when $v \equiv 0 \mod 12$ and $k = \frac{v}{4}$) and, furthermore, $k = \frac{1}{2}p + i$ where $i = 0, 1$ or 2. Finally, since $v \geq 36$ we have $p \geq 18$. Now apply Lemma 4.4 ($p = q$) or Lemma 4.3. $\square$

Theorem 4.7 gives the bottom quarter of the (anticipated) spectrum for $F(v)$. With regard to the values $k = \lceil \frac{v}{6} \rceil$, $v \equiv 0, 9$ or 12 mod 18, it is not difficult to see that the RRP construction, strictly employed, cannot work. For example, if $v = 36$, $k = 6$ then we would have to have $p > 12$ whereupon no $RRP(p, 6)$ exists. We do not know at present how to deal with these cases. In fact (as previously noted) we do not know of a single value for v in these classes mod 18 for which $\lceil \frac{v}{6} \rceil \in F(v)$.

5. Other Constructions for Maximal Sets

In this section we will briefly indicate some further constructions which will prove useful in studying this problem.

The first construction has already been given as Corollary 1.6 and, under optimum conditions, will yield the 'top one-third' of the spectrum for $F(v)$, where $v \equiv 3$ mod 6. Thus given $v \equiv 3$ mod 6 we write v as one of $3w$, $3w + 6$ or $3w + 12$ where $w \equiv 3$ modulo 6. Then apply Corollary 1.6 over all values of $k \in F(w)$. For example, we have $F(15) = \{4, 5, 6, 7\}$ (Lemma 3.1). We get therefore $F(45) \supseteq \{19, 20, 21, 22\}$, $F(51) \supseteq \{22, 23, 24, 25\}$ and $F(57) \supseteq \{25, 26, 27, 28\}$. For $v = 63$ we advance w to 21, employing Lemma 3.3 and so obtaining $F(63) \supseteq \{25, 26, 27, 28, 29, 30, 31\}$, and so on. In general we have the following inductive construction.

Theorem 5.1. *If $w \equiv 3 \mod 6$, $v = 3w + 6r (r = 0, 1$ or $2)$ and $k \in F(w)$ for $k \geq t$, then $w + k' \in F(v)$ for $k' \geq 3r + t$. In particular if $t = (w + 3)/6$ then $k'' \in F(v)$ for all $k'' \geq \frac{7v+9+12r}{18}$.*

Note that $(v-1)/2-(7v+9+12r)/18$ is roughly $v/9$, which represents one third of the (anticipated) spectrum for $F(v)$. The net result of Theorems 4.7 and 5.1 is that for $v \equiv 3$ mod 6 we can limit our attention to values of k between $v/4$ and $7v/18$.

In order to derive a result for $v \equiv 0$ mod 6 analogous to Theorem 5.1 we would need to know something about the spectrum for large 'holes' in Nearly Kirkman

Triple Systems (i.e. an analogue to Theorem 1.6 for $NKTSs$, with $w \equiv 0 \mod 6$ and $v = 3w$, $3w + 6$ or $3w + 12$).

The second construction, illustrated by the solutions for 10, 11 and 12 triangle-factors on 27 points (Lemma 3.5) uses the observation that if C is a collection of triangle-factors whose leave graph contains a component on $m \not\equiv 0 \pmod 3$ vertices, then C is a maximal set. In the simplest case, where the component is a K_m, it is easy to see that $m \leq \frac{1}{3}v$, where $k \geq \frac{1}{3}v$. This construction will be useful for those cases $k \geq \frac{v}{3}$ where the foregoing recursive construction does not apply. We will now apply this construction to prove the following.

Theorem 5.2. *Let $v \equiv 0 \mod 3$, $33 \leq v \leq 42$, and suppose that $\frac{1}{3}v \leq k \leq \frac{v-1}{2}$ and $k \not\equiv 0 \mod 3$. Then $k \in F(v)$.*

Proof. We begin with $v = 33$, constructing maximal sets for $k = 11, 13$ and 14:

<u>$k = 11$</u> Take the blocks of a resolvable $TD(3, 11)$.

<u>$k = 13$</u> Take the point set $(\mathbb{Z}_{13} \times \{1,2\}) \cup \{\infty_1, \infty_2, \cdots, \infty_7\}$ and develop the following triangle-factor modulo 13:

$$\begin{array}{lll} 0_1 2_1 7_1 & \infty_1 3_1 6_2 & \infty_5 10_1 12_2 \\ 0_2 2_2 7_2 & \infty_2 3_2 6_1 & \infty_6 10_2 12_1 \\ 1_1 4_1 5_1 & \infty_3 8_1 9_2 & \infty_7 11_1 11_2\,. \\ 1_2 4_2 5_2 & \infty_4 8_2 9_1 & \end{array}$$

<u>$k = 14$</u> Take the point set $(\mathbb{Z}_{14} \times \{1,2\}) \cup \{\infty_1, \infty_2, \cdots, \infty_5\}$ and develop the following triangle-factor modulo 14:

$$\begin{array}{lll} 9_1 11_1 12_1 & 2_1 5_2 7_1 & \infty_3 3_1 13_2 \\ 9_2 11_2 12_2 & 2_2 5_1 7_2 & \infty_4 3_2 13_1 \\ 0_1 1_2 6_1 & \infty_1 4_1 10_2 & \infty_5 8_1 8_2\,. \\ 0_2 1_1 6_2 & \infty_2 4_2 10_1 & \end{array}$$

For $k = 16$, take a $KTS(33)$. In each of the foregoing, the leave graph contains a K_m ($m \not\equiv 0 \mod 3$) as a component; where $k = 13, 14$ it occurs on the ∞s.

Now let $v = 36$, so that $k = 13, 14, 16$, or 17.

$\underline{k=13}$ Take the point set $(\mathbb{Z}_{13}\times\{1,2\})\cup\{\infty_1,\infty_2,\cdots,\infty_{10}\}$ and develop the following triangle-factor modulo 13:

$$\begin{array}{lll}
0_1 1_1 4_1 & \infty_3 2_1 9_2 & \infty_7 6_1 7_2 \\
0_2 1_2 4_2 & \infty_4 2_2 9_1 & \infty_8 6_2 7_1 \\
\infty_1 3_1 12_2 & \infty_5 5_1 10_2 & \infty_9 8_1 11_2 \\
\infty_2 3_2 12_1 & \infty_6 5_2 10_1 & \infty_{10} 8_2 11_1 \ .
\end{array}$$

$\underline{k=14}$ Here our point set is $(\mathbb{Z}_{14}\times\{1,2\})\cup\{\infty_1,\infty_2,\cdots,\infty_8\}$. Develop the following triangle-factor modulo 14:

$$\begin{array}{lll}
0_1 2_1 3_1 & \infty_1 7_1 13_2 & \infty_5 1_1 12_2 \\
0_2 2_2 3_2 & \infty_2 7_2 13_1 & \infty_6 1_2 12_1 \\
4_1 5_2 9_1 & \infty_3 6_1 11_2 & \infty_7 8_1 10_2 \\
4_2 5_1 9_2 & \infty_4 6_2 11_1 & \infty_8 8_2 10_1 \ .
\end{array}$$

For $k = 16$, construct a maximal set of 4 triangle-factors on each group in a resolvable $TD(3,12)$, and for $k = 17$ take an $NKTS(36)$.

We proceed now to $v = 39$, solving for $k = 13, 14, 16, 17, 19$.

$\underline{k=13}$ Take the blocks of a resolvable $TD(3,13)$.

$\underline{k=14}$ Our point set is $(\mathbb{Z}_{14}\times\{1,2\})\cup\{\infty_1,\infty_2,\cdots,\infty_{11}\}$. Develop the following triangle-factor modulo 14:

$$\begin{array}{llll}
6_1 7_1 10_1 & \infty_3 1_1 4_2 & \infty_7 8_1 12_2 & \infty_{11} 13_1 13_2 \ . \\
6_2 7_2 10_2 & \infty_4 1_2 4_1 & \infty_8 8_2 12_1 & \\
\infty_1 0_1 5_2 & \infty_5 2_1 3_2 & \infty_9 9_1 11_2 & \\
\infty_2 0_2 5_1 & \infty_6 2_2 3_1 & \infty_{10} 9_2 11_1 &
\end{array}$$

$\underline{k=16}$ Take the point set $(\mathbb{Z}_{16}\times\{1,2\})\cup\{\infty_1,\infty_2,\cdots,\infty_7\}$. Develop the following triangle-factor modulo 16:

$$\begin{array}{llll}
10_1 12_1 13_1 & 4_1 9_1 14_2 & \infty_3 1_1 2_1 & \infty_7 5_1 5_2 \ . \\
10_2 12_2 13_2 & 4_2 9_2 14_1 & \infty_4 1_2 2_1 & \\
7_1 11_1 15_2 & \infty_1 0_1 3_2 & \infty_5 8_1 6_2 & \\
7_2 11_2 15_1 & \infty_2 0_2 3_1 & \infty_6 8_2 6_1 &
\end{array}$$

$\underline{k=17}$ Take the point set $(\mathbb{Z}_{17}\times\{1,2\})\cup\{\infty_1,\infty_2,\cdots,\infty_5\}$, and develop the following triangle-factor modulo 17:

$$\begin{array}{llll}
0_1 1_1 8_2 & 2_1 6_1 12_1 & \infty_1 3_1 4_2 & \infty_5 7_1 7_2 \ . \\
0_2 1_2 8_1 & 2_2 6_2 12_2 & \infty_2 3_2 4_1 & \\
9_1 11_1 15_2 & 5_1 10_1 13_1 & \infty_3 14_1 16_2 & \\
9_2 11_2 15_1 & 5_2 10_2 13_2 & \infty_4 14_2 16_1 &
\end{array}$$

For $k = 19$ we take a $KTS(39)$.

Finally, we consider $v = 42$, taking in turn $k = 14, 16, 17, 19, 20$.

$\underline{k = 14}$ Take the blocks of a resolvable $TD(3, 14)$.

$\underline{k = 16}$ Our point set is $(\mathbb{Z}_{16} \times \{1,2\}) \cup \{\infty_1, \infty_2, \cdots, \infty_{10}\}$. Develop the following triangle-factor modulo 16:

$$\begin{array}{llll} 0_1 3_1 9_1 & \infty_1 1_1 8_2 & \infty_5 5_1 10_2 & \infty_9 13_1 14_2 \\ 0_2 3_2 9_2 & \infty_2 1_2 8_1 & \infty_6 5_2 10_1 & \infty_{10} 13_2 14_1 \; . \\ 6_1 7_1 11_1 & \infty_3 2_1 4_2 & \infty_7 12_1 15_2 & \\ 6_2 7_2 11_2 & \infty_4 2_2 4_1 & \infty_8 12_2 15_1 & \end{array}$$

$\underline{k = 17}$ Our point set is $(\mathbb{Z}_{17} \times \{1,2\}) \cup \{\infty_1, \infty_2, \cdots, \infty_8\}$. We develop the following triangle-factor modulo 17:

$$\begin{array}{llll} 0_1 2_1 7_1 & 10_1 16_1 15_2 & \infty_3 3_1 14_2 & \infty_7 11_1 13_2 \\ 0_2 2_2 7_2 & 10_2 16_2 15_1 & \infty_4 3_2 14_1 & \infty_8 11_2 13_1 \; . \\ 5_1 6_1 9_1 & \infty_1 1_1 4_2 & \infty_5 8_1 12_2 & \\ 5_2 6_2 9_2 & \infty_2 1_2 4_1 & \infty_6 8_2 12_1 & \end{array}$$

For $k = 19$ we can construct a triangle-factor on each group in a resolvable $3-GDD$ of type 6^7, while for $k = 20$ we take an $NKTS(42)$.

This completes the proof of Theorem 5.2. □

For the sake of completeness we will fill in some of the gaps left by Theorem 5.2. We will apply the same type of construction as that used in Theorem 5.2 but we'll have to be careful to ensure that the set of triangle-factors so produced really is a maximal set.

Lemma 5.3. *Let* $v \equiv 0$ *modulo 3,* $33 \leq v \leq 42$, *and suppose that* $\frac{1}{3}v \leq k \leq \frac{v-1}{2}$ *and* $k \equiv 0$ *modulo 3. Then* $k \in F(v)$, *except possibly for* $12 \in F(36)$.

Proof. We begin with $v = 33$, constructing maximal sets for $k = 12$ and $k = 15$.

$\underline{k = 12}$ Point set $(\mathbb{Z}_{12} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup (\{b\} \times \mathbb{Z}_2) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Develop the following triangle-factor modulo 12 (subscripts on as are developed modulo 3 and subscripts on bs are developed modulo 2).

$$\begin{array}{lll} 0_1 1_1 5_1 & a_0 10_1 10_2 & \infty_2 3_2 8_1 \\ 0_2 5_2 9_2 & a_1 4_2 6_2 & \infty_3 7_1 11_2 \\ b_0 6_1 9_1 & a_2 2_1 4_1 & \infty_4 7_2 11_1 \\ b_1 1_2 2_2 & \infty_1 3_1 8_2 & \end{array}$$

Consider now the leave graph. On $\mathbb{Z}_{12} \times \{1,2\}$ there remains pure difference 6 and mixed differences $\pm1, \pm2, \pm3$ and 6. The edges of pure difference 6 and mixed differences ±3 form a K_4-factor (on $\mathbb{Z}_{12} \times \{1,2\}$) and, furthermore, any triangle on $\mathbb{Z}_{12} \times \{1,2\}$ is contained in one of these K_4s. Hence there is no triangle-factor on the leave graph and so our 12 triangle-factors form a maximal set.

$\underline{k = 15}$ Point set $(\mathbb{Z}_{15} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3\}$. Develop the following triangle-factor modulo 15.

$0_1 1_1 5_1$	$3_1 9_1 0_2$	$\infty_1 11_1 1_2$
$4_2 6_2 9_2$	$13_1 5_2 13_2$	$\infty_2 4_1 8_2$
$2_1 11_2 12_2$	$7_1 10_1 3_2$	$\infty_3 14_1 2_2$
$12_1 10_2 14_2$	$6_1 8_1 7_2$	

The leave graph consists of a triangle, three pentagons (pure difference 6 on $\mathbb{Z}_{15} \times \{2\}$) and a 15-cycle (pure difference 7 on $\mathbb{Z}_{15} \times \{1\}$).

Now consider $v = 36$, with $k = 15$ (we do not yet have a construction for $k = 12$); here we simply construct a maximal set of 3 triangle-factors on each group of a resolvable $TD(3,12)$.

For $v = 39$ we have $k = 15$ and $k = 18$.

$\underline{k = 15}$ We take as our point set $(\mathbb{Z}_{15} \times \{1,2\}) \cup \{\infty_1, \infty_2, \cdots, \infty_9\}$. Develop the following triangle-factor modulo 15.

$0_1 4_1 9_1$	$\infty_2 1_2 2_1$	$\infty_7 11_1 14_2$
$0_2 4_2 9_2$	$\infty_3 3_1 7_2$	$\infty_8 11_2 14_1$
$5_1 6_1 8_1$	$\infty_4 3_2 7_1$	$\infty_9 13_1 13_2$
$5_2 6_2 8_2$	$\infty_5 10_1 12_2$	
$\infty_1 1_1 2_2$	$\infty_6 10_2 12_1$	

On $\mathbb{Z}_{15} \times \{1,2\}$ there remain pure difference 7 and mixed differences ±5, ±6, ±7. It is easy to see therefore that the leave contains no triangle-factor.

$\underline{k = 18}$ Our point set is $(\mathbb{Z}_{18} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3\}$. Develop the following triangle-factor modulo 18.

$0_1 1_1 8_1$	$7_1 1_2 5_2$	$\infty_1 5_1 12_2$
$11_1 13_1 17_1$	$10_1 3_2 9_2$	$\infty_2 2_1 17_2$
$3_1 6_2 7_2$	$12_1 15_1 2_2$	$\infty_3 14_1 14_2$
$4_1 9_1 0_2$	$16_1 4_2 11_2$	
$6_1 8_2 16_2$	$10_2 13_2 15_2$	

The leave graph consists of a triangle and nine 4-cycles.

Finally, we consider $v = 42$, constructing maximal sets for $k = 15$ and $k = 18$.

<u>$k = 15$</u> Our point set is $(\mathbb{Z}_{15} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup (\{b\} \times \mathbb{Z}_3) \cup (\{c\} \times \mathbb{Z}_5) \cup \{\infty\}$. Develop the following triangle-factor modulo 15 (subscripts on as and bs are developed modulo 3 and subscripts on cs are developed modulo 5):

$$\begin{array}{lll} 6_1 8_1 11_1 & b_1 6_1 14_2 & c_3 11_2 12_2 \\ a_0 12_1 4_2 & b_2 5_1 13_1 & c_4 14_1 0_1 \\ a_1 9_2 13_2 & c_0 9_1 2_2 & 5_2 8_2 10_2 \\ a_2 3_1 7_1 & c_1 1_2 7_2 & \infty 2_1 3_2 \, . \\ b_0 1_1 0_2 & c_2 4_1 10_1 & \end{array}$$

As all pure differences in $\mathbb{Z}_{15} \times \{1,2\}$ are exhausted, we clearly have a maximal set.

<u>$k = 18$</u> Take the point set $(\mathbb{Z}_{18} \times \{1,2\}) \cup \{\infty_1, \infty_2, \cdots, \infty_6\}$. Develop the following triangle-factor modulo 18.

$$\begin{array}{lll} 0_1 8_1 6_2 & 3_2 5_2 10_2 & \infty_3 9_1 13_2 \\ 0_2 8_2 6_1 & 12_1 15_1 16_1 & \infty_4 9_2 13_1 \\ 1_1 7_1 2_2 & 12_2 15_2 16_2 & \infty_5 14_1 17_2 \\ 1_2 7_2 2_1 & \infty_1 4_1 11_2 & \infty_6 14_2 17_1 \\ 3_1 5_1 10_1 & \infty_2 4_2 11_1 & \end{array}$$

There remain on $\mathbb{Z}_{18} \times \{1,2\}$ pure difference 9 and mixed differences 0, 8, 9 and 10. Now the edges of pure difference 9 and mixed differences 0 and 9 form a K_4-factor (on $\mathbb{Z}_{18} \times \{1,2\}$) and, furthermore, any triangle on $\mathbb{Z}_{18} \times \{1,2\}$ is contained in one of these K_4s. Hence the leave graph contains no triangle-factor.

This completes the proof of Lemma 5.3. □

Theorem 5.2 and Lemma 5.3 together give the top half of the spectrum for $F(v)$, $v = 33, 36, 39, 42$. Indeed it seems quite reasonable to suggest that the recursive construction presented earlier in this section, together with the construction illustrated by the two foregoing results, will lead to an algorithm for the general construction of maximal sets of size k for k in the interval $v/3 \le k \le (v-1)/2$. Many of the details remain to be worked out, however, and this we defer to a later study. (One such "detail" which is of interest in its own right is the construction of Nearly Kirkman Triple Systems with large holes (see the remarks following Theorem 5.1).)

We conclude this section by collecting what we have proven in regards to the spectrum $F(v)$ for $v = 33, 36, 39, 42$.

Theorem 5.4. *Let $v \equiv 0 \bmod 3$, $33 \leq v \leq 42$. Then $k \in F(v)$ for all $\lceil \frac{v}{6} \rceil \leq k \leq \frac{(v-1)}{2}$ with the possible exceptions of $v = 33$ and $k = 6, 9, 10$; $v = 36$ and $k = 6, 12$; $v = 39$ and $k = 12$; and $v = 42$ and $k = 13$.*

Proof. For $\lceil \frac{v}{6} \rceil \leq k \leq \lceil \frac{v-1}{4} \rceil$ see Theorem 4.7, and for $\frac{v}{3} \leq k \leq \frac{v-1}{2}$ see Theorem 5.2 and Lemma 5.3. For $v = 36$ and $k = 10, 11$ see Lemma 4.4. For $v = 39$ and $k = 11$ use Lemma 4.3 (with $p = 18$ and $q = 21$) and, finally, for $v = 42$ and $k = 12$ apply Lemma 4.4A. □

6. Conclusion

In this paper we have initiated the study of the problem of determining the spectrum for maximal sets of triangle-factors on v points. The authors are certain that this spectrum will contain the interval $\lceil \frac{v}{6} \rceil < k \leq \frac{(v-1)}{2}$. Indeed we have proven this for $\lceil \frac{v}{6} \rceil < k \leq \frac{v}{4}$ and we have given good grounds for believing this to be true for $\frac{v}{3} \leq k \leq \frac{(v-1)}{2}$. There remains the interval $\frac{v}{4} < k < \frac{v}{3}$, for which a new idea appears to be needed. Additionally, there remains $k = \lceil \frac{v}{6} \rceil$, $v \equiv 0, 9$ or $12 \bmod 18$; we know of not a single example of such a maximal set, nor do we know of any good reason why such a maximal set should not exist. We do know only that $\lceil \frac{v}{6} \rceil \notin F(v)$ for $v = 9, 12$ and 18. Whether or not $6 \in F(33)$ also remains as an interesting open problem.

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APPENDIX

Pair-types of Restricted Resolvable Designs $RRP(p, \frac{1}{2}(p+1))$

The following table is used in the proof of Lemma 4.5. The parameter t here is $(p-3)/6$.

Table I

$t-4 \pmod{12}$	t	Pair-Type of $RRP(p, \frac{1}{2}(p+1))$
0	≥ 4	$(p/3)^3$
1	≥ 5	$((p-3)/2)^2 3^1$
2	6	$18^2 3^1$
	≥ 18	$((p-9)/2)^2 3^3$
3	7	15^3
	≥ 19	$((p-15)/2)^2 3^5$
4	8	$21^2 3^3$
	≥ 20	$((p-9)/2)^2 9^1$
5	9	$24^2 3^3$
	21	$63^2 3^1$
	≥ 33	$((p-3)/2)^2 3^1$
6	10	21^3
	22	$63^2 3^3$
	≥ 34	$((p-3)/2)^2 3^1$
7	11	$27^2 3^5$
	23	$63^2 3^5$
	≥ 35	$((p-15)/2)^2 15^1$
8	12	$21^3 3^4$
	24	$69^2 9^1$
	≥ 36	$((p-9)/2)^2 3^3$

Table I (continued)

$t-4 \pmod{12}$	t	Pair-Type of $RRP(p, \frac{1}{2}(p+1))$
9	1	3^3
	13	27^3
	25	$75^2 3^1$
	37	$111^2 3^1$
	≥ 49	$((p-9)/2)^2 3^3$
10	2	3^5
	14	$39^2 3^3$
	26	$75^2 3^3$
	38	$111^2 3^3$
	≥ 50	$((p-21)/2)^2 21^1$
11	3	3^7
	15	$27^3 3^4$
	27	$75^2 3^5$
	39	$111^2 3^5$
	≥ 51	$((p-15)/2)^2 3^5$

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