Various super-simple designs with block size four

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ABSTRACT: In this note the existence of a $(v; \rho_2; 4, 2)$ BTD , for $\rho_2 = 0$, 1 and 2, in which any pair of blocks intersect in at most two elements, is proved for any admissible v.

1 Introduction and definitions

A *balanced ternary design* is a collection of multi-sets of size k, chosen from a v-set in such a way that each element occurs 0, 1 or 2 times in any one block, each pair of non-distinct elements, $\{x, x\}$, occurs in ρ_2 blocks of the design and each pair of distinct elements, $\{x, y\}$, occurs λ times throughout the design. We denote these parameters by $(v; \rho_2; k, \lambda)$ BTD. It is easy to see that each element must occur singly in a constant number of blocks, say ρ_1 blocks, and so each element occurs altogether $r = \rho_1 + 2\rho_2$ times. Also if b is the number of blocks in the design, then

$$vr = bk \qquad \text{and} \qquad \lambda(v-1) = r(k-1) - 2\rho_2.$$

(For further information [3] should be consulted.)

A BTD is called *simple* if it contains no repeated blocks.

A $(v; \rho_2; 4, \lambda)$ BTD is said to be *super-simple* if any pair of its blocks have at most two elements in common, where repetition of elements is counted. For example, the blocks $xxyz$ and $xxst$ are said to have two elements in common. Obviously, any super-simple BTD is a simple BTD. In [7] Gronau and Mullin introduce super-simple $(v; 0; 4, \lambda)$ BTDs (which are of course balanced incomplete block designs $(v, 4, \lambda)$) and in [9] Kejun proves that super-simple $(v; 0; 4, 3)$ BTDs exist if and only if $v \equiv 0$ or 1 (mod 4), $v \geqslant 8$.

In this note we concentrate on the cases $\rho_2 = 0$, 1 and 2, $k = 4$ and $\lambda = 2$. Indeed, we shall prove the following results.

MAIN THEOREM (1) There exists a super-simple $(v, 4, 2)$ BIBD if and only if $v \equiv 1 \pmod 3$ and $v \neq 4$ ([7], Theorem A).
(2) There exists a super-simple $(v; 1; 4, 2)$ BTD if and only if $v \equiv 0 \pmod 6$.
(3) There exists a super-simple $(v; 2; 4, 2)$ BTD if and only if $v \equiv 2 \pmod 3$, $v \geqslant 11$.

Since Theorem A of [7] uses Theorem 3.1 of that paper, and Theorem 3.1 of [7] is not correct as it stands, in Section 2, we shall give a correct proof for Theorem A of [7], which is part (1) of the main theorem. Nevertheless, we shall use some of the results of [7].

In Sections 3 and 4, we deal with super-simple $(v; \rho_2; 4, 2)$ BTDs, with $\rho_2 = 1$ and 2, respectively. It has been shown (see Donovan [6]) that a $(v; \rho_2; 4, 2)$ BTD, $\rho_2 = 1$ and 2, exists for all admissible v. However, these were not necessarily all simple.

Most of the techniques we use here involve certain group divisible designs and frames. A *group divisible design*, GDD$(K, \lambda, M; v)$ is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, where the v-set is partitioned into disjoint subsets (called groups) of size $m \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in λ blocks. If $M = \{m\}$ and $K = \{k\}$ we write GDD$(k, \lambda, m; v)$. A group divisible design, with element set X, group set G and block set B, is also denoted by GDD(X, G, B). In this paper, a *transversal design* TD(k, n) is a GDD$(k, 1, n; kn)$.

A BTD *with hole*, or *frame*-BTD, is a collection of multi-sets (blocks) of size k chosen from a v-set V so that the following conditions hold:

(i) $\{\infty_i \mid i = 1, 2, ..., h\} = H$ is a subset of V called a *hole*;

(ii) any element in $V \setminus H$ occurs 0, 1 or 2 times per block, and precisely 2 times in ρ_2 blocks;

(iii) each element of H occurs at most once in any block;

(iv) any pair xy, where x and y are distinct elements, not both in H, occurs λ times altogether in the blocks.

We write the parameters of a frame-BTD as $(v[h]; \rho_2; k, \lambda)$. Of course a BTD is a frame with $h = 0$.

Analogously, we call a transversal design, a group divisible design and a frame-BTD *super-simple* if any two of their blocks have at most two elements in common.

We shall now summarize the main results we use for our constructions.

THEOREM 1.1 ([5])

For all integers m, λ and v, a necessary and sufficient condition for the existence of a group divisible design GDD$(4, \lambda, m; v)$ is that $(\lambda, m, v) \neq (1, 2, 8)$ or $(1, 6, 24)$, and that

$$v \equiv 0 \pmod{m}, \quad \lambda(v - m) \equiv 0 \pmod{3}, \quad \lambda v(v - m) \equiv 0 \pmod{12}$$

and $v \geqslant 4m$ or $v = m$.

THEOREM 1.2 ([10])

Let (X, G, B) be a "master" GDD of index unity and let $w : X \to \mathbb{Z}^+ \cup \{0\}$ be a weighting of the GDD. For every $x \in X$, let S_x be $w(x)$ "copies" of x. Suppose that for each block $b \in B$, a GDD$(\bigcup_{x \in b} S_x, \{S_x : x \in b\}, A_b)$ of index λ is given. Let $X^* = \bigcup_{x \in X} S_x$, $G^* = \{\bigcup_{x \in g} S_x : g \in G\}$ and $B^* = \bigcup_{b \in B} A_b$. Then (X^*, G^*, B^*) is a GDD of index λ.

It is easy to see that if all small (input) GDDs in Theorem 1.2 are super-simple then the resulting GDD is also super-simple.

THEOREM 1.3 ([6])
If there exists a $GDD(k, \lambda, \{v-h, (v'-h)^*\}; (n-1)(v-h)+v'-h)$ *and a frame-*$BTD(v[h]; \rho_2; k, \lambda)$ *and a BTD with parameters* $(v'; \rho_2; k, \lambda)$*, then there exists a BTD with parameters* $((n-1)(v-h)+v'; \rho_2; k, \lambda)$.

Obviously, if we apply Theorem 1.3 with a super-simple GDD together with a super-simple frame-BTD and a super-simple BTD then the resulting BTD is also super-simple.

2 The case $\rho_2 = 0$

In this section, using some results of [7], we prove that there exists a super-simple $(v, 4, 2)$ BIBD ($(v; 0; 4, 2)$ BTD) for all $v \equiv 1 \pmod 3$ and $v \neq 4$. It is known (see Hanani [8]) that $(v, 4, 2)$ BIBDs exist if and only if $v \equiv 1 \pmod 3$. Obviously, there does not exist a super-simple (4,4,2) BIBD.

LEMMA 2.1 *There exists a super-simple* $GDD(4, 2, 12; 12n)$ *for all integers* $n \geqslant 5$.

Proof. By Theorem 1.1 there exists a $GDD(4, 1, 6; 6n)$ for all integers $n \geqslant 5$. Moreover there exists a super-simple $GDD(4, 2, 2; 8)$ with groups $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$ and blocks 1357, 1368, 1458, 1467, 2358, 2367, 2457 and 2468. Now apply Theorem 1.2 with a $GDD(4, 1, 6; 6n)$ as a "master" GDD together with a super-simple $GDD(4, 2, 2; 8)$. The resulting GDD is a super-simple $GDD(4, 2, 12; 12n)$. ∎

COROLLARY 2.2 *There exists a super-simple* $(12n+1, 4, 2)$ *BIBD for all integers* $n \geqslant 5$ *or* $n = 1$.

Proof. First note that any (v, k, λ) BIBD is also a $(v[1]; 0; k, \lambda)$ frame-BTD. Now apply Theorem 1.3 with a super-simple $GDD(4, 2, 12; 12n)$, which exists by Lemma 2.1, together with a super-simple $(13, 4, 2)$ BIBD, which exists (consider base blocks 0 1 3 9 and 0 1 5 11 (mod 13)). The result is a super-simple $(12n+1, 4, 2)$ BIBD, where $n \geqslant 5$. ∎

LEMMA 2.3 (see also [7], Theorem 3.2)
Let $m \geqslant 4$, $m \neq 6$, $m \neq 10$ *and* $0 \leqslant n \leqslant m$ *be integers. Then there exists a super-simple* $GDD(4, 2, \{3m, 3n\}; 12m+3n)$.

Proof. Start with a transversal design $TD(5, m)$ which exists for $m \geqslant 4$, $m \neq 6$ and $m \neq 10$. Delete $m-n$ points of the last group to obtain a $GDD(\{4, 5\}, 1, \{m, n\}; 4m+n)$. Now apply Theorem 1.2 with the following as the ingredient designs:

(i) for the blocks of size 4, use a super-simple $GDD(4, 2, 3; 12)$ with groups $G_i = \{i, i+4, i+8\}$, $0 \leqslant i \leqslant 3$, and base blocks 0 2 3 5 and 0 1 6 7 (short orbit), which are cycled under the permutation (0 1 2 ... 11);

(ii) for the blocks of size 5 use the following two group divisible designs T_1 and T_2 with $\lambda = 1$, block size 4 and groups $G_i = \{(i, j) \mid 0 \leqslant j \leqslant 2\}$, $0 \leqslant i \leqslant 4$ (see [7]):

$$T_1 = \{((0,0), (1,1), (2,1), (3,0)) (\mathrm{mod}(5,3))\}$$

$$T_2 = \{((0,0),(1,2),(2,2),(3,0))(\mathrm{mod}(5,3))\}.$$

The result is a super-simple GDD$(4,2,\{3m,3n\};12m+3n)$. ∎

COROLLARY 2.4 (see also [7], Corollary 3.2.2 part 2)
Let $m \geqslant 4$, $m \neq 6$, $m \neq 10$ and $0 \leqslant n \leqslant m$. If there exists a super-simple $((3m+f)[f];\rho_2;4,2)$ frame-BTD and a super-simple $(3n+f;\rho_2;4,2)$ BTD, then there exists a super-simple $(12m+3n+f;\rho_2;4,2)$ BTD.

LEMMA 2.5 (see also [7], Theorem A)
If there exists a super-simple $(v,4,2)$ BIBD for all admissible $v \leqslant 136$, then there exists a super-simple $(v,4,2)$ BIBD for all $v \equiv 1$ (mod 3), $v \neq 4$.

Proof. Let $w \equiv 1$ (mod 3), $w \geqslant 34$. Then by Corollary 2.4 we can construct the desired designs of the orders belonging to $W(w) = \{4(w-1)+7, 4(w-1)+10, \ldots, 4(w-1)+19\}$, which are just 5 consecutive numbers of type 1 mod 3. Since $4(w-1)+19 = 4((w+3)-1)+7$ and the gap between two consecutive numbers of type 1 mod 3 has length 3, $\bigcup_{w \geqslant 34} W(w)$ covers all of the remaining orders. ∎

Now we examine small cases. Indeed, we show that for all $v \equiv 1$ (mod 3), $7 \leqslant v \leqslant 136$, there exists a super-simple $(v,4,2)$ BIBD. So part (1) of the main theorem follows with this information and Lemma 2.5.

LEMMA 2.6 *If $v \in \{7,10,13,16,19,22,25,28,31,34,37,40,43,46,79,82\}$, then there exists a super-simple $(v,4,2)$ BIBD.*

Proof. See [7]. ∎

LEMMA 2.7 *If $v \equiv 1$ (mod 3), $49 \leqslant v \leqslant 136$ and $v \neq 79$ or 82, then there exists a super-simple $(v,4,2)$ BIBD.*

Proof. For $v \in \{61,73,85,97,109,121,133\}$ apply Corollary 2.2. For $v = 64$ we proceed as follows. Adjoin seven new points to a resolvable 2-$(15,3,1)$ design to obtain a pairwise balanced design on 22 points which contains one block of size 7 and all the other blocks of size 4 (see [7]). Delete a point which occurs on the block of size 7 to obtain a GDD$(4,1,\{3,6\};21)$. Since there exists a super-simple GDD$(4,2,3;12)$ (see Lemma 2.3), we can apply Theorem 1.2 to find a super-simple GDD$(4,2,\{9,18\};63)$. Now apply Theorem 1.3 with this GDD together with a super-simple $(10,4,2)$ BIBD and a super-simple $(19,4,2)$ BIBD (which exist by Lemma 2.6). The result is a super-simple (64,4,2) BIBD. For $v = 52$, 88 and 100 see Table 1. This table gives base blocks for these designs (short orbits are marked with an asterisk). These designs were found using the program *autogen* (see [1]).

52	0	1	3	5	0	3	7	12	0	6	13	30	0	6	21	37
	0	8	19	36	0	8	20	38	0	9	20	34	0	10	23	33
	(0	1	26	27)*												
88	0	1	9	41	0	2	57	51	0	2	76	35	0	3	45	71
	0	3	70	18	0	4	77	17	0	4	20	26	0	5	10	19
	0	7	19	42	0	7	31	56	0	8	30	58	0	10	34	61
	0	11	36	59	0	13	29	67	(0	1	44	45)*				
100	0	1	46	74	0	2	91	72	0	2	41	32	0	3	93	59
	0	3	80	69	0	4	78	60	0	4	84	22	0	5	43	29
	0	5	36	24	0	6	12	27	0	7	17	54	0	8	23	68
	0	8	29	65	0	13	48	61	0	14	47	63	0	17	42	75
	(0	1	50	51)*												

Table 1

For the remaining cases, we use Corollary 2.4 according to Table 2.

v	m	n	v	m	n	v	m	n
49	4	0	91	7	2	118	8	7
55	4	2	94	7	3	124	9	5
58	4	3	103	7	6	127	9	6
67	5	2	106	8	3	130	9	7
70	5	3	112	8	5	136	9	9
76	5	5	115	8	6			

Table 2

3 The case $\rho_2 = 1$

In this section we shall prove that there exists a super-simple $(v; 1; 4, 2)$ BTD for all integers $v \equiv 0 \pmod 6$. It is easy to see that the condition $v \equiv 0 \pmod 6$ is necessary for the existence of a $(v; 1; 4, 2)$ BTD.

LEMMA 3.1 *There exists a super-simple* $(12n; 1; 4, 2)$ *BTD for all integers* $n \geqslant 5$.

Proof. Apply Theorem 1.3 with a super-simple GDD$(4, 2, 12; 12n)$, which exists by Lemma 2.1 for all integers $n \geqslant 5$, together with a super-simple $(12; 1; 4, 2)$ BTD, which exists (see Table 3). ∎

LEMMA 3.2 *There exists a super-simple* $(12n + 6; 1; 4, 2)$ *BTD for all integers* $n \geqslant 5$.

Proof. Apply Theorem 1.3 with a super-simple GDD$(4, 2, 12; 12n)$, a super simple $(12; 1; 4, 2)$ BTD (see Table 3) and a super-simple $(18[6]; 1; 4, 2)$ frame-BTD (see Table 3). ∎

So far, we have proved that there exists a super-simple $(v; 1; 4, 2)$ BTD for all $v \equiv 0 \pmod 6$ and $v \geqslant 60$ or $v = 12$. The remaining cases are $v = 6$, 18, 24, 30, 36, 42, 48 and 54. For $v = 6$ or 18, see [4]. For $v = 36$, consider initial blocks 0 0 1 3, 0 2 6 13, 0 4 13 25, 0 5 17 22, 0 6 15 22 and 0 8 16 26 (mod 36). The other remaining cases are settled by the following lemmas.

LEMMA 3.3 *There exists a super-simple $(v;1;4,2)$ BTD for all $v \equiv 0$ or $6 \pmod{24}$.*

Proof. Apply Theorem 1.2 with a GDD$(4,1,3;w)$, which exists by Theorem 1.1 for all $w \equiv 0$ or $3 \pmod{12}$, $w \geqslant 12$, together with a super-simple GDD$(4,2,2;8)$ (see Lemma 2.1). The result is a super-simple GDD$(4,2,6;v)$, where $v \equiv 0$ or $6 \pmod{24}$, $v \geqslant 24$. Now apply Theorem 1.3 with this GDD and a $(6;1;4,2)$ BTD. ■

LEMMA 3.4 *There exists a super-simple $(v;1;4,2)$ BTD, for all $v \equiv 6 \pmod{18}$, $v \geqslant 2$.*

Proof. Apply Theorem 1.2 with a GDD$(4,1,2;6n+2)$, which exists by Theorem 1.1 for all integers $n \geqslant 2$, together with a super-simple GDD$(4,2,3;12)$ (see Lemma 2.3). The result is a super-simple GDD$(4,2,6;18n+6)$, where $n \geqslant 2$. Now apply Theorem 1.3 with this GDD and a $(6;1;4,2)$ BTD. ■

12	0024	119b	221a	3318	4418	551a	6601	7701
	8805	9905	aa03	bb03	2357	2369	245b	2678
	289b	3479	3456	469a	47ab	567b	68ab	789a
18[6]	$\infty_1 33b$	$\infty_1 22b$	$\infty_1 6aa$	$\infty_1 678$	$\infty_1 017$	$\infty_1 458$	$\infty_1 149$	$\infty_1 059$
	$\infty_2 77b$	$\infty_2 00b$	$\infty_2 689$	$\infty_2 156$	$\infty_2 348$	$\infty_2 129$	$\infty_2 24a$	$\infty_2 35a$
	$\infty_3 88b$	$\infty_3 55b$	$\infty_3 679$	$\infty_3 246$	$\infty_3 147$	$\infty_3 039$	$\infty_3 02a$	$\infty_3 13a$
	$\infty_4 99b$	$\infty_4 44b$	$\infty_4 78a$	$\infty_4 15a$	$\infty_4 237$	$\infty_4 018$	$\infty_4 026$	$\infty_4 356$
	$\infty_5 66b$	$\infty_5 11b$	$\infty_5 89a$	$\infty_5 04a$	$\infty_5 258$	$\infty_5 239$	$\infty_5 347$	$\infty_5 057$
	$\infty_6 abb$	$\infty_6 79a$	$\infty_6 046$	$\infty_6 136$	$\infty_6 257$	$\infty_6 038$	$\infty_6 128$	$\infty_6 459$

Table 3

4 The case $\rho_2 = 2$

In this section we shall prove that there exists a super-simple $(v;2;4,2)$ BTD for all integers $v \equiv 2 \pmod 3$ and $v \geqslant 11$. Note that the necessary condition for the existence of a $(v;2;4,2)$ BTD is $v \equiv 2 \pmod 3$ and $v \geqslant 11$ (see [6]).

LEMMA 4.1 *There exist a super-simple $(12n+2;2;4,2)$ BTD and a super-simple $(12n+2[2];2;4,2)$ frame-BTD for all integers $n \geqslant 1$.*

Proof. Apply Theorem 1.3 with a super-simple GDD$(4,2,12;12n)$, a super-simple $(14[2];2;4,2)$ frame-BTD (see [2]) and a super-simple $(14;2;4,2)$ BTD (see [2]). The result is a super-simple $(12n+2;2;4,2)$ BTD, where $n \geqslant 5$ or $n=1$. For $v=26$ see [2], and for $v=38$ and 50 see the Appendix. Similarly, we can construct a super-simple $(12n+2[2];2;4,2)$ frame-BTD, for $n \geqslant 5$. For the remaining values see the Appendix. ■

LEMMA 4.2 *If there exists a super-simple $(v;2;4,2)$ BTD for all admissible v with $11 \leqslant v \leqslant 248$ then there exists a super-simple $(v;2;4,2)$ BTD for all $v \equiv 2 \pmod 3$, $v \geqslant 11$.*

Proof. Let $v \equiv 2 \pmod 3$ and $v \geqslant 251$. Then $v = 3(4m+n)+2$, where $m \equiv 0 \pmod 4$ and $3 \leqslant n \leqslant 18$. Now apply Lemma 2.3, Lemma 4.1 and Corollary 2.4. The result is a super-simple $(v;2;4,2)$ BTD. ■

LEMMA 4.3 *There exists a super-simple* $(v;2;4,2)$ *BTD for all* $v \equiv 2 \pmod 3$ *and* $11 \leqslant v \leqslant 248$.

Proof. For $v \equiv 2 \pmod{12}$ apply Lemma 4.1. For $v = 11$, 17, 23 and 29 see [2]. For $v = 41$ see [6]. For $v = 35$ we may use a super-simple $(35[11];2;4,2)$ frame-BTD (see [6]) and a super-simple $(11;2;4,2)$ BTD. For $v \in \{20,32,44,47,53,56,65,68,71,77,80,83,89,92,95,101,104\}$ see the Appendix. Some of these designs were found using the program *autogen* (see [1]). For $v \in \{137,140,143,149,152\}$ we proceed as follows. First consider that there exists a super-simple GDD$(4,2,3;18)$ with groups $G_i = \{i, i+6, i+12\}$, $0 \leqslant i \leqslant 5$, and base blocks 0 1 3 8, 0 1 4 14 and 0 2 9 11 (short orbit) (mod 18). Secondly, apply a method similar to that described in Lemma 2.3 with a TD$(6,8)$ to obtain a GDD$(\{5,6\},1,\{8,n\};40+n)$, where $0 \leqslant n \leqslant 8$. Now apply Theorem 1.2 with this GDD, together with a super-simple GDD$(4,2,3;15)$ and a super-simple GDD$(4,2,3;18)$. The result is a GDD$(4,2,\{24,3n\};120+3n)$. Finally, apply Theorem 1.3 together with a super-simple $(26[2];2;4,2)$ frame-BTD or a super-simple $(35[11];2;4,2)$ frame-BTD. For the remaining cases, we use Corollary 2.4 according to Table 4 (see the Appendix and Lemma 4.1 for $((w+2)[2];2;4,2)$ frame-BTDs, where $w = 42$, 54 or $w = 12n$ and $n \geqslant 1$). This completes the proof. ■

v	m	n	frame-BTD used	v	m	n	frame-BTD used
59	4	3	$(14[2];2;4,2)$	107	8	3	$(26[2];2;4,2)$
113	8	5	$(26[2];2;4,2)$	116	8	6	$(26[2];2;4,2)$
119	8	7	$(26[2];2;4,2)$	125	8	6	$(35[11];2;4,2)$
128	8	7	$(35[11];2;4,2)$	131	8	8	$(35[11];2;4,2)$
155	12	3	$(38[2];2;4,2)$	161	12	5	$(38[2];2;4,2)$
164	12	6	$(38[2];2;4,2)$	167	12	7	$(38[2];2;4,2)$
173	12	9	$(38[2];2;4,2)$	176	12	10	$(38[2];2;4,2)$
179	12	11	$(38[2];2;4,2)$	185	14	5	$(44[2];2;4,2)$
188	14	6	$(44[2];2;4,2)$	191	14	7	$(44[2];2;4,2)$
197	14	9	$(44[2];2;4,2)$	200	14	10	$(44[2];2;4,2)$
203	14	11	$(44[2];2;4,2)$	209	14	13	$(44[2];2;4,2)$
212	14	14	$(44[2];2;4,2)$	215	16	7	$(50[2];2;4,2)$
221	16	9	$(44[2];2;4,2)$	224	16	10	$(50[2];2;4,2)$
227	16	11	$(44[2];2;4,2)$	233	16	13	$(50[2];2;4,2)$
236	16	14	$(44[2];2;4,2)$	239	16	15	$(50[2];2;4,2)$
245	18	9	$(56[2];2;4,2)$	248	18	10	$(56[2];2;4,2)$

Table 4

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Appendix

Hole elements for frame-BTDs are denoted by x and y; short starters are marked with an asterisk. For a frame-BTD on $v[2]$ elements, blocks are cycled under the permutation $(x\ y)(0\ 1\ 2\ \ldots\ (v-3))$, while blocks in BTDs on v elements are cycled under the permutation $(0\ 1\ 2\ \ldots\ (v-1))$.

20	0 0 1 3	0 0 7 12	0 2 6 11	(0 4 10 14)*
26[2]	x 0 0 3	0 0 4 13	0 1 6 7	0 2 7 16
	(y 0 8 16)*	(0 2 12 14)*		
32	0 0 2 10	0 0 4 13	0 1 7 8	0 3 12 17
	0 3 14 20	(0 5 16 21)*		
38	0 0 2 5	0 0 4 11	0 1 7 22	0 3 13 26
	0 6 14 28	0 8 17 26	(0 1 19 20)*	
38[2]	0 0 1 4	x 0 0 13	0 2 9 28	0 3 10 24
	0 5 11 20	0 5 22 30	(0 2 18 20)*	(y 0 12 24)*
44	0 0 1 5	0 0 6 15	0 7 16 30	0 2 10 27
	0 2 13 26	0 3 10 21	0 4 12 32	(0 3 22 25)*
44[2]	x 0 0 3	0 0 9 17	0 1 2 6	0 4 12 22
	0 5 15 31	0 6 18 29	0 7 14 27	(0 2 21 23)*
	(y 0 14 28)*			
47	0 0 9 25	0 0 6 17	0 1 4 12	0 1 3 27
	0 2 7 34	0 4 14 32	0 5 12 33	0 8 18 31
50	0 0 8 23	0 0 12 30	0 1 2 4	0 3 9 16
	0 5 11 37	0 5 19 33	0 7 17 28	0 9 19 35
	(0 4 25 29)*			
50[2]	0 0 1 4	0 5 27 35	0 2 5 11	0 6 20 40
	0 7 17 36	0 7 25 38	0 9 25 36	x 0 0 15
	(0 2 24 26)*	(y 0 16 32)*		
53	0 0 13 23	0 0 17 25	0 5 24 39	0 4 11 12
	0 1 3 5	0 3 9 29	0 6 16 37	0 7 18 38
	0 9 21 35			
56	0 0 13 38	0 0 17 27	0 4 11 12	0 1 3 5
	0 3 9 23	0 6 22 30	0 7 21 37	0 9 20 41
	0 10 22 41	(0 5 28 33)*		
56[2]	x 0 0 3	0 0 15 23	0 9 16 28	0 4 10 17
	0 1 2 22	0 4 36 41	0 5 16 30	0 6 14 26
	0 9 19 30	(0 2 27 29)*	(y 0 18 36)*	
65	0 0 1 3	0 0 4 6	0 5 10 27	0 7 14 38
	0 8 16 40	0 9 18 40	0 10 21 47	0 11 23 46
	0 12 28 48	0 13 26 46	0 14 35 50	
68	0 0 42 49	0 0 10 40	0 1 13 51	0 2 66 31
	0 3 9 21	0 3 7 23	0 5 11 48	0 5 13 27
	0 8 23 52	0 9 24 41	0 11 25 47	(0 1 34 35)*
71	0 0 68 47	0 0 20 35	0 1 64 9	0 1 38 22
	0 2 54 11	0 2 6 12	0 4 11 34	0 5 23 49
	0 5 30 44	0 10 29 42	0 12 26 43	0 13 31 46
77	0 0 35 61	0 0 1 63	0 2 66 15	0 2 74 38
	0 3 55 22	0 4 8 25	0 5 11 23	0 6 34 65
	0 7 31 39	0 7 34 44	0 9 28 57	0 9 30 54
	0 10 27 47			

80	0	0	24	26	0	0	59	63	0	1	20	6	0	2	16	31
	0	3	70	77	0	4	73	38	0	5	27	50	0	8	16	36
	0	9	28	51	0	9	36	48	0	10	32	43	0	12	30	55
	0	13	31	46	(0	1	40	41)*								
83	0	0	52	45	0	0	61	50	0	1	2	17	0	2	48	68
	0	3	26	62	0	3	42	76	0	4	16	73	0	4	13	41
	0	5	11	23	0	5	13	40	0	6	25	53	0	8	27	51
	0	9	29	58	0	14	32	53								
89	0	0	11	8	0	0	55	62	0	1	68	24	0	1	48	44
	0	2	53	75	0	2	58	17	0	3	16	63	0	4	64	21
	0	5	10	19	0	6	12	32	0	7	37	57	0	9	28	61
	0	10	35	59	0	12	35	50	0	13	31	49				
92	0	0	30	67	0	0	60	64	0	1	87	76	0	2	63	12
	0	2	49	16	0	3	80	59	0	3	23	41	0	4	10	73
	0	5	24	48	0	7	18	27	0	7	21	57	0	8	34	42
	0	9	31	57	0	13	40	53	0	15	37	54	(0	1	46	47)*
95	0	0	60	31	0	0	86	36	0	1	76	19	0	1	80	74
	0	2	81	58	0	2	69	43	0	3	50	84	0	3	24	30
	0	4	67	55	0	4	11	24	0	5	10	32	0	7	15	53
	0	8	33	56	0	10	43	61	0	12	41	58	0	13	30	55
101	0	0	38	83	0	0	84	48	0	1	10	60	0	1	8	52
	0	2	90	97	0	2	72	35	0	3	81	37	0	3	82	12
	0	4	28	14	0	5	66	46	0	5	11	26	0	8	29	76
	0	12	39	73	0	13	36	58	0	15	39	69	0	16	32	58
	0	19	46	71												
104	0	0	15	62	0	0	3	96	0	1	50	25	0	2	21	57
	0	2	37	85	0	4	36	82	0	4	80	43	0	5	28	91
	0	5	77	71	0	6	84	93	0	7	14	23	0	10	40	69
	0	10	43	74	0	12	34	63	0	12	39	56	0	13	44	58
	0	16	34	54	(0	1	52	53)*								

(Received 12/5/93)