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Solutions to the Oberwolfach problem for orders up to 100

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Abstract

A complete set of solutions for all instances of the Oberwolfach problem for orders $61 \leq n \leq 100$ is presented.

1 Introduction

A *k-factor* in a graph G is a k-regular spanning subgraph of G. A *k-factorization* of G is a collection of edge-disjoint k-factors whose edge-sets partition the edge set of G. If G has a k-factorization it is also called *k-factorable*.

Let F be a 2-factor; it consists of t disjoint cycles of lengths $p_1, p_2, \ldots, p_t$. In 1967, G. Ringel posed the famous Oberwolfach problem OP(F) which asks whether, for any 2-factor F of order n, the complete graph K_n when n is odd, or $K_n \setminus I$ (i.e., the complete graph with a 1-factor I removed) when n is even, is 2-factorable into 2-factors, each isomorphic to F. If the 2-factor F consists of t disjoint cycles of lengths $p_1 \geq p_2 \geq \ldots \geq p_t$, we also use the notation OP$(p_1, p_2, \ldots, p_t)$ in place of OP(F).

There are only four known instances for which the Oberwolfach problem does not have a solution, cf. [14]: $OP(3,3)$, $OP(5,4)$, $OP(5,3,3)$ and $OP(3,3,3,3)$. OP(F) has been intensively studied and solved for several classes of 2-factors, in particular: when F is bipartite [5, 10], F is uniform (all cycles have the same lengths) [3, 4, 11], for $t = 2$ [16], when F contains a sufficiently long cycle [7], for some instances with short cycles only, cf. [13]. Moreover, OP(F) has been completely solved for orders $n = 2q$, where q is any prime congruent to 5 (mod 8) [2], and for some values of prime q congruent to 1 (mod 16) [6]. Asymptotic evidences (for large n without any lower bound) have been provided [9, 12], although the problem still remains widely open.

Solutions to the Oberwolfach problem were produced for orders $n \leq 17$ [1], $18 \leq n \leq 40$ [8], and recently for $41 \leq n \leq 60$ [15]. The aim of this paper is to provide solutions for the complete set of all 40,119,909 instances for orders $61 \leq n \leq 100$.

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2 Constructions

Among numerous variations of standard difference methods, the following modification of the well known Bose's method of pure and mixed differences turned out to be efficient to produce feasible solutions for each case in a reasonable computational time. Since the number of 2-factors is, depending on the parity of n, either $\lfloor \frac{n-1}{2} \rfloor$ or $\lfloor \frac{n-2}{2} \rfloor$, it seems to be convenient to assume that a solution being constructed admits as an automorphism a permutation containing exactly r fixed points and two cycles, each of length $m = \frac{n-r}{2}$, where $r = 1$ if n is odd and $r = 2$ otherwise.

Let P and R be a 2-element and an r-element set, respectively, $P = \{1, 2\}$, $R = \{1, 2, \ldots, r\}$. Let $V = V' \cup V''$ where $V' = \mathbb{Z}_m \times P$ and $V'' = \{\infty\} \times R$. A standard notation x_i is used for the pair (x, i). For any two vertices $x_i \neq y_j$ of V', the differences arising from this pair may be of two kinds:
(1) if $i = j$ then $\pm(x - y)$ are *pure* differences of type i
(2) if $i \neq j$ then $\pm(x - y)$ are *mixed* differences.
A pure difference of any type may be equal to any nonzero element of $\mathbb{Z}_m$ while a mixed difference may be equal to any element of $\mathbb{Z}_m$. Moreover, for any $x_i \in V'$ and any $\infty_j \in V''$ we get an *infinity* difference of type (i, j).

Let G be a regular graph of order $n = 2m + r$ and degree $2m$ such that $V(G) = V = V' \cup V''$ and G admits as an automorphism the permutation $\alpha = (0_1, 1_1, \ldots, (m-1)_1)(0_2, 1_2, \ldots, (m-1)_2)$, where all vertices in V'' are fixed points of α. We say that a 2-factorization $\mathcal{F}$ of G is $(2, r)$-*rotational* if there exists a 2-factor F of G such that, among pairs determined by edges of F, every nonzero element of $\mathbb{Z}_m$ occurs at most once as a pure difference of type i for each $i \in P$, every element of $\mathbb{Z}_m$ occurs at most once as a mixed difference, moreover for each $i \in P, j \in R$ there is at most one edge with infinity difference of type (i, j) and there is no edge induced by V''. Then F is the *base* 2-*factor* for $\mathcal{F} = \{F, \alpha F, \alpha^2 F, \ldots, \alpha^{m-1} F\}$.

Depending on the residue class of n modulo 4, these cases are considered separately. If $n \equiv 3 \pmod 4$ then $G = K_n$ and $r = 1$. If $n \equiv 0$ or $2 \pmod 4$ then $G = K_n \setminus I$ and $r = 2$, where I is a 1-factor of G and $\{\infty_1, \infty_2\} \in I$. If $n \equiv 9 \pmod{12}$ and $p_1 = 3$ then $G = K_n \setminus F'$ and $r = 3$, where F' is a 2-factor of G that consists of 3-cycles only and one of them is induced by V''. For all remaining cases when $n \equiv 1 \pmod 4$, some modifications to the above method are necessary and then 2-factorizations which are constructed are not $(2, r)$-rotational. Nevertheless, the priority was to reduce the number of different constructions which could be used. With respect to the large number of instances to verify, in order to simplify computations and to significantly shorten output files, the main assumption was made to have every solution represented just by a single base 2-factor.

For the completeness of results, solutions for the cases when the Oberwolfach problem has been settled before, are also included. The complete set of files is available from the author at `http://home.agh.edu.pl/~meszka/op.html`. Although the authors of [15] claim they have constructed solutions for each instance and each order $41 \leq n \leq 60$, in the files they have provided online many instances are missing.

Thus complete files for these orders are attached, too. Every text file (separate for each $41 \le n \le 100$) contains solutions for all instances for a given order n; files are compressed using 7z archive format. Each line in a file corresponds to one instance; it has the format $OP(p_1, p_2, \ldots, p_t) : C_1, C_2, \ldots, C_t$, where each C_i contains consecutive vertices of a cycle of length p_i in the base 2-factor F, $i = 1, 2, \ldots, t$.

2.1 $n \equiv 3 \pmod 4$

Let $m = \frac{n-1}{2}$ and $V = \mathbb{Z}_m \times \{1, 2\} \cup \{\infty\}$ be the vertex set of K_n. Then m is odd. To get all remaining 2-factors it is enough to apply α^i to the base 2-factor F, for every $i = 1, 2, \ldots, m$.

2.2 $n \equiv 0 \pmod 4$

Let $m = \frac{n-2}{2}$ and $V = \mathbb{Z}_m \times \{1, 2\} \cup \{\infty_1, \infty_2\}$ be the vertex set of K_n. Then m is odd. Let I be a 1-factor with the edge set $\{\{i_1, i_2\} : i \in Z_m\} \cup \{\infty_1, \infty_2\}$. Thus, in the base 2-factor F, the mixed difference 0 is excluded. The permutations α^i, $i = 1, 2, \ldots, m$, applied to F, produce all remaining 2-factors.

2.3 $n \equiv 2 \pmod 4$

Let $m = \frac{n-2}{2}$ and $V = \mathbb{Z}_m \times \{1, 2\} \cup \{\infty_1, \infty_2\}$ be the vertex set of K_n. Then m is even. Let I be a 1-factor with the edge set $\{\{i_j, (i + \frac{m}{2})_j\} : i = 0, 1, \ldots, \frac{m}{2} - 1,\ j = 1, 2\} \cup \{\infty_1, \infty_2\}$. In this way the base 2-factor F does not contain edges of pure difference $\frac{m}{2}$ of both types. Similarly to the above, the permutations α^i, $i = 1, 2, \ldots, m$, are applied to F to get all remaining 2-factors.

2.4 $n \equiv 1 \pmod 4$

Since $m = \frac{n-1}{2}$ is even, in order to use pure differences $\frac{m}{2}$, constructions used previously have to be modified. Let $V = \mathbb{Z}_m \times \{1, 2\} \cup \{\infty\}$ be the vertex set of K_n. Although α is not assumed to be an automorphism of $\mathcal{F}$ anymore and two base 2-factors F and F' are needed to generate $\mathcal{F}$, it is possible to easily transform F to F' and to get in this way a 2-factorization $\mathcal{F}$ represented just by a single 2-factor F. To do this, four cases are considered separately:

(1) $p_1 \ge 5$: To construct F, all pure, mixed and infinity differences except for the mixed difference $\frac{m}{2}$ are used. Moreover, it is required that C_{p_1} contains the path $(0_1, \frac{m}{2}_1, \frac{m}{2}_2, 0_2)$. Let F' denote a 2-factor obtained from F by replacing the edges $\{0_1, \frac{m}{2}_1\}$, $\{0_2, \frac{m}{2}_2\}$ with $\{0_1, \frac{m}{2}_2\}$, $\{0_2, \frac{m}{2}_1\}$. This transformation does not change the length of C_{p_1}. Instead of pure differences $\frac{m}{2}$ of both types, mixed difference $\frac{m}{2}$ is used twice in F'. Then $\mathcal{F} = \{F, \alpha F, \alpha^2 F, \ldots, \alpha^{\frac{m-2}{2}} F, \alpha^{\frac{m}{2}} F', \alpha^{\frac{m+2}{2}} F', \ldots, \alpha^{m-1} F'\}$.

(2) $p_1 = 4$ and $n \equiv 1 \pmod 8$: Notice that $p_t = 3$. It is assumed that $C_{p_1} = (0_1, 2_2, \frac{m}{2}_1, \infty)$ and $C_{p_t} = (0_2, 1_1, \frac{m}{2}_2)$. Let F' be a 2-factor obtained from F by

replacing C_{p_1}, C_{p_t} with cycles $(1_1, 0_2, \infty, \frac{m}{2}_2)$, $(0_1, \frac{m}{2}_1, 2_2)$. Notice that both these pairs of cycles contain the same vertices and differ in the use of the pure difference $\frac{m}{2}$ (of type 1 missing in the first pair and of type 2 missing in the second) and repeated infinity differences (type 1 in the first pair and type 2 in the second). Then $\mathcal{F} = \{F, \alpha F, \alpha^2 F, \ldots, \alpha^{\frac{m-2}{2}} F, \alpha^{\frac{m}{2}} F', \alpha^{\frac{m+2}{2}} F', \ldots, \alpha^{m-1} F'\}$.

(3) $p_1 = 4$ and $n \equiv 5 \pmod 8$: A parity argument determines that the above construction (2) cannot be used. Let F' be a 2-factor obtained from F by applying a permutation β such that $\beta(i_1) = i_2$, $\beta(i_2) = ((i + \frac{m}{2})_1)$ and $\beta(\infty) = \infty$, for each $i = 0, 1, \ldots, m-1$. Then $\mathcal{F} = \{F, \alpha F', \alpha^2 F, \alpha^3 F', \ldots, \alpha^{m-2} F, \alpha^{m-1} F'\}$.

(4) $p_1 = 3$: Then $n \equiv 9 \pmod{12}$. Let $m' = \frac{n-3}{2}$. In this case a solution that is $(2,3)$-rotational is constructed. To construct F, all pure, mixed and infinity differences except for pure differences $\pm\frac{m'}{3}$ of both types, are used. Let $F^* = \{\{i_1, (\frac{m'}{3} + i)_1, (\frac{2m'}{3} + i)_1\}, \{i_2, (\frac{m'}{3} + i)_2, (\frac{2m'}{3} + i)_2\} : i = 0, 1, \ldots, \frac{m'}{3} - 1\} \cup \{\infty_1, \infty_2, \infty_3\}$. Then $\mathcal{F} = \{F, \alpha F, \alpha^2, \alpha^{m'-1} F, F^*\}$.

For any order n, the list of all instances $OP(p_1, p_2, \ldots, p_t)$ simply contains all partitions of n into parts, none less than 3. Enumerations of such partitions, for each $41 \leq n \leq 100$, are included in Table 1. In the case of each instance, purely combinatorial methods were used to construct a required 2-factorization. A construction of a base 2-factor F was split into two stages: in the first edges of F were partitioned just according to their type, and in the second vertices were labeled. To accomplish both steps efficiently, several randomized searches were applied. This approach allowed to generate all solutions in average computational time 0.6 second (for $n = 61$) and 8.8 second (for $n = 97$) per instance on one core of the Intel Xeon E5-2680v3 2.5 GHz processor. Due to the large number of instances to verify, a cluster of HPE ProLiant XL730f Gen9 servers, each with two such processors and 12 cores per processor, was used.

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n	i	n	i	n	i	n	i
41	2075	42	2438	43	2842	44	3323
45	3872	46	4510	47	5237	48	6095
49	7056	50	8182	51	9465	52	10945
53	12625	54	14578	55	16779	56	19323
57	22210	58	25519	59	29269	60	33581
61	38438	62	44004	63	50305	64	57480
65	65585	66	74831	67	85241	68	97084
69	110441	70	125577	71	142627	72	161955
73	183669	74	208233	75	235858	76	267016
77	302008	78	341474	79	385714	80	435525
81	491365	82	554102	83	624363	84	703263
85	791483	86	890414	87	1001014	88	1124831
89	1263105	90	1417812	91	1590370	92	1783200
93	1998184	94	2238095	95	2505329	96	2803342
97	3134927	98	3504321	99	3915113	100	4372211

Table 1: The number i of instances for order n

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