Young Raising Operators and Q-functions

M. A. Salam
Departement of Mathematics and Computing
University of Central Queensland
Rockhamton, Australia
and
B. G. Wybourne
Institüt Fizyki
Universytet Mikaloja Kopernika
ul. Grudziqdzka 5/7
87-100 Torun', Poland

Abstract

A number of properties of Young raising operators as applied to S-functions and Schur Q-functions are noted. The order of evaluating the action of inverse raising operators is found to require careful specification and the maximum power of the operators δ_{ij} is determined. The operation of the inverse raising operator on a partition λ is found to be the same as for its conjugate $\tilde{\lambda}$. A new definition of the Shifted Lattice Property that can efficiently remove all the dead tableaux in the analogue of the Littlewood-Richardson rule for Q-functions is introduced. A simple combinatorial analogue of raising and inverse raising operators is given that in turn simplifies the computation of the Kronecker products of Schur Q-functions.

1 Introduction

Young raising operators play an important role in the representation theory of the symmetric group S_n [**1**,**10**] and also in the theory of the projective representations of S_n via Schur Q-functions [**2**,**7**,**8**]. Schur Q-functions are a specialisation of the Hall-Littlewood polynomials [**3**] and have recently been shown to be relevant to the bilinear Kadomtsev-Petviashvii (KP) equations in (2+1)-dimensions [**4**,**5**]. In this article we report a number of properties and results related to the applications of

Young raising operators to Schur Q-functions and relevant to the elucidation of the properties of Schur Q-functions .

The expansion of a Q-function in terms of S-functions and vice versa as discussed in sections 6 and 7 is of particular importance. The efficiency of the algorithm for the calculation of the inner product of Schur Q-functions reported in [8] depends on the efficiency of rapid expansion of Q-functions and S-functions. In sections 6 and 7 we give very economical algorithms for such expansions.

The operation of the Young raising operator δ_{ij} on a partition $\lambda \equiv (\lambda_1 \cdots \lambda_i \cdot \lambda_j \cdots \lambda_n)$ increases λ_i and decreases λ_j by 1 provided $i < j$. Young raising operators are widely used in the operations of generalised S-functions $S_\lambda(-1)$ and Schur Q-functions Q_λ as shown in Eq.(1). The generalised S-function, S_λ, will be represented by $\{\lambda\}_q$, defined by

$$\{\lambda\}_q = \prod_{i<j} (1 - \delta_{ij})\, q_\lambda.$$

Here we shall discuss their properties in this context which have not been discussed before.

One can expand a generalised S-function in terms of Q-functions using the raising operator as follows:

$$\{\lambda_1\lambda_2 \cdots \lambda_n\}_q = \prod_{1\le i<j\le n} (1 + \delta_{ij})Q_\lambda, \tag{1}$$

and one can expand a Q-function in terms of generalised S-functions using the inverse raising operator as follows:

$$Q_\lambda = \prod_{1\le i<j\le n} [1 + \sum_t (-1)^t \delta_{ij}^t]\{\lambda\}_q. \tag{2}$$

Sagan [6] and Worley [11] have developed a combinatorial theory of shifted tableaux in the description of Schur Q-functions . These tableaux play a similar role to that of ordinary Young tableaux of S-functions. A shifted diagram is a diagonally adjusted Young diagram with the restriction that the $(i+1)$-th row does not exceed the i-th row. This condition ensures that partitions are restricted to those involving distinct parts only. Let $\mathbf{P}'$ denote the ordered alphabet $\{1' < 1 < 2' < 2 \cdots\}$. The letters $1'$,$2'$ $\cdots$ are said to be *marked* and we denote an *unmarked* version of any $a \in \mathbf{P}'$ by $|a|$. Let **DP** represent partitions into distinct parts only, then for each $\lambda \in \mathbf{DP}$ there is an associated shifted diagram defined by

$$\mathbf{D}'_\lambda = \{(i,j) \in Z^2 : i \le j \le \lambda_i + i - 1,\ 1 \le i \le \ell(\lambda)\}.$$

A shifted tableau $\mathbf{T}$ of shape λ is an assignment $\mathbf{T} : \mathbf{D}'_\lambda \rightarrow \mathbf{P}'$ satisfying the following conditions:

1. $\mathbf{T}(i,j) \le \mathbf{T}(i+1,j)$, $\mathbf{T}(i,j) \le \mathbf{T}(i,j+1)$;
2. *Each column has at most one* k $(k = 1, 2, \cdots)$;
3. *Each row has at most one* k' $(k' = 1', 2' \ldots)$.

If $|\mathbf{T}(i,j)| = k$ then the tableau $\mathbf{T}$ is said to have a content

$$\gamma = (\gamma_1, \gamma_2, ...) \text{ and } x^{\mathbf{T}} = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots .$$

We can define a generating function $Q_\lambda = Q_\lambda(x)$ in the variables $x_1, x_2 \cdots$ for each $\lambda \in \mathbf{DP}$ by

$$Q_\lambda = \sum_{\mathbf{T}:\mathbf{D}'_\lambda \to \mathbf{P}'} x^{\mathbf{T}} \tag{3}$$

where the summation is over all tableaux $\mathbf{T}$.

2 Order of the inverse raising operators

It is generaly assumed that the order of the operators is not important. Actual applications of (1) and (2) show that the order of raising operators is not important indeed but that of inverse raising operators used in (2) is. We observe that the operators in (2) should be set in such a way that from right to left the values of j are in non-ascending order and the values of i are in descending order for a given value of j. As an example, for a three part partition λ, the equation (2) becomes

$$Q_\lambda = [1 + \sum_t (-1)^t \delta_{12}^t][1 + \sum_t (-1)^t \delta_{13}^t][1 + \sum_t (-1)^t \delta_{23}^t]\{\lambda_1\lambda_2\lambda_3\}_q \tag{4}$$

3 Maximum power of the operators δ_{ij}

Regarding the maximum value of t, we observe that the list on the right side of the equation (2) contains S-functions, so the maximum value of t is that for which λ_j becomes $(j-n)$. Hence it is not dependent on the value of λ_j as given by Thomas [**10**] but on the position of λ_j. As an example, if we apply $[1+\sum_t(-1)^t\delta_{12}^t]$ on $\{321\}$ then the maximum value of t will be 3 whereas according to Thomas it should not be greater than 2. Thus

$$\begin{aligned}(1 - \delta_{12} + \delta_{12}^2 - \delta_{12}^3)\{321\}_q &= \{321\}_q - \{411\}_q + \{501\}_q - \{6-11\}_q, \\ &= \{321\}_q - \{411\}_q + \{6\}_q, \end{aligned} \tag{5}$$

where the second line of (5) is achieved by using the modification rules for S-functions.

4 Raising operator and conjugate partitions

Another important property of the raising operator used in equation (1) which simplifies its computation is that it gives the same set of Q-functions if applied to $Q_{\tilde{\lambda}}$ where $\tilde{\lambda}$ is the conjugate partition of λ:

$$\prod_{1\le i<j\le n} (1 + \delta_{ij}) Q_\lambda = \prod_{1\le i<j\le n} (1 + \delta_{ij}) Q_{\tilde{\lambda}} \tag{6}$$

As an example the Q-function content of $\{5\}$ is the same as that of $\{1^5\}$. It is clear from the S_n irreps analogue that the characters of positive classes of conjugate representations are same. Hence the expansion of conjugate representations in terms of spin irreps will also be the same.

5 Shifted lattice property

A shifted analogue of the lattice permutation is given by Stembridge [9]. Given a shifted tableaux $\mathbf{T}$ a word $w(\mathbf{T}) = w_1 w_2 \cdots w_n$ is a sequence obtained by reading the rows of $\mathbf{T}$ from left to right (rather than right to left), starting with the last row (rather than the top row). Let $w^r = w_n w_{n-1} \cdots w_1$ denote the reverse of w and let $\hat{w} = \hat{w}_1 \cdots \hat{w}_n$ denote the word obtained by inverting the marks of w, i.e. $\hat{2} = 2'$ and $\hat{2'} = 2$. Let $n_i(w, j)$ denote the number of occurrences of the letter i among $w_1 \cdots w_j$ and $n_i(w, 0) = 0$. An extended word of $\mathbf{T}$ is the sequence defined by $e(\mathbf{T}) = w^r \hat{w}$. The tableau $\mathbf{T}$ is said to satisfy the *shifted lattice property* if the extended word $e = e_1 \cdots e_{2n}$ satisfies the following conditions for all $i \geq 1$ and $0 \leq j < 2n$:

$$n_i(e, j) = n_{i-1}(e, j) \text{ implies } \begin{cases} e_{j+1} \neq i, i' & 0 \leq j < n \\ e_{j+1} \neq i, (i-1)' & n \leq j < 2n. \end{cases} \tag{7}$$

We can modify Stembridge's definition of the shifted lattice property as follows.

DEFINITION 1 *The tableaux* $\mathbf{T}$ *is said to satisfy the shifted lattice property if the reversed word* $w^r = w_n w_{n-1} \cdots w_1$ *satisfies the following conditions for all* $i \geq 1$, $0 < j \leq n$ *and* $0 \leq k < n$.

(i) $n_i(w^r, k) = n_{i-1}(w^r, k)$ *implies* $w^r_{k+1} \neq i, i'$,

(ii) $n_{(i-1)'}(w^r, j) - n_{i'}(w^r, j) = \nu_{(i-1)} - \nu_i$ *implies* $w^r_{j-1} \neq i', (i-1)$, *where* ν *is the content of the tableaux* $\mathbf{T}$.

The second condition in (7) arises when

$$n_i(e, n) + n_{i'}(e, n) - n_{i'}(e, 2n - j) = n_{(i-1)}(e, n) + n_{(i-1)'}(e, n) - n_{(i-1)'}(e, 2n - j),$$

whereas we know that $n_{|i|}(e, n) = \nu_i$ and $\nu_{i-1} \geq \nu_i$, hence

$$n_{(i-1)'}(e, 2n - j) - n_{i'}(e, 2n - j) = \nu_{i-1} - \nu_i.$$

Thus to satisfy the second condition of (7) $e_{2n-j-1} \neq i', (i-1)$.
There are two advantages of this definition. First we do not need to form extended words as required by Stembridge. Secondly during the formation of the tableaux from the top right corner which is the reverse direction of the word, the dead tableaux can easily be discarded as soon as they arise.

6 Expansion of a Q-function

A Q-function can be expanded in terms of generalised S-functions $\{\lambda\}_q$ using the inverse raising operators as given by (2). A large number of non-standard partitions are generated in this process which are standardised using the modification rules.
We can expand a Q-function in a more efficient way by making use of the following expression:

$$Q_\mu = \sum_\lambda b_\mu^\lambda \{\lambda\}_q, \tag{8}$$

where the coefficients b_μ^λ are evaluated as follows:

$$b_\mu^\lambda = \sum_{i=2}^{l(\mu)} \sum_{n_i=p}^{q} [C(t) - C(s)\delta_i] (-1)^x, \tag{9}$$

where $q = \sum_{k=i}^{l(\mu)}(\mu_k - \lambda_k)$, $l(\mu)$ is the length of the partition (μ), p is the maximum of 0 and $\mu_i - \lambda_i$, $t = \lambda_1 - \mu_1 - n_2$, $s = \lambda_2 - \mu_2 - n_2 - 1$, $x = \sum_{j=2}^{l(\mu)} n_j$ and δ_i is equal to 1 if $\sum_{j=i+1}^{l(\mu)} \mu_j > \lambda_i$ and zero otherwise. $C(z)$ is a binomial coefficient defined by

$$C(z) = \sum_\rho \prod_{k=2}^{l(\rho)} \begin{pmatrix} N' - k + 1 \\ N(\rho_k) \end{pmatrix},$$

where N' is the number of i values such that $n_i \geq \rho_k$, $N(\rho_k)$ is the number of ρ_k's of equal weight, ρ is a partition of z such that $\sum_i^{l(\rho)} \rho_i \leq \sum_i^{l(\rho)} n_i$ and $C(z) = 0$ for $z < 0$.
The above expression is specifically designed to calculate the coefficients of the S-functions appearing in the expansion of a Q-function in an alternative way to using the inverse raising operators. Eq.(9) is suggested by the action of inverse raising operators and the modification rules for the S-functions. It has been suggested and tested by comparing the results of implementing both formulae (9) and the inverse raising operators on the computer. A large number of cases for the partitions μ of length $l(\mu) \leq 5$ and weight $|\mu| \leq 20$ have been tested. It is very difficult for a computer to handle the situation arising from the action of inverse raising operators on a partition μ of length $l(\mu) > 5$, whereas the algorithm of (9) can easily calculate the coefficient of any partition appearing in the expansion of a Q-function corresponding to the partition μ of length $l(\mu) \leq 9$.
A simple example of $\lambda \equiv 41$ and $\mu \equiv 32$ will illustrate the working of (9).
$i = 2$

$\mu_2 - \lambda_2 = 1$, $p = 1$, $n_2 = 1$ and $q = 1$.
$t = 4 - 3 - 1 = 0$ and $s = 1 - 2 - 1 - 1 = -3$.
Then $\rho \equiv 1$, $N' = 1$ and $N(\rho_{k=2}) = 1$.
Hence $C(t) = 1$, $C(s) = 0$, $\delta_i = 0$ and $x = 1$.
Thus $b_{32}^{41} = (1 - 0)(-1) = -1$.

Though Eq.(9) looks complicated, it has several advantages as stated below:

- It enables us to calculate the coefficient of a particular S-function without making the whole expansion. This simplification is very useful in the calculation of the inner product of Q-functions.
- Application of the modification rules on a rather large list of partitions is avoided.
- Its computer algorithm is extremly efficient for partitions of length greater than 3 where the inverse raising operators produce a very large number of dead and non-standard partitions.

In order to simplify further the computation we can easily set the highest and the lowest partitions arising in (8).

DEFINITION 2 *A partition* $\nu = (\nu_1 \nu_2 \cdots \nu_i)$ *is lower than* $\mu = (\mu_1 \mu_2 \cdots \mu_j)$ *if for all* $1 \leq k \leq j$, $\sum_k \nu_k \leq \sum_k \mu_k$ *and* $|\mu| = |\nu|$.

This definition is different from Macdonald's [3] definition of a lower partition and is more appropriate in this case.

THEOREM 1 *The highest partition* $\{\lambda\}$ *appearing in the expansion of a* Q*-function* Q_μ *in terms of* S*-functions is* $\{n\}$*, where* n *is the weight of the partition* (μ)*.*

Proof

Equation (9) and the properties of inverse raising operators observed in the previous sections lead to the above conclusion.

COROLLARY 1 *The coefficient*

$$b_\mu^n = \begin{cases} -1 & \text{for odd } \sum_{i=2}^{l(\mu)}(i-1)\mu_i \\ +1 & \text{for even } \sum_{i=2}^{l(\mu)}(i-1)\mu_i \end{cases} \tag{10}$$

Proof

The above result can be concluded from (9). It appears from Eq.(9) that for $\lambda \equiv n$, the term $[C(t) - C(s)\delta_i]$ vanishes except for $i = l(\mu)$. In that case it is 1 and $x = \sum_{i=2}^{l(\mu)}(i-1)\mu_i$.

THEOREM 2 *The lowest partition* $\{\lambda\}$ *in the expansion of a* Q*-function* Q_μ *in terms of* S*-functions is* $\lambda \equiv \mu$ *.*

Proof

It is obvious from the nature of inverse raising operators.

COROLLARY 2 *The coefficient*

$$b_\mu^\mu = 1.$$

Proof

Since every $\lambda_i - \mu_i$ is 0, the equation (9) immediately gives the above result. This method is far more efficient and powerful than the inverse raising operators. A great advantage of this method is that the coefficient b_μ^λ of a particular λ can easily be calculated without making the whole expansion.

7 Expansion of an S-function

A generalised S-function $\{\lambda\}_q$ can be expanded in terms of Q-functions using the raising operators as given by (1). This is a very cumbersome method and generates a large number of non standard partitions which are standardised at the end using modification rules.
We can write (1) as follows:

$$\{\mu\}_q = \sum_\lambda g_{\lambda\mu} Q_\lambda, \tag{11}$$

where $g_{\lambda\mu}$ is the number of shifted tableaux of unshifted shape μ and content λ such that

1. $w = w(S)$ satisfies the shifted lattice property,
2. The leftmost i of $|w|$ is unmarked in w for $1 \leq i \leq l(\lambda)$.

The coefficients $g_{\lambda\mu}$ are easily computable using the techniques developed in [8]. Stembridge [9] has used the same coefficients in the product of a basic spin representation and an ordinary irrep of S_n.
Similar to the expansion of a Q-function, we can set the highest and the lowest partitions arising in (11) using the properties of shifted tableaux of unshifted shape.

DEFINITION 3 *The rank of a partition (ρ) is the maximum value of i for which $\rho_i \geq i$.*

THEOREM 3 *The highest partition (λ) in the expansion of an S-function S_μ in terms of Q-functions is*

$$\lambda_i = \mu_i + \tilde{\mu}_i - 2i + 1 \quad for\ 1 \leq i \leq r(\mu), \tag{12}$$

where $\tilde{\mu}$ is the conjugate of μ and $r(\mu)$ is the rank of μ.

Proof
By theorem 1 of [8] and the fact that only the leftmost 1 of the first row can be marked, we can not make a second entry of 1 in any other row. Hence the maximum number of 1's can be placed in the first row and the first column only. Similarly the maximum number of 2's can be placed in the remaining places of the second row and the second column and so on.

COROLLARY 3 *The highest partition (λ) in the expansion of an S-function S_μ in terms of Q-functions is the same as for $S_{\tilde{\mu}}$.*

Proof
It can easily be concluded by conjugating both sides of (12).

COROLLARY 4 *For the highest partition (λ), the coefficient*

$$g_{\lambda\mu} = 1.$$

Proof

It is clear from the proof of theorem 3 that there is only one possible tableau for the highest partition.
In order to work out the lowest (λ) in (11) we note that if $\mu \in \mathbf{DP}$ then the lowest $\lambda \equiv \mu$ otherwise if $\tilde{\mu} \in \mathbf{DP}$ then the lowest $\lambda \equiv \tilde{\mu}$. If both μ and $\tilde{\mu}$ have repeated parts then we can use the following algorithm.

Algorithm 1

1. If $\mu_1 < l(\mu)$ then $\lambda \equiv \tilde{\mu}$ otherwise $\lambda \equiv \mu$.
2. Reading from right to left, for any $\lambda_{i+1} \geq \lambda_i$ interchange them such that

$$\lambda_1, \ldots \lambda_i, \lambda_{i+1}, \ldots \lambda_n \to \lambda_1, \ldots \lambda_{i+1} + 1, \lambda_i - 1, \ldots \lambda_n.$$

3. Repeat step 2 till a partition of distinct parts is obtained.

As an example, for $\mu \equiv 433111$

$$\lambda \equiv \tilde{\mu} \equiv 6331,$$

using step 2

$$\lambda \equiv 6421.$$

8 Conclusion

The properties of Young raising operators discussed in sections 1 and 2 remove all the ambiguities and the property observed in section 3 simplifies the computation of equation (1). We have redefined the Shifted Lattice Property in section 4 which is more efficient and simplifies the computational problems. In sections 5 and 6 we have given alternatives to (1) and (2) which are more powerful and easily computable.

References

1. Littlewood D. E., *Proc. London Math. Soc.*, (3) **11** (1961) 485-498.
2. Luan Dehuai and Wybourne B. G., *J. Phys. A: Math. Gen.*, **14** (1981) 327-348.
3. Macdonald I. G., *Symmetric Functions and Hall Polynomials* (1979) (Oxford University Press)
4. Nimmo J. J. C., *J. Phys. A: Math. Gen.*, **22** (1989) 3213-3221.
5. Nimmo J. J. C., *J. Phys. A: Math. Gen.*, **23** (1990) 751-760.
6. Sagan B. E., *J. Combin. Theory. Ser. A* **45** (1987) 62-103.

7. Salam M. A. and Wybourne B. G., *J. Phys. A: Math. Gen.,* **22** (1989) 3771-3778.

8. Salam M. A. and Wybourne B. G., *J. Math. Phys.* **31** (1990) 1310.

9. Stembridge J. R., *Advances in Mathematics* **74** (1989) 109-153.

10. Thomas G. P., *Can. J. Math.*, Vol. XXXIII, No. 1 (1981) 49-54.

11. Worley D. R., Ph.D. Thesis, M. I. T. (1989).

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