ABOUT VERTEX-CRITICAL NON-BICOLORABLE HYPERGRAPHS

Claude BERGE
C.N.R.S., Paris

ABSTRACT

The hypergraphs whose chromatic number is ≤ 2 ("bicolorable" hypergraphs) were introduced by E.W. Miller [13] under the name of "set-systems with Property B ". This concept appears in Number Theory (see [5], [10]). It is also useful for some problems in positional games and Operations Research (see [3], [4], [7]); different results have been found under the form of inequalities involving the sizes of the edges, the number of vertices, etc...(see [6], [11], [12]).

A non-bicolorable hypergraph which becomes bicolorable when any of its edges is removed is called "edge-critical", and several of its properties can be found in the literature ([2], [4], [14]). In this paper, instead of edge-critical hypergraphs, we study the vertex-critical hypergraphs; the applications are more numerous, and it seems that somewhat stronger results could imply the famous "four-color theorem".

I Vertex-critical hypergraphs and the four-color problem

Let $H = (E_1, E_2, ..., E_m)$ be a hypergraph which is *simple* (i.e. $E_i \supseteq E_j$ implies $i = j$). Denote by $X = \{x_1, x_2,..., x_n\}$ its vertex-set, and for $A \subseteq X$, denote by H/A the partial hypergraph $H/A = (E \,/\, E \in H , E \subseteq A)$ (this family can be empty). We denote also by $H\text{-}H(x_i)$ the hypergraph obtained from H by removing all the edges which contain the vertex x_i (and all the vertices which become of degree 0).

Let $\chi(H)$ denote the chromatic number of H, i.e. the least number of colors needed to color the vertices so that no edge is monochromatic (except, of course, the edges of cardinality one, or "loops"). The hypergraph H is *edge-critical* (with respect to the non-bicolorability) if $\chi(H) > 2$ and

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H-E is bicolorable for every $E \in H$. A hypergraph H is *vertex-critical* if $\chi(H) > 2$ and $H-H(x)$ is bicolorable for every vertex x.

Clearly, every hypergraph which is not bicolorable has a partial hypergraph which is edge-critical, and every edge-critical hypergraph is also vertex-critical. Furthermore, every hypergraph which is not bicolorable contains a set A of vertices such that the hypergraph H/A is vertex-critical.

Some classical examples of edge-critical hypergraphs are: the finite projective plane with 7 points, the complete r-uniform hypergraph K^r_{2r-1} of order 2r-1, the Lovász hypergraph L_r, the complement of L_3, etc...(see [2], Chap.2). Seymour [14] has characterized the edge-critical hypergraphs having as many vertices as edges (by association with strongly connected directed graphs without even circuits).

Number Theory provides several examples of vertex-critical hypergraphs which are not edge-critical : Consider the "triangle hypergraph" K^T_n, that is the hypergraph whose vertices are the edges of the complete graph K_n and whose edges are the triangles of K_n. Since the Ramsey number R(3,3) is 6, we have $\chi(K^T_6) = 3$ and $\chi(K^T_5) = 2$. The hypergraph K^T_6 is vertex-critical: if the vertices of K_6 are *a,b,c,d,e,f*, and if the edge *af* is removed, the other edges can be colored with two colors without producing a monochromatic triangle (for example with blue : *ab*, *bc* , *bf* ,*ae*, *ed*; *ef*, *cd* ; with red : *ac*, *ad*, *bd*, *be*, *ce*, *cf*, *df*). Nevertheless, it is easy to check that the hypergraph K^T_6 is not edge-critical.

The well known theorem of van der Waerden ("*If the natural numbers are split into two classes, then for every k at least one class contains an arithmetic progression of k terms,*") can be generalized as follows: If A_k is a finite set of integers such that in every bicoloring of A_k at least one color class contains an arithmetic progression of k terms, and if A_k is minimal, then the arithmetic progressions of k terms define a vertex-critical hypergraph (which is not necessarly edge-critical).

It is well known that *every planar graph is four-colorable* (K. Appel and W. Haken, 1979) but the proof involves too many hours of computer time to be checked directly by mathematical reasoning. The concept of vertex-critical hypergraph suggest a new approach, based on the results of the following sections and on the specific properties of the odd cycles in a planar graph.

For a simple graph G, let H(G) denote the hypergraph on V(G) whose edges are the minimal odd cycles of G; these cycles are elementary and chordless. The hypergraph H(G) is simple. We have :

PROPOSITION. *A graph G is four-colorable if and only if the hypergraph H(G) is bicolorable.*

If H(G) admits a bicoloring (A,B), then the subgraphs G_A and G_B have no odd cycles, and consequently, they admit respectively a bicoloring (A_1, A_2) and a bicoloring (B_1, B_2). Clearly, (A_1, A_2, B_1, B_2) is a four-coloring for G, and G is four-colorable.

The converse is obvious.

COROLLARY. *A graph G which is not four-colorable contains a subgraph G_A such that the hypergraph $H(G_A)$ is vertex-critical.*

This follows from the equality : $H(G_A) = H(G) / A$.

If G is a planar graph, the hypergraph H(G) has many specific topological properties, and some of them should imply that H(G) is not vertex-critical. On the other hand, it would be interesting to complete the statement of Sterboul's Conjecture so that this statement imply direcly the four-color theorem.

2 Deeply bicolorable hypergraphs

Let x and y be two vertices of the hypergraph $H = (E_1, E_2, \ldots, E_m)$. We say that x is *dependent on* y, and we write x -> y , if every edge containing x contains also y. A vertex of degree 0 or 1 is always a dependent vertex.

THEOREM 1. *Let* H *be a hypergraph and let* A *be the set of dependent vertices; if* H/X-A *is bicolorable, then every bicoloring of* H/X-A *can be extended to a bicoloring of* H.

Assume that H / X-A has already been colored with two colors, say red and blue, so that no edge $E \subseteq X-A$ is monochromatic; we shall assign one of the two colors to each uncolored vertex so that no edge of H is monochromatic.

The directed graph G on X defined by the arcs (x,y) such that x->y is transitive, and consequently each terminal component is either a singleton {y} with $y \notin A$ or a symmetric complete subgraph with all its vertices in A. By a famous theorem of König, a transitive graph has a kernel, which is obtained by picking up one vertex in each terminal strongly connected component. Let S be a kernel of G. Color arbitrarly with blue each vertex in S which has not yet been colored. Then assign to each vertex $x \in A-S$ a color different from the color of one of its successors in S. Thus, every edge which meets A is bichromatic; since every edge which does not meet A is also bichromatic, a bicoloring of H has been obtained.

Q.E.D.

COROLLARY 1. *A vertex-critical hypergraph H contains no dependent vertices.*

If the set of dependent vertices is a non-empty set A, then H/X-A is bicolorable (or empty), and by the theorem 1, H is also bicolorable. A contradiction.

COROLLARY 2. *The hypergraph H of the maximal cliques in a triangulated (chordal) graph G is bicolorable.*

Let G be a minimal triangulated graph such that the associated hypergraph H is not bicolorable. Since G is triangulated, there exists a vertex which belongs to only one maximal clique, and this vertex is necessarly a dependent vertex for H. Hence, by the corollary 1, H is not vertex-critical. This contradicts the minimality of G.

A *cycle* of the hypergraph H is an alternating sequence $(x_1, E_1, x_2, E_2, x_3, ..., E_k, x_{k+1})$ such that $k \geq 2$, all the edges E_i are distinct, all the vertices x_j are distinct (except $x_{k+1} = x_1$), and $E_i \supseteq \{x_i, x_{i+1}\}$ for $i = 1,2,...,k$. Fournier and Las Vergnas proved in [8] that *if every odd cycle has three edges with a non-empty intersection, then the hypergraph is bicolorable*. To understand the exact scope of this result, consider a hypergraph H such that one of the (induced) subhypergraphs obtained from H by removing successively a remaining dependent vertex is bicolorable. From the results above, we see that H is bicolorable, and we shall call it a *deeply bicolorable* hypergraph. By analogy with the theorem of Kirchhoff about bicolorable graphs, we state the theorem as follows:

THEOREM 2. A hypergraph H and all its partial hypergraphs are deeply bicolorable if and only if every odd cycle of H has three edges with a non-empty intersection.

Proof: 1° Let x and y be two vertices of a hypergraph H whose odd cycles have the property; it suffices to show that if x is a vertex dependent on y, the subhypergraph H' obtained by removing x is also bicolorable.

For $i \leq m$, put $E_i' = E_i - \{x\}$, and let $\sigma' = (x_1, E_1', x_2, ..., E_k', x_1)$ be an odd cycle of H'. If x->y and $y \notin E_1, E_2, ..., E_k$, then $x \notin E_1, E_2, ..., E_k$; so $E_i' = E_i$. Then σ' is also an odd cycle for H which has three edges, say E_p, E_q, E_r, with a non-empy intersection, or :

$$E_p' \cap E_q' \cap E_r' \neq \emptyset \qquad (1)$$

Now if $x \in E_p \cap E_q \cap E_r$, then $y \in E_p' \cap E_q' \cap E_r'$, so we have also (1).

In all cases, the cycle σ' has three edges with a non-empty intersection, and from the theorem of Fournier and Las Vergnas, H' is bicolorable. Hence, H is deeply bicolorable.

2° Assume now that H has an odd cycle $\sigma = (x_1, E_1, x_2, ..., E_k, x_1)$ without three edges having a non-empty intersection ; we may assume that its length k is minimum, and we shall show first that that *any two non-*

consecutive edges of the cycle are disjoint (which is trivial for $k \leq 3$, so we assume $k>3$).

Otherwise, we have, say, $E_1 \cap E_i \neq \emptyset$ for some i with $3 \leq i \leq k-1$. Let $a \in E_1 \cap E_i$. Clearly, $a \neq x_1, x_2, \ldots, x_k$, and σ can be decomposed into two cycles :

$$\sigma' = (a, E_1, x_2, E_2, \ldots, x_i, E_i, a)$$
$$\sigma'' = (a, E_i, x_{i+1}, E_{i+1}, \ldots, E_k, x_1, E_1, a).$$

Their lengths being respectively $i \leq k-1$ and $k-i+2 \leq k-1$, and one of them being odd, this contradicts the minimality of the cycle σ.

The cycle σ defines by its edges a partial hypergraph H' of H ; after removing successively each remaining dependent vertex, H' becomes a graph C_{2p+1} (chordless cycle of length $2p+1$ odd), which is not bicolorable. So, one partial hypergraph of H is not deeply bicolorable.

Q.E.D.

3 Other properties of vertex-critical hypergraphs

Let H be a vertex- critical hypergraph, and let A be its incidence matrix, with m columns (representing the edges E_i) and n rows (representing the vertices x_j). The following property has been proved by Seymour [14] for edge-critical hypergraphs.

THEOREM 3. *Let H be a vertex-critical hypergraph with n vertices and m edges. Then $m \geq n$, and at least one of the $n \times n$ subdeterminants of the incidence matrix A is $\neq 0$.*

Assume that the theorem is false. Then there exists a n-dimensional vector $\mathbf{y} = (y_1, y_2, \ldots, y_n) \neq 0$ such that $A^*\mathbf{y} = 0$. The vertex-set of H is the union of $X^+ = \{ x_j \mid j \leq n ; y_j > 0 \}$, $X^- = \{ x_j \mid j \leq n ; y_i < 0 \}$ and $X^o = \{ x_j \mid j \leq n ; y_j = 0 \}$. Since $\mathbf{y} \neq 0$, we have $X^+ \neq \emptyset$, $X^- \neq \emptyset$; we have also $X^o \neq \emptyset$, because otherwise H would admit a bicoloring (X^+, X^-); which contradicts that H is vertex-critical.

Since $X^o \neq X$, the partial hypergraph H/X^o admits a bicoloring (Y,Z), and each edge of H meets both $X^+ \cup Y$ and $X^- \cup Z$; so $\chi(H) \leq 2$. A contradiction.

The following properties are consequences of a result due to Fournier and Las Vergnas for edge-critical hypergraphs.

THEOREM 4. *Let* H *be a vertex-critical hypergraph on* X, *and let* $x_0 \in X$. *Then there exists an odd cycle* $(x_1, E_1, x_2, E_2, x_3, \ldots, E_k, x_1)$ *such that :*

(1) $x_2 = x_0$;

(2) $E_i \cap E_j = \emptyset$ *if* E_i *and* E_j *are two non-consecutive edges ;*

(3) $E_1 \cap E_2 = \{x_0\}$.

The theorem 1' in [8] asserts that in an edge-critical hypergraph H' with $E_0 \in H'$ and $x_0 \in E_0$, *there is an odd cycle* $(x_1, E_1, \ldots, x_1)$ *satisfying* (1), (2), (3) with $E_1 = E_0$. Clearly, a vertex-critical hypergraph H on X contains an edge-critical hypergraph H' with the same vertex-set X ; if $x_0 \in X$, and if we take for E_0 any edge of H' which contains x_0 and apply to H' the theorem 1', we get the statement of the theorem 4.

THEOREM 5. *Let* H *be a vertex-critical hypergraph; there exists an odd cycle* $(x_1, E_1, \ldots, E_k, x_1)$ *such that :*

(1') $|E_p \cap E_q \cap E_r| = 0$ $(p<q<r\leq k)$;

(2') $|E_i \cap E_{i+1}| = 1$ $(i=1,2,\ldots,k-1)$;

(3') $|E_1 \cap E_k| \geq 1$.

This result was proved in [8] only for edge-critical hypergraphs, but the extension is obvious.

COROLLARY. *Let* H *be a hypergraph having no two intersecting edges of size* ≥ 4, *and no odd cycle* $(x_1, E_1, x_2, \ldots, E_k, x_1)$ *satisfying :*

(i) $|E_i \cap E_j| = 0$ *if* E_i *and* E_j *are two non-consecutive edges ;*

(ii) $|E_i \cap E_{i+1}| = 1$ $(i= 1,2,\ldots,k-1)$;

(iii) $|E_1 \cap E_k| \geq 1$.

Then H *is bicolorable.*

Proof: Suppose that such a hypergraph H is not bicolorable. Let H' be a partial hypergraph of H which is vertex-critical. From Theorem 5, H' contains an odd cycle $\sigma = (x_1, E_1, \ldots, E_k, x_1)$ satisfying (1'),(2'),(3'), and we may assume that the cycle σ is of minimal length k. Since σ

cannot satisfy (i), (ii) and (iii), the cycle σ has two non-consecutive edges which meet, say, E_1 and E_i, with $3 \le i \le k-1$. Let $a \in E_1 \cap E_i$. From (1') we have $a \ne x_1, x_2,.., x_k$. If $|E_1| \le 3$ then $E_1 = \{ a, x_1, x_2\}$ and therefore $E_1 \cap E_i = \{a\}$; if $|E_1| \ge 4$, then $|E_i| \le 3$, which implies that $E_i = \{a, x_i, x_{i+1}\}$, and therefore $E_1 \cap E_i = \{a\}$. As in the proof of the theorem 2, we see that this vertex a separates the cycle σ into two smaller cycles, and one of them is odd. This contradicts the minimality of the cycle σ.

This result is related to a conjecture posed by Sterboul [15] in 1973 : *A hypergraph with no odd cycle satisfying (i), (ii), (iii) is bicolorable.* In order to imply the four color theorem, we would rather suggest that if every odd cycle satisfying (i), (ii) and (iii) is *well covered* in some sense, the hypergraph is bicolorable.

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