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Uniquely completable and critical subsets of the integer addition table

Aurora Callahan

Department of Molecular Biology, Cell Biology, and Biochemistry
Brown University, Providence, RI, U.S.A.

Emma R. Hasson

Department of Mathematics, Brandeis University
Waltham, MA, U.S.A.

Kaethe Minden*

Department of Science, Mathematics and Computing
Bard College at Simon's Rock, Great Barrington, MA, U.S.A.

M. A. Ollis

Marlboro Institute for Liberal Arts and Interdisciplinary Studies
Emerson College, Boston, MA, U.S.A.

Xinyue Zhu

Department of Science, Mathematics and Computing
Bard College at Simon's Rock, Great Barrington, MA, U.S.A.

Abstract

A partial latin square is uniquely completable if there is exactly one latin square in which it is contained. A uniquely completable partial latin square is a critical set if removing any entry renders it no longer uniquely completable. These are well-studied concepts for finite latin squares; we offer the first consideration of them for infinite latin squares. We focus on the addition table of the integers. Results include the construction of critical sets of densities 1/4 and 95/176 and of infinitely many densities between these values, a chain of uniquely completable partial latin squares with empty intersection, a family of uniquely completable partial latin squares that contain no critical sets, and a partition of the addition table of the integers into three critical sets.

* Corresponding author: `kminden@simons-rock.edu`

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1 Introduction

There has been much work done with partial latin squares in the finite case, with particular emphasis on when they are uniquely completable and critical (definitions and references below). The most extensively studied squares are the addition tables of $\mathbb{Z}_n$, the integers modulo n. Here we extend this study to the infinite, considering unique completability and criticality of partial latin squares contained in the addition table of the integers $\mathbb{Z}$.

We prove results that are analogous to some of those from the finite case and others that have no finite version or are simply false for finite situations.

A finite *latin square* is an $n \times n$ array with n symbols arranged such that each symbol appears once in each row and once in each column. See [9] for a comprehensive account of the theory of latin squares.

An *infinite latin square on* $\mathbb{Z}$ has symbol set $\mathbb{Z}$ and rows and columns also indexed by $\mathbb{Z}$. As in the finite case, every integer must appear exactly once in each row and once in each column. Similarly to the finite case, an infinite latin square on $\mathbb{Z}$ can be thought of as a set of triples $\{(x, y, z) : x, y, z \in \mathbb{Z}\}$, where the first coordinate is the column index, the second coordinate is the row index, and the third coordinate is the entry, or symbol, contained in the cell in that row and column. Note that in this notation the positions of row and column indices are the reverse of that usually used for finite latin squares. We use this order because we shall consider our squares with cells indexed by the integer lattice embedded in $\mathbb{R}^2$ and so we may talk about x- and y-coordinates with the usual meaning for $\mathbb{R}^2$.

In particular, the *integer addition square*, which we denote $L_{\mathbb{Z}}$, is given by

$$L_{\mathbb{Z}} = \{(x, y, x + y) : x, y \in \mathbb{Z}\}.$$

Infinite latin squares are less well studied than finite squares. See [3, 4, 6, 8, 10, 11, 13, 15] for some examples. More generally, for a brief survey of infinite design theory, a broader category that includes the study of infinite latin squares, see [5].

A *partial latin square* is an array with empty cells allowed that has each symbol at most once in each row and at most once in each column. Again, we often think of partial latin squares as sets of triples and describe them as subsets of non-partial latin squares when appropriate.

Example 1.1. For $(a, b) \in \mathbb{Z}^2$, let the *quartered partial square* $\blacksquare_{(a,b)}$ be given by

$$\blacksquare_{(a,b)} = \{(x, y, x + y) : x < a,\ y \le b\} \cup \{(x, y, x + y) : x \ge a,\ y > b\} \subseteq L_{\mathbb{Z}}.$$

We will be focused on the particular instance of this class of partial latin squares centered at the origin, namely $\blacksquare_{(0,0)}$, which we abbreviate as $\blacksquare$. That is, $\blacksquare$ contains the non-axis integer lattice entries in the first and third quadrants along with the entries on the negative x-axis and the positive y-axis. The central part of $\blacksquare$ is given in Figure 1.

A partial latin square P is *uniquely completable* if there is exactly one latin square L with $P \subseteq L$. A uniquely completable partial latin square P is *critical*

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \\
 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

Figure 1: The partial square ▚, with the origin denoted by $\odot$.

$$
\begin{array}{cccccc}
0 & 1 & 2 & \cdot & \cdot & \cdot \\
1 & 2 & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\cdot & \cdot & \cdot & \cdot & 3 & 4
\end{array}
\qquad\qquad
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \cdot \\
1 & 2 & 3 & 4 & \cdot & \cdot \\
2 & 3 & 4 & \cdot & \cdot & \cdot \\
3 & 4 & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Figure 2: Two critical sets in L_6

if it is also minimal, in the sense that removing any single entry from P results in a partial square that is no longer uniquely completable. These are standard definitions for finite latin squares and it is natural to apply them without modification to the infinite case.

Let M_n be the addition table of the integers modulo n. For every n, there is a critical set of M_n with $\lfloor n^2/4 \rfloor$ entries. It is known that there are no critical sets with fewer entries than this when n is even or when $n < 10$, and it is conjectured that this is the case for all latin squares of order n. In [12] it is shown that for sufficiently large n a critical set for a latin square of order n must have at least $n^2/10^4$ entries.

At the other extreme, there is a critical set of M_n with $n(n-1)/2$ entries for every n; it is conjectured to be the one with the most filled cells. For squares of order n other than M_n, critical sets with more entries are known, but it is conjectured that there is no square of order n that has a critical set smaller than $\lfloor n^2/4 \rfloor$. For $n = 6$, critical sets with $\lfloor n^2/4 \rfloor = 9$ and $n(n-1)/2 = 15$ entries are given in Figure 2.

In [1] the question of whether it is possible to partition a latin square into critical sets is considered. They show that M_n can be partitioned into four critical sets for all $n > 1$ and give examples of other small latin squares that can be partitioned into two, three or four critical sets.

More details, history, context and references concerning critical subsets of M_n

and other finite latin squares may be found in [2, 14].

We consider infinite versions of some of these results and questions. In the next two sections we see that the quartered partial square ▚ is critical and define two further families of uniquely completable partial squares, the diagonal partial squares ◪$_{(a,b)}$ and the bowtie partial squares ⧓$_{(a,b)}$, that we use throughout the paper.

In Sections 4 and 5 we give examples of ways in which the infinite situation differs from the finite one. In Section 4 we build a chain of uniquely completable partial latin squares with each contained in the previous one such that intersection of the whole chain is empty and in Section 5 we show that a uniquely completable square may have no critical subsquares.

In Section 6 we introduce "density", a natural notion of how full a partial latin square is, in the sense of how much space it fills up in the ambient coordinate system. For finite squares, we can rephrase some of the work on the smallest and largest critical subsets of M_n as saying that all known critical subsets of M_n have density ρ in the range $1/4 \leq \rho < 1/2$ and that this is conjectured to be the range in which all critical sets of M_n lie. For $L_{\mathbb{Z}}$ we give critical subsets with densities 1/4, 3/8 and infinitely many values for ρ in the range $1/2 \leq \rho \leq 95/176$ (including the endpoints). In Section 7 we show that it is possible to decompose $L_{\mathbb{Z}}$ into three critical subsets, one of density 1/2 and two of density 1/4.

Finally, we collect various questions that arise from this work in Section 8.

2 Unique Completability

In this section we show that three classes of partial latin squares are uniquely completable to $L_{\mathbb{Z}}$: the quartered squares ▚$_{(a,b)}$, introduced in Example 1.1, the diagonal squares ◪$_{(a,b)}$, and the bowtie squares ⧓$_{(a,b)}$, defined below. It is useful to notate these partial squares with some indication as to the shape, since, in $L_{\mathbb{Z}}$, the unique completability (and indeed, the criticality) of the squares is translation invariant. That is, roughly speaking, the shape is what matters, not the exact location. We will first make this precise.

The *translation* of a (partial) latin square $L = \{(x, y, z) : x, y, z \in \mathbb{Z}\}$ by (a, b), where $(a, b) \in \mathbb{Z}^2$, is given by $L + (a, b) = \{(x + a, y + b, z + a + b) : (x, y, z) \in L\}$. Of course $L_{\mathbb{Z}}$, considered as a latin square, is invariant under translations (up to symbol names). This invariance is inherited by uniquely completable subsets of $L_{\mathbb{Z}}$.

Another "invariance" property of unique completability of latin subsquares of $L_{\mathbb{Z}}$ is transposition. The *transpose* of a (partial) latin square $L = \{(x, y, z) : x, y, z \in \mathbb{Z}\}$ is given by $L^{\mathrm{T}} = \{(y, x, z) : (x, y, z) \in L\}$.

Clearly both the translation and the transpose of an infinite latin square are still infinite latin squares.

Lemma 2.1. *Let P be a uniquely completable subset of $L_{\mathbb{Z}}$. Then both the transpose P^T and the translation $P + (a, b)$, for any $(a, b) \in \mathbb{Z}^2$, are also uniquely completable to $L_{\mathbb{Z}}$.*

$$
\begin{array}{cccccccccc}
 & & & & & & & & & \iddots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 7 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & \\
\cdot & \cdot & \cdot & \cdot & \mathbf{0} & 1 & 2 & 3 & 4 & \cdots \\
\cdot & \cdot & \cdot & -2 & -1 & 0 & 1 & 2 & 3 & \\
\cdot & \cdot & -4 & -3 & -2 & -1 & 0 & 1 & 2 & \\
\cdot & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & \\
-8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & \\
\iddots & & & & \vdots & & & & & \ddots
\end{array}
$$

Figure 3: The diagonal partial square ◪ centered at the origin, with the origin denoted by a bolded 0.

Proof. As P is uniquely completable to $L_{\mathbb{Z}}$ we have that P^T is uniquely completable to $L_{\mathbb{Z}}^T$, which is $L_{\mathbb{Z}}$.

If $P+(a,b) \subseteq L$ for some infinite Latin square L, then $P \subseteq L+(-a,-b)$. By the unique completability of P we have $L+(-a,-b) = L_{\mathbb{Z}}$ and then by the invariance of $L_{\mathbb{Z}}$ under translation we have $L = L_{\mathbb{Z}}$. $\square$

Next we define a class of partial latin subsquares which we refer to as *diagonal squares*. For $(a,b) \in \mathbb{Z}^2$ let

$$◪_{(a,b)} = \{(x,y,x+y) : y-b \leq x-a\}.$$

This gives a class of partial latin subsquares. Of particular interest is the one centered at the origin, namely:

$$◪_{(0,0)} = \{(x,y,x+y) : y \leq x\}.$$

Again we abbreviate the one centered at the origin as $◪ = ◪_{(0,0)}$. That is, ◪ contains all entries on and below the line $y = x$. The central part of ◪ is given in Figure 3.

For $(a,b) \in \mathbb{Z}^2$ let

$$⧓_{(a,b)} = \{(x,y,x+y) : y \geq b \text{ and } y < -x+a\} \cup \{(x,y,x+y) : y < b \text{ and } y \geq -x+a\}.$$

In particular, we are interested in $⧓ = ⧓_{(0,0)}$, which contains the negative entries in the second quadrant (including cells on the negative x-axis) and the non-negative entries in the fourth quadrant (excluding cells on the non-negative x-axis). We refer to this class of partial squares as *bowtie partial squares*. The central part of ⧓ is given in Figure 4.

Lemma 2.2. *The quartered, diagonal, and bowtie partial latin squares* ▚, ◪, *and* ⧓ *are uniquely completable to* $L_{\mathbb{Z}}$.

$$
\begin{array}{ccccccccccccc}
\ddots & & & & & & & & & & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -4 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdots & -5 & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \\
 & & & & & & & & & & & & \ddots
\end{array}
$$

Figure 4: The bowtie partial square ⧓, with the origin denoted by $\odot$.

Proof. First, note that each partial square is a subset of $L_{\mathbb{Z}}$, so if they are uniquely completable then they are uniquely completable to $L_{\mathbb{Z}}$.

Consider completing ▚. The only symbol we may place in position $(0,0)$ is 0, as row 0 contains integers less than 0 and column 0 contains integers greater than 0. Having added $(0,0,0)$, similar arguments successively force us to add $(1,0,1)$, $(2,0,2),\ldots$, completing row 0. We are now forced to add $(0,-1,-1)$ and then $(1,-1,0),(2,-1,1),\ldots$, completing row -1. Rows of index -2 and less follow in the same way, as do rows of index 1 and above, completing ▚ to $L_{\mathbb{Z}}$.

Next, consider completing ◪. Each cell with coordinates of the form $(x,x+1)$ is forced to contain $2x+1$ as column x contains all symbols less than $2x+1$ and row $x+1$ contains all symbols greater than $2x+1$. Once these entries are added, we have the translated partial square $◪_{(0,1)}$. The same argument then obtains $◪_{(0,2)}$, $◪_{(0,3)}$ and indeed $◪_{(0,n)}$ for all $n \in \mathbb{N}$, completing ◪ to $L_{\mathbb{Z}}$.

Finally, consider completing ⧓. We must put 0 in row 0 and the only available slot is in column 0, as the other cells are either filled or in a column that already contains 0. So, add $(0,0,0)$. Having added $(0,0,0)$, similar arguments successively force us to add $(1,0,1),(2,0,2)\ldots$, completing row 0. We are now forced to add $(-1,1,0)$ for similar reasons to those for $(0,0,0)$ and then $(0,1,1),(1,1,2),\ldots$, completing row 1. Rows of index 2 and above follow in the same way, as do rows -1 and less, completing ⧓ to $L_{\mathbb{Z}}$. $\square$

By Lemma 2.1, this means that for any coordinate (a,b) on the integer lattice, the partial latin squares $▚_{(a,b)}$, $⧓_{(a,b)}$, and $◪_{(a,b)}$ are also uniquely completable to $L_{\mathbb{Z}}$.

By Lemma 2.1, we also have that for any $(a,b) \in \mathbb{Z}^2$, the transpose of the bowtie partial square, namely $⧓^{\mathrm{T}}_{(a,b)} = ⧗_{(a,b)}$, is also uniquely completable to $L_{\mathbb{Z}}$.

In the next section we show that ▚, ⧓, and $⧗ = ⧗_{(0,0)}$ are critical.

3 Latin Trades and Criticality

First we note that criticality of partial subsquares of $L_{\mathbb{Z}}$ is translation and transpose invariant.

Lemma 3.1. *Let P be a critical subset of $L_{\mathbb{Z}}$. Then both the transpose P^T and the translation $P+(a,b)$, for any $(a,b) \in \mathbb{Z}^2$, are also critical.*

Proof. Straightforward and analogous to proof of Lemma 2.1 □

With finite latin squares, "latin trades" are the most powerful available tool for proving criticality of sets. We find that their infinite analogs are similarly powerful.

Let L and L' be distinct latin squares and let $T \subseteq L$ and $T' \subseteq L'$ with $T \cap T' = \emptyset$. If $L \setminus T = L' \setminus T'$ then T is a *latin trade* or *latin interchange* and T' is its *disjoint mate* (and vice versa).

There is a trivial method of finding trades that works in both the infinite and finite cases: take all instances of a pair of symbols. A disjoint mate is found by swapping them. The same can be done with either rows or columns in place of symbols. In $L_{\mathbb{Z}}$ these give trades two of whose symbols, row indices and column indices are all unbounded. In fact, in $L_{\mathbb{Z}}$ latin trades necessarily have this property. This follows from [8] where it is shown that a torsion-free abelian group does not have a finite trade (although torsion-free non-abelian ones may); Lemma 3.2 gives a direct proof.

Lemma 3.2. *Let T be a latin trade of $L_{\mathbb{Z}}$. Then at least two of the row indices, column indices and symbols of T are unbounded above and at least two of the row indices, column indices and symbols of T are unbounded below.*

Proof. As the (x,y)-entry of $L_{\mathbb{Z}}$ is given by $x+y$, if the symbols are unbounded above then at least one of the row or column indices must also be unbounded above. So, assume that the symbols are bounded above. Let c be the largest symbol used and take $(a,b,c) \in T$. Let T' be a disjoint mate of T.

As T is a trade, we must have $(a,b',c') \in T$, where $c' < c$ (hence $b' < b$) and $(a,b',c) \in T'$. As c may not appear twice in a row in $(L_{\mathbb{Z}} \setminus T) \cup T'$, we must have $(a',b',c) \in T$. As $b' < b$, we have $a' = c - b' > a$. Hence the row indices are unbounded above. A similar argument shows that the column indices are also unbounded above.

The analogous argument shows that two of the row indices, column indices and symbols of T are unbounded below. □

The following two lemmas demonstrate how trades are used with respect to unique completability and criticality. The arguments for these results in the finite case (see [7, 14]) apply equally well to the infinite case.

Lemma 3.3. *Let T be a latin trade of a latin square L and let $P \subseteq L$. If $T \cap P = \emptyset$ then P is not uniquely completable (and so not a critical set).*

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 0 & \mathit{1} & 2 & \mathit{3} & 4 & 5 & 6 & 7 & 8 & \\
 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\cdots & -2 & -1 & \mathbf{0} & \mathit{1} & 2 & \mathit{3} & 4 & 5 & 6 & \cdots \\
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \\
 & -4 & -3 & -2 & -1 & 0 & \mathit{1} & 2 & \mathit{3} & 4 & \\
 & & & & & \vdots & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 0 & \mathit{3} & 2 & \mathit{1} & 4 & 5 & 6 & 7 & 8 & \\
 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\cdots & -2 & -1 & \mathbf{0} & \mathit{3} & 2 & \mathit{1} & 4 & 5 & 6 & \cdots \\
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \\
 & 4 & -3 & -2 & -1 & 0 & \mathit{3} & 2 & \mathit{1} & 4 & \\
 & & & & & \vdots & & & & &
\end{array}
$$

Figure 5: The staircase trade $S_{1,0,2}$ in $L_{\mathbb{Z}}$ and the resulting square obtained by replacing it with $S'_{1,0,2}$. The trade entries are italicised and blue; the origin is bold.

Proof. P can be completed to both L and $(L \setminus T) \cup T'$, where T' is a disjoint mate of T. $\square$

Lemma 3.4. *Let P be a uniquely completable subset of L. If for each $e \in P$ there is a latin trade T with $T \cap P = \{e\}$ then P is a critical set.*

Proof. Let e be an arbitrary element of P. We have that $T \cap (P \setminus \{e\}) = \emptyset$ and so $P \setminus \{e\}$ is not uniquely completable by Lemma 3.3. Hence, P is critical. $\square$

In this section we introduce two trades in $L_{\mathbb{Z}}$ that have a regular structure and we use them to prove criticality of ▚ and ⧅.

Let h be a positive integer and for any pair of integers (a, b) let

$$S_{a,b,h} = \{(a - ht, b + ht, a + b) : t \in \mathbb{Z}\} \cup \{(a - ht, b + ht + h, a + b + h) : t \in \mathbb{Z}\} \subseteq L_{\mathbb{Z}}.$$

This is a trade; its disjoint mate is given by

$$S'_{a,b,h} = \{(a - ht, b + ht, a + b + h) : t \in \mathbb{Z}\} \cup \{(a - ht, b + ht + h, a + b) : t \in \mathbb{Z}\} .$$

Note that for any "height" h, the trade $S_{a,b,h}$ has two entries per row and per column and uses exactly two symbols, namely $a+b$ and $a+b+h$; whereas $S'_{a,b,h}$ is constructed by switching the two symbols in all of these positions. Call a trade with this structure a *staircase* trade. Figure 5 illustrates the staircase trade $S_{1,0,2} = S_{-1,2,2}$.

Here (a, b) designate the coordinates of a kind of "starting position" for the trade, and there are many ways to label one trade with this notation. It is helpful to label the trades by the position rather than the symbol. It is also worth mentioning that the "staircase" structure in these staircase trades can go only in one direction, which is downhill (going in the positive x-axis direction) or negatively sloped. This is because of the structure of $L_{\mathbb{Z}}$ and the fact that the staircase is essentially swapping some instances of pairs of symbols, and in the other direction, this would not be the case.

One use of staircase trades is to show that quartered squares are critical.

Theorem 3.5. *The quartered partial latin square ▚ is a critical set.*

$$\begin{array}{ccccccccccc}
\ddots & & & & & \vdots & & & & & \iddots \\
& \mathit{0} & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
& \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
& \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
& -\mathit{3} & \cdot & \cdot & \mathit{0} & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \\
& -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
& -6 & -5 & -4 & -\mathit{3} & \cdot & \cdot & \mathit{0} & \cdot & \cdot & \\
& -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & & \ddots
\end{array}$$

Figure 6: The staircase trade $S_{-1,-2,3}$ and the partial latin square ▚. The trade entries that do not intersect ▚ are italicised and blue; the one that does is italicised and purple.

Proof. We have that ▚ is uniquely completable by Lemma 2.2. To show that it is critical, we show that for each entry of ▚ there is a staircase trade that intersects ▚ only in that entry, using Lemma 3.4. Suppose $(a, b, a+b) \in$ ▚ and consider the staircase trade $S_{a,b,-(a+b)}$. The two symbols in $S_{a,b,-(a+b)}$ are $a+b$ and 0 and all entries lie in the second and fourth quadrants (possibly including the x-axis), except for $(a, b, a+b)$. Hence

$$S_{a,b,-(a+b)} \cap \text{▚} = \{(a, b, a+b)\}.$$

Figure 6 illustrates the trade when $(a, b, c) = (-1, -2, -3)$. □

Let h be a positive integer and for any pair of integers (a, b) let

$$H_{a,b,h} = \{(a+ht, b, a+b+ht) : t \in \mathbb{Z}\} \cup \{(a+ht, b+h, a+b+ht+h) : t \in \mathbb{Z}\}$$

and let

$$V_{a,b,h} = \{(a, b+ht, a+b+ht) : t \in \mathbb{Z}\} \cup \{(a+h, b+ht, a+b+ht+h) : t \in \mathbb{Z}\}.$$

Each of these is a trade. The disjoint mates are given by

$$H'_{a,b,h} = \{(a+ht, b, a+b+ht+h) : t \in \mathbb{Z}\} \cup \{(a+ht, b+h, a+b+ht) : t \in \mathbb{Z}\}$$

and

$$V'_{a,b,h} = \{(a, b+ht, a+b+ht+h) : t \in \mathbb{Z}\} \cup \{(a+h, b+ht, a+b+ht) : t \in \mathbb{Z}\}$$

respectively. To see this for $H_{a,b,h}$, note that the affected cells lie in two rows and for each row they use exactly the integers that are congruent to $a \pmod{h}$. The entries used in the rows are lined up so that each column has either zero or two affected

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & \mathit{-1} & 0 & \mathit{1} & 2 & \mathit{3} & 4 & \mathit{5} & 6 & \mathit{7} & \\
\cdots & -2 & -1 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
 & \mathit{-3} & -2 & \mathit{-1} & 0 & \mathit{1} & 2 & \mathit{3} & 4 & \mathit{5} & \\
 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \\
 & & & & & \vdots & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & \mathit{-3} & 0 & \mathit{-1} & 2 & \mathit{1} & 4 & \mathit{3} & 6 & \mathit{5} & \\
\cdots & -2 & -1 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
 & \mathit{-1} & -2 & \mathit{1} & 0 & \mathit{3} & 2 & \mathit{5} & 4 & \mathit{7} & \\
 & 4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \\
 & & & & & \vdots & & & & &
\end{array}
$$

Figure 7: The horizontal ladder trade $H_{1,-1,2}$ in $L_{\mathbb{Z}}$ and the resulting square obtained by replacing it with $H'_{1,-1,2}$. The trade entries are italicized and blue; the origin is bold.

entries; those columns with two entries have those entries exchanged. The argument for $V_{a,b,h}$ is similar, with the roles of rows and columns switched.

Call a trade of the form $H_{a,b,h}$ a *horizontal ladder trade* and one of the form $V_{a,b,h}$ a *vertical ladder trade*. Figure 7 illustrates the ladder trade $H_{1,-1,2}$.

Ladder trades are helpful in showing that bowtie squares are critical.

Theorem 3.6. *The bowtie partial latin square* $\boxbslash$ *is a critical set.*

Proof. We show that for each entry of $\boxbslash$ there is a vertical ladder trade that intersects $\boxbslash$ only in that entry.

Let $(a, b, a + b) \in \boxbslash$ with $a > 0$. Consider the vertical ladder trade $V_{0,b,a}$. The two columns of the trade are 0 and a. There are no entries of $\boxbslash$ in column 0. The entries of the trade in column a are congruent to $a + b \pmod{a}$. As the entries of $\boxbslash$ in column a are $0, 1, \dots a - 1$ there is exactly one entry included in the ladder trade: $(a, b, a + b)$. A similar argument works for $a < 0$, using the trade $V_{0,b,|a|}$.

Figure 8 illustrates the trade when $(a, b, a + b) = (3, -1, 2)$. □

$$
\begin{array}{cccccccccccc}
\ddots & & & & & \vdots & & & \vdots & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \mathit{2} & \cdot & \cdot & \mathit{5} & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdots & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \mathit{-1} & 0 & 1 & \mathit{2} & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \mathit{-4} & \cdot & \cdot & \mathit{-1} & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \\
 & & & & & \vdots & & & \vdots & & & \ddots
\end{array}
$$

Figure 8: The ladder trade $V_{0,-1,3}$ and the partial latin square $\boxbslash$. The trade entries that do not intersect $\boxbslash$ are italicized and blue; the one that does is italicized and purple.

Corollary 3.7. *For any pair of integers (a, b), the partial subsquares* ⧓$_{(a,b)}$*,* ⧒$_{(a,b)}$*, and* ▚$_{(a,b)}$ *of $L_{\mathbb{Z}}$ are critical sets.*

Proof. This follows immediately from Lemma 3.1 and Theorems 3.5 and 3.6. $\square$

4 Infinite chains of uniquely completable subsets

In the finite case there is a conceptually straightforward (if not usually easy to implement) method to find a critical subset of any uniquely completable set. If the set itself is not critical then there is an element that may be removed and the resulting set is also uniquely completable. Remove it. Continuing the procedure we must find a critical set at some point as the empty set is not uniquely completable and is only finitely many steps away.

This argument does not hold for the infinite case. Is it possible that we can keep removing elements indefinitely? Yes. For example, in $L_{\mathbb{Z}}$ consider removing the elements

$$(0, 0, 0), (0, 1, 1), (0, 2, 2), \ldots$$

one at a time. At each stage the partial square is clearly still uniquely completable. In this section we show the perhaps more surprising result that there can be an infinite chain of uniquely completable sets with each contained in the previous one and with empty intersection.

Another natural question is whether it is possible for a uniquely completable set to have no critical subset. In the next section, we show that the answer to the second question is also yes.

Theorem 4.1. *There is a sequence $Q_0, Q_1, \ldots$ of uniquely completable sets of $L_{\mathbb{Z}}$ with the properties that $Q_{i+1} \subseteq Q_i$ for each $i \geq 0$ and $\bigcap_{i=0}^{\infty} Q_i = \emptyset$.*

Proof. Consider $Q_i =$ ◪$_{(i,0)}$ for each $i \geq 0$. As ◪ is uniquely completable, we have that Q_i is uniquely completable for all i by Lemma 2.1. The filled cells of Q_i are exactly those that lie on or below the line $y = x - i$, so $Q_{i+1} \subseteq Q_i$ and $\bigcap_{i=0}^{\infty} Q_i = \emptyset$. $\square$

5 A uniquely completable set with no critical subset

In this section we study a family of variations of the bowtie critical set ⧓ in order to show that a uniquely completable set might not have a critical subset in the infinite case.

Let T be a subset of $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let Q_T be the subsquare of $L_{\mathbb{Z}}$ given by the union of the sets $\{(x, y, x + y) : x < 0 \text{ and } 0 \leq y < -x\}$, $\{(1, t - 1, t) : t \in T\}$ and $\{(x, y, x + y) : x > 1 \text{ and } -x \leq y < 0\}$. Equivalently,

$$Q_T = (\text{⧓} \setminus \{(1, -1, 0)\}) \cup \{(1, t - 1, t) : t \in T\}.$$

$$
\begin{array}{ccccccccccccc}
\ddots & & & & & & & & & & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \\
 & -4 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdots & -5 & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \\
 & & & & & & & & & & & & \ddots
\end{array}
$$

Figure 9: The partial square $Q_{\{3,4\}}$ with the origin denoted by $\odot$.

That is, we obtain Q_T from $\boxbslash$ by removing the symbol 0 in column 1 and adding each $t \in T$ to column 1 in the appropriate position. For example, Figure 9 shows $Q_{\{3,4\}}$.

Whether Q_T is uniquely completable depends on T. For example, when the complement of T in $\mathbb{N}$ is finite we have:

Lemma 5.1. *If $\mathbb{N} \setminus T$ is finite then Q_T is uniquely completable.*

Proof. Suppose that the complement of T in $\mathbb{N}$ is finite. Any symbol $s \notin T$ with $s > 0$ must appear in a non-negative row of column 1, as s already appears in every negative row. The number of such s is equal to the number of available gaps in column 1. Now consider where 0 can go in column 1. It cannot be in a non-negative row as there is no space; it cannot be in a row less than -1 as 0 already appears in those rows. Therefore 0 is in row -1. This means that we have completed all of the entries of $\boxbslash$ and hence the set is uniquely completable to $L_{\mathbb{Z}}$ by Lemma 2.2. $\square$

When the complement of T in $\mathbb{N}$ is infinite the situation is trickier. Say that T is *sequential* if for every $k \in \mathbb{N}$ there is a sequence of k consective integers in T; that is, T is sequential if it contains runs of arbitrarily long length. If $\mathbb{N} \setminus T$ is finite then T is necessarily sequential.

Our goal is to show that sequentiality of T characterizes the unique completability of Q_T. We start with an example that illustrates the main points of the argument.

Example 5.2. Suppose $T = \{1, 2, 4, 7, 8, \ldots\}$ and $\mathbb{N} \setminus T = \{c_1, c_2, \ldots\}$ is infinite with $c_1 < c_2 \cdots$. We attempt to construct a square that is not $L_{\mathbb{Z}}$ that contains Q_T.

In such an alternative completion, the symbol 0 must appear in row -1 in a column other than 1, otherwise we have all of $\boxbslash$ which we know to be uniquely completable to $L_{\mathbb{Z}}$. For this example, we consider the consequences of putting it in column -2.

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & 3 & 4 & \cdot & \cdot & \cdot & \mathit{\color{blue}8} & 9 & \\
 & 2 & 3 & \cdot & \cdot & \cdot & \mathit{\color{blue}7} & 8 & \\
 & 1 & 2 & \cdot & \cdot & \cdot & \mathbf{\color{red}5} & 7 & \\
 & 0 & 1 & \cdot & \cdot & \cdot & \mathbf{\color{red}3} & 6 & \\
 & -1 & 0 & \cdot & \cdot & \cdot & \mathit{\color{blue}4} & 5 & \\
\cdots & -2 & -1 & \mathbf{\color{red}3} & 1 & 2 & \mathbf{\color{red}0} & 4 & \cdots \\
 & -3 & -2 & -1 & 0 & 1 & \mathit{\color{blue}2} & 3 & \\
 & -4 & -3 & -2 & -1 & \mathbf{0} & \mathit{\color{blue}1} & 2 & \\
 & -5 & -4 & \mathbf{\color{red}0} & -2 & -1 & \mathbf{\color{red}-3} & 1 & \\
 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & \\
 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & \\
 & -8 & -7 & \mathbf{\color{red}-3} & -5 & -4 & \mathbf{\color{red}-6} & -2 & \\
 & & & & \vdots & & & &
\end{array}
$$

Figure 10: A partially filled square from Example 5.2 where the elements of T in column 1 are italicised and in blue and all elements that differ from $L_{\mathbb{Z}}$ are bold and red. The origin is also bold.

$$
\begin{array}{cccccc}
5 & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{6} & 10 \\
4 & \mathit{6} & \mathit{5} & \mathit{7} & 8 & 9 \\
3 & \mathit{5} & \mathit{6} & \mathit{4} & 7 & 8 \\
2 & \mathit{4} & \mathit{3} & \mathit{6} & 5 & 7 \\
1 & \mathit{2} & \mathit{4} & \mathit{5} & 3 & 6
\end{array}
\qquad\qquad
\begin{array}{cccccc}
5 & * & * & * & \mathit{9} & 10 \\
4 & \mathit{6} & \mathit{5} & \mathit{7} & 8 & 9 \\
3 & \mathit{5} & \mathit{6} & \mathit{4} & 7 & 8 \\
2 & \mathit{4} & \mathit{3} & \mathit{6} & 5 & 7 \\
1 & \mathit{2} & \mathit{4} & \mathit{5} & 3 & 6
\end{array}
$$

Figure 11: The region $\{(x, y) : -3 \leq x \leq 2,\ 4 \leq y \leq 8\}$ for the two scenarios of Example 5.2. On the left, $9 \notin T$ and everything works smoothly. On the right, $9 \in T$ and one of the cells marked with an asterisk must contain a 6. Entries in columns -2 through 1 that are new since Figure 10 are italicised and blue.

Complete columns with index less than -2 and greater than 1 exactly as in $L_{\mathbb{Z}}$. Fill the grid from row 2 downwards making only the switches from $L_{\mathbb{Z}}$ implied by exchanging the entries of the vertical ladder trade $V_{-2,2,3}$ with its disjoint mate. Next, fill the gaps in column 1 by putting c_i in the row $c_{i+1}-1$ (that is, the space where the symbol c_{i+1} is in $L_{\mathbb{Z}}$). At this point we have the partially completed square shown in Figure 10.

We fill the gaps in the remaining rows in increasing order of index. This is successful for rows up to index 7. Whether we are able to extend to row 8 depends on whether $9 \in T$. If $9 \notin T$, we may place 6 in cell $(1, 8)$ and proceed successfully. However, if $9 \in T$ we must place 6 in one of cells $(-2, 8)$, $(-1, 8)$ or $(0, 8)$, which is forbidden as we have used 6 in each of these columns already. Figure 11 illustrates the two situations.

The next lemma shows that when T is not sequential then we may complete Q_T

using the approach of the scenario in Example 5.2 with $9 \notin T$. The subsequent lemma shows that if T is sequential then we necessarily hit an issue of the type encountered with $9 \in T$ in Example 5.2.

Lemma 5.3. *If T is not sequential then Q_T is not uniquely completable.*

Proof. Let $\mathbb{N} \setminus T = \{c_1, c_2, \ldots\}$, where $c_1 < c_2 \cdots$ and $c_{i+1} - c_i \leq M$ for all i. Choose $c \in \mathbb{N} \setminus T$ with $c \geq M$. We complete Q_T to a square that is not $L_{\mathbb{Z}}$ as follows.

1. Fill all columns with index less than $-c+1$ or greater than 1 with their usual $L_{\mathbb{Z}}$ entries.
2. For each c_i with $c_i < c$, put c_i in row $c_i - 1$ of column 1 (that is, in its usual $L_{\mathbb{Z}}$ position).
3. In row $c - 1$, put c in column $-c + 1$ and 0 in column 1. Fill the remainder of this row and all rows with lower index using the ladder trade of width c induced by these entries.
4. For each c_i with $c_i > c$, put c_i in row $c_{i+1} - 1$ of column 1 (that is, everything not in T that has not already been assigned a slot in column 1 gets shifted up to the next gap).
5. Fill the remaining rows one at a time in increasing order of index. There are two situations for row b:

   $b + 1 \in T$: We have the symbols $\{b - c + 1, b - c + 2, \ldots, b\}$ to place. This is possible as there are at least q columns where $b - c + q$ has not yet appeared for each q.

   $b + 1 \notin T$: We have the symbols $\{b - c + 1, b - c + 2, \ldots, b + 1\} \setminus \{z\}$ to place, where $z \in \mathbb{N} \setminus T$ (and we know that $b - c + 1 \leq z < b + 1$ because $\mathbb{N} \setminus T$ has bounded gaps, bounded by $c \geq M$). This is possible as there are again at least q columns where a required $b - c + q$ has not yet appeared for each $q \leq c$, and c columns where $b + 1$ has not yet appeared.

This gives the required latin square. $\square$

Lemma 5.4. *If T is sequential then Q_T is uniquely completable to $L_{\mathbb{Z}}$.*

Proof. We attempt alternative completions of Q_T and see that we run into a contradiction.

First, consider the placement of 0 in row -1. It cannot be in column 1 as this would give all of the entries of $\boxbslash$, which uniquely completes to $L_{\mathbb{Z}}$. So, it must be placed in a column with index less than 1. Choose an arbitrary such column index a. The remainder of the argument considers rows with non-negative index. We work through them in increasing order.

In row 0, the two cells in columns 0 and 1 must contain symbols $\{0, 1\}$. If $1 \in T$ then we are forced to put 0 in column 0; in this case, if $a = 0$ then we immediately reach a contradiction. The remainder of the row—that is, columns with index at least 2—is now forced, and these cells contain the same symbols as they do in $L_{\mathbb{Z}}$.

Continue row-by-row while the row index is less than $|a|$. In row y we must fill the $y + 2$ columns $-y$ to 1 with symbols $\{0, 1, \dots y + 1\}$. If this is ever impossible then we are done, so assume it is possible. Again, the columns with index at least 2 are forced to have the same entries as they do in $L_{\mathbb{Z}}$.

In row $|a|$ we must likewise fill the $|a|+2$ columns a to 1 with symbols $\{0, 1, \dots |a|+1\}$. Here there is the additional constraint that 0 cannot be placed in column a. The remainder of the row is forced to match $L_{\mathbb{Z}}$. Again assuming this is possible, we have now used $|a| + 2$ copies of the symbol 0 in the $|a| + 2$ rows with indices -1 to $|a|$.

In row $|a| + 1$, we are forced to place 0 in column $a - 1$. We must then place the symbols $\{1, 2, \dots |a| + 2\}$ in columns a to 1, and then the remainder of the row is forced. This continues: in row $|a|+y$, the entries in columns with index less than a or greater than 1 are forced to match $L_{\mathbb{Z}}$ and we must place symbols $\{y, y+1, \dots, |a|+y + 1\}$ in columns a to 1.

As T has unbounded gaps, there must be arbitrarily larges values s such that $s \notin T$ and $\{s + 1, s + 2. \dots, s + |a| + 1\} \subseteq T$. Choose one with $s > |a|$. Assume the above process reaches row $s - 1$ (which has a gap where s would go in column 1) without a contradiction. We know that 0 has appeared in column 1 with row index less than $s - 1$. Therefore, when placing the symbols $\{s - |a| - 1, s - |a|, \dots, s\}$ in row $s - 1$, we know that at least one of the values $\{s - |a| - 1, s - |a|, \dots, s - 1\}$ has not yet appeared in column 1 and therefore must do so now: there is no opportuntity to do so for the next $|a| + 1$ rows, at which point these symbols will have no further opportunity to appear in columns a to 1.

Now consider the $|a|+2$ rows indexed $s-1$ to $s+|a|$. In each of them the symbol s must appear somewhere in the $|a| + 1$ columns indexed a to 0. But a symbol cannot appear $|a|+2$ times in $|a|+1$ columns, so we are guaranteed to reach a contradiction.

Hence Q_T is uniquely completable to $L_{\mathbb{Z}}$ □

Combining Lemmas 5.3 and 5.4 we characterize exactly when Q_T is uniquely completable and, as a consequence, deliver the promise of the section heading.

Theorem 5.5. *The partial square Q_T is uniquely completable if and only if T is sequential.*

Corollary 5.6. *The uniquely completable partial latin square $Q_{\mathbb{N}}$ has no critical subset.*

Proof. As $\mathbb{N}$ is sequential, $Q_{\mathbb{N}}$ is uniquely completable by Lemma 5.4. We can not remove an entry with column index other than 1 while maintaining unique completability by the argument of the proof of Theorem 3.6. Hence any critical subset of $Q_{\mathbb{N}}$ must be of the form Q_T for some $T \subseteq \mathbb{N}$.

As a critical set is uniquely completable, by Theorem 5.5 we know that T must be sequential. However, for any $t \in T$, the set $T \setminus \{t\}$ is also sequential. Hence $Q_{T\setminus\{t\}}$ is also uniquely completable and Q_T is not critical. $\square$

6 Density, and More Critical Sets

In the finite case, asking how many entries a uniquely completable or critical set may have for a given n has been a fruitful question in the study of partial latin squares. In this section we consider the analagous questions in the infinite setting.

It is possible that a partial latin square of order n with n entries cannot be completed: consider the example

$$\{(x, 1, x) : 1 \leq x < n\} \cup \{(2, n, n)\}.$$

A similar construction gives an infinite partial latin square that cannot be completed:

$$\{(x, 0, x) : x \neq 0\} \cup \{(0, 1, 0)\}.$$

Contrarily, any partial latin square of order n with at most $n - 1$ entries can be completed to at least one latin square of order n. The analogous infinite result in this case is that every infinite partial latin square with finitely many entries can be completed to at least one infinite latin square. One way to see this is to complete the given symbols to a finite latin square of some order n and then, assuming without loss of generality that the symbols used are $\{0, 1, \ldots, n - 1\}$, construct an infinite square using this square and its translates.

Given a partial latin subsquare P of $L_{\mathbb{Z}}$, define $\sigma_n(P)$ to be the number of entries $(x, y, x+y)$ that have $-n \leq x, y \leq n$, and define $\rho_n(P)$ to be $\sigma_n(P)/(2n+1)^2$. So ρ_n is the fraction of the square box of side length $2n + 1$ centered at the origin that is filled. Let $\rho^+(P) = \limsup_{n\to\infty} \rho_n(P)$ be called the *upper density* of P and $\rho^-(P) = \liminf_{n\to\infty} \rho_n(P)$ be called the *lower density* of P. Let $\rho(P) = \lim_{n\to\infty} \rho_n(P)$, if it exists; call this the *density* of P.

One can imagine successively larger and larger square boxes, centered at the origin in the integer lattice. To find density, find the proportion of filled-in entries over all possible entries from $L_{\mathbb{Z}}$ in these larger and larger boxes, and take the limit. First note that choice of "origin" is insignificant for subsquares of $L_{\mathbb{Z}}$.

Theorem 6.1. *Density is independent of the choice of origin for infinite partial subsquares.*

Proof. Given a Latin subsquare P, fix one of the origins without loss of generality to be $(0, 0, 0)$. Let the other arbitrary origin be $(a, b, a + b)$.

For large n two square boxes of side length $s = 2n+1$, one centered at each origin, overlap in a rectangular region of $(s-|a|)(s-|b|)$ cells. There is an "L-shape" region contained in the first box but not the second, and an identical L-shape (rotated 180°)

contained in the second but not the first. Each of these regions has $s|a| + s|b| - |a||b|$ cells.

Therefore the difference in the number of entries in the two boxes is at most $s|a| + s|b| - |a||b|$, which occurs when one L-shape's cells are all filled and the others are all empty. Hence the difference in densities from the two calculations is at most

$$\lim_{s\to\infty} \frac{s|a| + s|b| - |a||b|}{s^2} = 0.$$

$\square$

Moreover, density is translation and transpose invariant for partial subsquares of $L_{\mathbb{Z}}$.

Lemma 6.2. *Let P be a partial subsquare of $L_{\mathbb{Z}}$. Then $\rho(P) = \rho(P^{\mathrm{T}})$ and for any $(a,b) \in \mathbb{Z}^2$, we have that $\rho(P) = \rho(P + (a,b))$.*

We can compute densities of the partial subsquares considered thus far.

Lemma 6.3. *For any $(a,b) \in \mathbb{Z}^2$, the density of the quartered partial latin square ▚$_{(a,b)}$ is $1/2$.*

Proof. Consider the quartered partial square ▚. The numbers of filled entries in the square box of side length $2n+1$ centered at the origin is $2n^2 + 2n$. Therefore, $\rho($▚$) = \lim_{n\to\infty} \frac{2n^2+2n}{(2n+1)^2} = 1/2$. By Lemma 6.2 we obtain the same value for ▚$_{(a,b)}$. $\square$

Lemma 6.4. *For any $(a,b) \in \mathbb{Z}^2$, the density of the bowtie square ◩$_{(a,b)}$ is $1/4$, as is the density of the transposed bowtie square ◪$_{(a,b)}$.*

Proof. Consider the bowtie partial square ◩. The numbers of filled entries in the square box of side length $2n+1$ centered at the origin is $n^2 + n$. Therefore, $\rho($◩$) = \lim_{a\to\infty} \frac{n^2+n}{(2n+1)^2} = 1/4$. By Lemma 6.2 we obtain the same value for ◩$_{(a,b)}$ and ◪$_{(a,b)}$. $\square$

Clearly 0 is the minimum density of a partial subsquare of $L_{\mathbb{Z}}$, and 1 is the greatest. Can there be critical subsets of any density in between? In Section 6.1 we give critical subsquares with other densities. Then in Section 6.2 we give an example of a uniquely completable partial subsquare of $L_{\mathbb{Z}}$ in which the upper and lower densities are not equal.

6.1 A family of critical sets with myriad densities

We define a family of partial subsquares of $L_{\mathbb{Z}}$ with a range of densities close to $1/2$. These squares consist of two wings, R_m^+ and R_m^-, which are mirror images of each other in the line $y = -x$. For an integer $m > 2$ define $M = m^2 - m - 1$. The edges of the wing R_m^+ are given by lines through the origin with slopes $-1/m$ and M and those for R_m^- are found by taking the reflection:

$$R_m^+ = \left\{(x, y, x+y) : x > 0 \text{ and } -\tfrac{1}{m}x < y \le Mx\right\} \subseteq L_{\mathbb{Z}},$$

$$\begin{array}{cccccccccccc}
 & & & & & & & & & & & \iddots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 10 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & 9 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 7 & 8 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \odot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & 2 & 3 & 4 & \\
\cdot & \cdot & \cdot & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & 3 & \\
\cdot & \cdot & -6 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \\
\cdot & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \\
-10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & & & \ddots
\end{array}$$

Figure 12: The partial subsquare R_2, with the origin denoted by $\odot$.

$$R_m^- = \left\{(x, y, x+y) : y < 0 \text{ and } My \leq x < -\tfrac{1}{m}y\right\} \subseteq L_{\mathbb{Z}}.$$

Set $R_m = R_m^+ \cup R_m^-$. See Figures 12 and 13 for for visual depictions of R_2 and R_3 as partial latin subsquares of $L_{\mathbb{Z}}$.

Theorem 6.5. *For $m \geq 2$, the partial square R_m is a critical subsquare of $L_{\mathbb{Z}}$.*

Proof. We show the unique completability of R_m first, and then the minimality (using Lemma 3.4).

Unique Completeness: The first entry that is always immediately determined is $(0, 0, 0)$, since all of the R_m squares include the negative y-axis and the positive x-axis. Next we show that there is only one way to fill in each entry in quadrant IV. This is enough, as one would now have a superset of ◪, which by Lemma 2.2 is uniquely completable.

Assume you have been able to uniquely complete the unfilled entries of the gap in quadrant IV, row by row, and are now attempting to fill the entry in coordinate $(x, y) \in \mathbb{Z}^2$. Note that it must be the case that $1 \leq \lceil -y/m \rceil \leq x \leq -my$ and $-1 \geq \lfloor -x/m \rfloor \geq y \geq -mx$ since the unfilled entries are in this gap.

Now we take inventory of what symbols in $\mathbb{Z}$ are available to us for this entry. We know that our choice cannot include anything in the closed interval $[My + y, x + y - 1]$, based off what is already in our current row to the left. Moreover, to the right, we know we any entries in the interval $[-my + y + 1, \infty)$ are taken, and based off what is below, we cannot have anything in $(-\infty, -mx + x - 1]$. Finally, from the entries above, anything in the closed interval $[x + y + 1, Mx + x]$ is excluded.

$$
\begin{array}{cccccccccccccc}
 & & & & & & & & & & & & & \iddots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & 9 & 10 & 11 & 12 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 7 & 8 & 9 & 10 & 11 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 6 & 7 & 8 & 9 & 10 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & 9 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & 8 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & 7 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \odot & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\cdot & -6 & -5 & -4 & -3 & -2 & -1 & \cdot & \cdot & \cdot & 3 & 4 & 5 & \\
-8 & -7 & -6 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
-9 & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
-10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
-11 & -10 & -9 & -8 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & & \vdots & & & & & & & \ddots &
\end{array}
$$

Figure 13: The partial subsquare R_3, with the origin denoted by $\odot$.

Letting s be the symbol we are looking to add, we have:

$$s \notin (-\infty, -mx+x-1] \cup [My+y, x+y-1] \cup [x+y+1, Mx+x] \cup [-my+y+1, \infty).$$

However, recall that $M = m^2 - m - 1$. This means that

$$s \notin (-\infty, -(m-1)x-1] \cup [m(m-1)y, x+y-1] \cup [x+y+1, m(m-1)x] \cup [-(m-1)y+1, \infty).$$

Moreover notice that since $1 \leq \lceil -y/m \rceil \leq x$ we have that $-y \leq mx$, and multiplying by $m-1 > 0$ gives

$$-(m-1)y \leq (m-1)mx.$$

Similarly, using that $-1 \geq \lfloor -x/m \rfloor \geq y$ we obtain $-x \geq my$ and multiplying by $m-1$ here gives

$$-(m-1)x \geq (m-1)my.$$

This means that $s \notin (-\infty, x+y-1] \cup [x+y+1, \infty)$, so the missing entry must be $x+y$ as desired. $\square$

Minimality: To show that R_m is critical, we show that for an arbitrary point $(a, b, a+b) \in R_m$, there is a latin trade intersecting R_m precisely at that point. In fact we may assume without loss of generality that $(a, b, a+b) \in R_m^-$, since any trade we have for R_m^- for a point $(a, b, a+b)$ can be reflected to the point $(-b, -a, -b-a) \in R_m^+$.

In particular we find staircase trades. See Figure 18 for the trade we find for $m = 2$ and the point $(0, -2, -2)$ and Figure 19 for the trade found for $m = 3$ with the point $(4, -1, 3)$.

To find the appropriate staircase trade for $(a, b, a+b) \in R_m^-$ proceed as follows. View R_m in the xy-plane restricted to the integers, and the boundaries of R_m are essentially defined by four lines. Lines of fixed symbol have slope -1. Staircase trades consist of points on two such (parallel) lines with slope -1.

See where the line $y = -x + a + b$ intersects with the "boundary" lines defining the left wing: $y = \frac{1}{M}x$ and $y = -mx$ (the left wing is actually bounded by the equations $y = \lceil \frac{1}{M}x \rceil - 1$ and $x = \lfloor -\frac{1}{m}y \rfloor$ in $\mathbb{R}^2$ so these lines are estimates). Let $(x_1, y_1, a+b) \notin R_m$ be the first point in $L_{\mathbb{Z}}$ on the line $y = -x + a + b$ not in R_m to the right of/below R_m^-. Similarly let $(x_2, y_2, a+b) \notin R_m$ be the first point in $L_{\mathbb{Z}}$ on the line $y = -x + a + b$ to the left of/above R_m^- with $x_2 < x_1$ and $y_2 > y_1$. Let $h_1 = x_1 - a = b - y_1$, and let $h_2 = a - x_2 = y_2 - b$.

We may approximate the values of the distances h_1 and h_2 by intersecting the boundary lines of the left wing with the line $y = -x + a + b$. Indeed it follows that:

$$h_1 = \left\lceil \frac{b + ma}{1 - m} \right\rceil \tag{1}$$

and

$$\frac{a - Mb}{M+1} < h_2 \leq \frac{a - Mb}{M+1} + 1. \tag{2}$$

From here we break up into cases based first on how h_1 and h_2 compare, and then on whether $a > 0$ or not. In all cases, we use staircase trades of the form $S_{a,b,d}$, where $d \geq 0$, is chosen so that $d \geq h_1, h_2$. The reason for this is because of the following claim.

Claim 6.5.1. *Let $d \geq h_1, h_2$. Then the staircase trade $S_{a,b,d}$ intersects R_m only in $(a, b, a+b)$ if and only if the two distinct points $(a, b+d, a+b+d)$ and $(a+d, b, a+b+d)$ lie outside of R_m.*

Proof (of Claim). In order to ensure that a staircase trade intersects R_m exactly at one point, it is enough to focus on making sure one of the lines in the trade intersects R_m once, and the other one never does. Beyond these intersection points, we can then guarantee we never intersect R_m again. (Indeed, lines make up the trade and the boundaries of R_m and can only intersect once.) By making $d \geq h_1, h_2$, we have already guaranteed that the line with constant symbol $a+b$ intersects R_m only at the point $(a, b, a+b)$, so all that is left is to show the other line defining the trade never intersects R_m. □

The best case scenario would be if we could use the larger of h_1, h_2 as the height of our staircase trade. However, some peculiarities come up and we have to find a potentially larger height in some cases. We give some figures illustrating the situation for each of the cases – we may use h_2 as the height in the first part of Case 1, and we may use h_1 in Case 2(b).

Recall that $(a, b, a+b)$ is assumed to be in R_m^-, so by definition, $b < 0$ and $Mb \leq a < -\frac{1}{m}b$.

Case 1: $h_2 \geq h_1$**.** Consider whether or not $(a, y_2, a+y_2) = (a, b+h_2, a+b+h_2) \in R_m$.

If not, use the staircase trade S_{a,b,h_2}. This works by Claim 6.5.1 because it implies that $(a+h_2, b, a+b+h_2) \notin R_m$. To see this, we need to verify, by the definition of R_m and the positioning of this point, that $a + h_2 \geq -\frac{1}{m}b$ (true since it is for $a+h_1$ and $h_2 \geq h_1$) and $-\frac{1}{m}(a+h_2) \geq b$. The last inequality is equivalent to saying $h_2 \leq -a-mb$. But using inequality (2) this follows from the fact that $\frac{a-Mb}{M+1} + 1 \leq -a - mb$, which is true because

$$b \leq -\frac{m^2-m+1}{m^3-2m^2+m+1}a - \frac{m^2-m}{m^3-2m^2+m+1}.$$

This follows from the fact that $(a, b, a+b)$ is in the left wing, which by definition means $b < -ma$. Indeed the slope $-\frac{m^2-m+1}{m^3-2m^2+m+1} > -m$ and the y-intercept $-\frac{m^2-m}{m^3-2m^2+m+1} > -1$ for $m \geq 2$.

Otherwise, we need to find $(x_0, y_0, a+b)$ where $y_0 > y_2$ and $x_0 < x_2$ such that $(a, y_0, a+y_0) \notin R_m$. Then use the staircase trade S_{a,b,y_0-b}.
Using Claim 6.5.1, we are done if we can establish $(a+y_0-b, b, a+y_0) \notin R_m^+$ (it cannot be in R_m^- since $h_1 \leq h_2 = y_2 - b < y_0 - b$).

Case 1(a): $a > 0$**.** Notice that in this case, by the definition of R_m^- (in particular $b < 0$ and $a > 0$ are integers), $b = -ma - n$ for some $n \in \mathbb{N}$. Let $y_0 = Ma+1$. See Figure 14 for the situation. We have that $-b \geq ma + \frac{1}{m-1}$ since $-b > ma$ and $a, b \in \mathbb{Z}$. From this it follows from our value of b above (and since $M = m^2 - m - 1$) that $b \leq -\frac{1}{m}(a+y_0-b)$, which means that $(a+y_0-b, b, a+y_0) \notin R_m^+$, as desired.

Case 1(b): $a \leq 0$**.** See Figure 15. Let $y_0 = \lfloor \frac{a-1}{M} \rfloor$. In this case it is easy to see that S_{a,b,y_0-b} avoids intersecting with R_m^+ since entries there are always positive, whereas both entries $a+b$ and $a+y_0$ in the trade are less than or equal to 0.

Case 2: $h_1 > h_2$**.** Again we split into cases depending on a.

Case 2(a): $a \leq 0$**.** Here we may use the staircase trade $S_{a,b,h}$ where h is a positive integer and

$$-mb - a \geq h \geq \max\left(\frac{b+ma}{1-m}, \frac{a-Mb}{M} + 1\right).$$

Note that it will always be true that $-mb - a \geq \frac{b+ma}{1-m}$ since $Mb \leq a$. So such an h exists since

$$(-mb-a) - \left(\frac{a-Mb}{M}\right) > 1.$$

This is because as $a < 0$, we have that $Mb(1-m) > M + a(M+1)$, since the left side is positive and the right side is negative. From this our desired inequality follows.

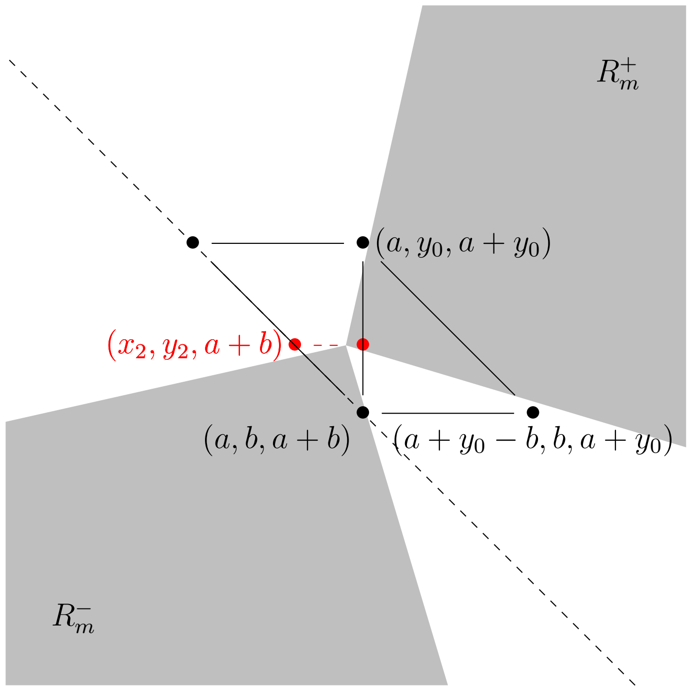

Figure 14: Illustration of Case 1(a) of criticality proof.

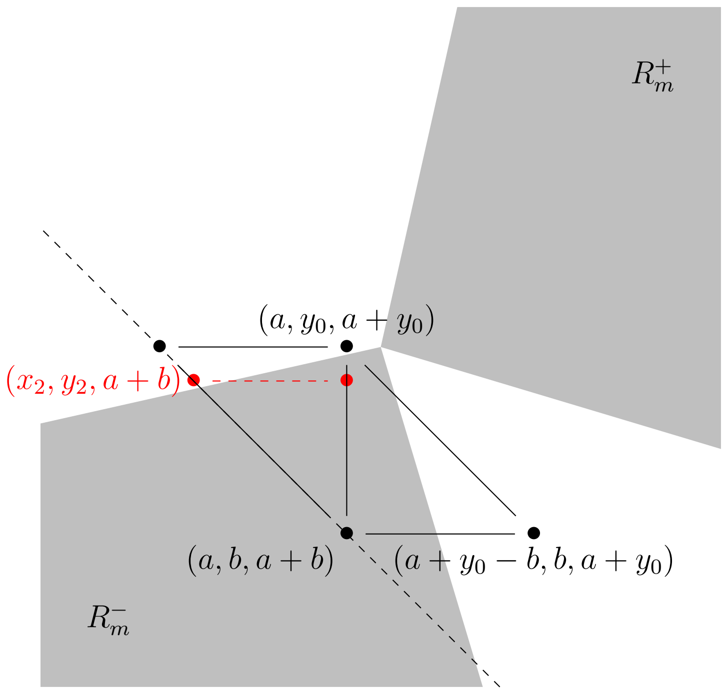

Figure 15: Illustration of Case 1(b) of criticality proof.

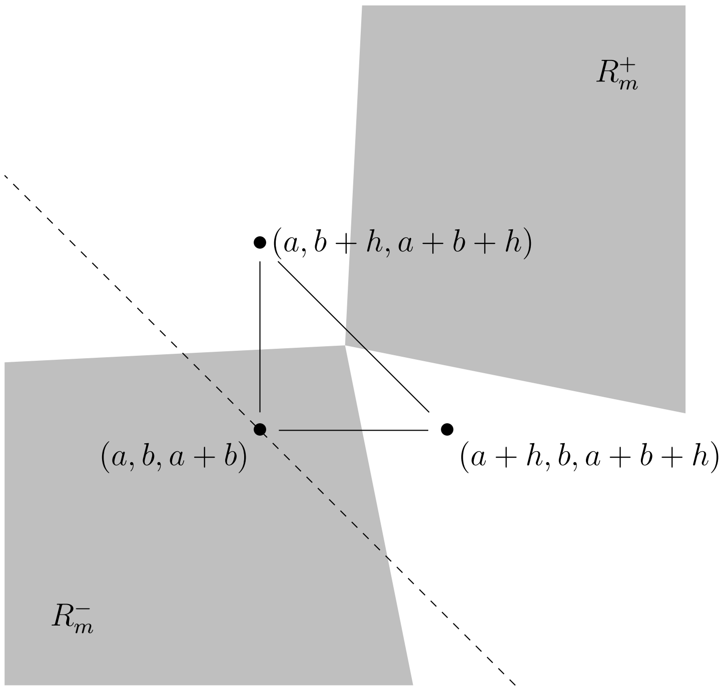

Figure 16: Illustration of Case 2(a) of criticality proof.

We then have ensured that $(a+h, b, a+b+h)$ and $(a, b+h, a+b+h)$ are not in R_m using this value of h. First of all it has to be the case that $h \geq h_1 > h_2$ based on our definition h and the case we are in. This tells us $(a+h, b, a+b+h) \notin R_m^-$. (Note $(a, b+h, a+b+h) \notin R_m^+$ since $a \leq 0$.) It immediately follows from our bounds on h that the first point $(a+h, b, a+b+h)$ is in the gap between the left wing and the right wing, namely, $b \leq -\frac{1}{m}(a+h)$, and the second point is above the left wing, namely $a < M(b+h)$. See Figure 16 for a sketch of the situation. By Claim 6.5.1 the trade $S_{a,b,h}$ works as desired.

Case 2(b): $a > 0$. It turns out this can only happen if $m = 2$. In this case we may use the staircase trade S_{a,b,h_1}. The points $(x_1, b, a+b+h_1), (a, y_1, a+b+h_1) \notin R_2$, namely we have that $b \leq -\frac{1}{2}(a+h_1)$ and $b+h_1 \leq -\frac{a}{2}$. These follow from plugging in $m = 2$ into (1) (we see that $h_1 = -b - 2a$) and using that $b \leq a$. See Figure 17 for a sketch of the situation. □

Theorem 6.6. *For $m \geq 2$, $m \in \mathbb{N}$, the density of R_m is given by*

$$\rho(R_m) = \frac{2m^3 - m^2 - 4m - 1}{4m^3 - 4m^2 - 4m}.$$

Proof. For all n we have $\sigma_n(R_m^+) = \sigma_n(R_m^-)$; that is, the number of entries in each wing in a square region centered at the origin is the same. We compute $\sigma_n(R_m^+)$ in three parts and multiply by 2 to get $\sigma_n(R_m)$. Suppose that $n = mq + r$ with $0 \leq r < m$ and that $n = MQ + R$ with $0 \leq R < M$.

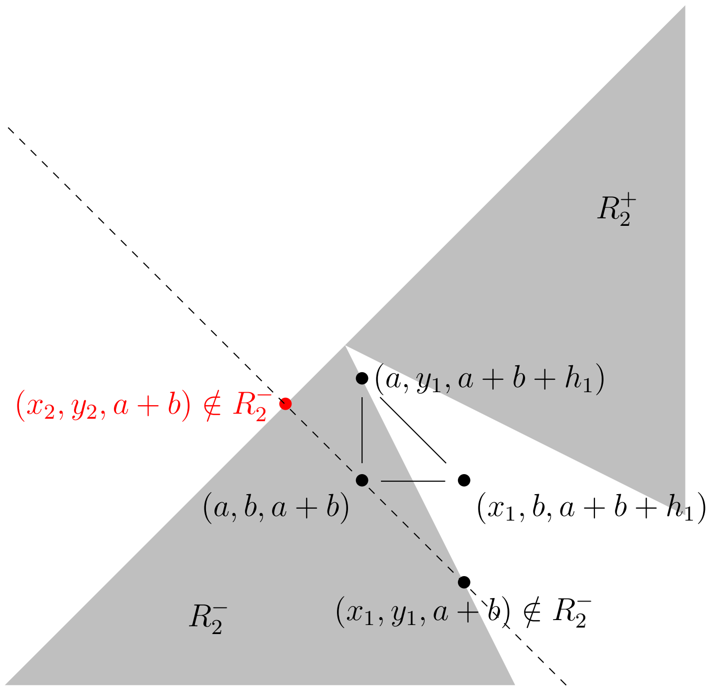

Figure 17: Illustration of Case 2(b) of criticality proof.

$$
\begin{array}{cccccccccccc}
 & & & & & & & & & & & \iddots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 10 & \\
\cdot & \mathit{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & 9 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 7 & 8 & \\
\cdot & -\mathit{2} & \cdot & \mathit{0} & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
\cdot & \cdot & \cdot & -\mathit{2} & \cdot & \mathit{0} & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & 2 & 3 & 4 & \\
\cdot & \cdot & \cdot & -4 & -3 & -\mathit{2} & \cdot & \mathit{0} & \cdot & \cdot & 3 & \\
\cdot & \cdot & -6 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \\
\cdot & -8 & -7 & -6 & -5 & -4 & -3 & -\mathit{2} & \cdot & \mathit{0} & \cdot & \\
-10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & & \ddots &
\end{array}
$$

Figure 18: Illustration of R_2 criticality with $a = 0$ and $b = -2$. The staircase trade $S_{0,-2,2}$ is italicised, with entries not in the partial square in blue and the point of intersection in purple.

$$
\begin{array}{ccccccccccccccc}
\ddots & & & & & & & \vdots & & & & & & & \iddots \\
 & \mathit{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 11 & 12 & 13 & 14 & 15 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 10 & 11 & 12 & 13 & 14 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 9 & 10 & 11 & 12 & 13 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & 9 & 10 & 11 & 12 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 7 & 8 & 9 & 10 & 11 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 6 & 7 & 8 & 9 & 10 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & 9 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & 7 & \\
\cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \odot & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
 & \mathit{-7} & -6 & -5 & -4 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \mathit{3} & 4 & 5 & \\
 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -11 & -10 & -9 & -8 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -12 & -11 & -10 & -9 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -13 & -12 & -11 & -10 & -9 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \\
 & -14 & -13 & -12 & -11 & -10 & -9 & -8 & -7 & -6 & \cdot & \cdot & \cdot & \cdot & \\
 & -15 & -14 & -13 & -12 & -11 & -10 & -9 & -8 & -7 & \cdot & \cdot & \cdot & \cdot & \\
 & -16 & -15 & -14 & -13 & -12 & -11 & -10 & -9 & -8 & -7 & \cdot & \cdot & \cdot & \\
 & -17 & -16 & -15 & -14 & -13 & -12 & -11 & -10 & -9 & -8 & \mathit{-7} & \cdot & \cdot & \\
 & -18 & -17 & -16 & -15 & -14 & -13 & -12 & -11 & -10 & -9 & \cdot & \cdot & \cdot & \\
 & -19 & -18 & -17 & -16 & -15 & -14 & -13 & -12 & -11 & -10 & -9 & \cdot & \cdot & \\
\iddots & & & & & & & \vdots & & & & & & & \ddots
\end{array}
$$

Figure 19: llustration of R_3 criticality with $a = 4$ and $b = -1$; the entries of the staircase trade $S_{-6,-1,10}$ are italicized, with those not in the partial square in blue and the point of intersection in purple.

First, consider the contribution to $\sigma_n(R_m^+)$ for the x-axis. All of the positive x-axis is included, which gives us n entries. Second, consider the part of R_m^+ in quadrant IV. We have

$$m \cdot \frac{q(q-1)}{2} + qr$$

entries. Similarly, for quadrant I we have

$$n^2 - M \cdot \frac{Q(Q-1)}{2} - QR$$

entries. Sum these three terms and multiply by 2 to get the value

$$\sigma_n(R_m) = 2\left(n + m \cdot \frac{q(q-1)}{2} + qr + n^2 - M \cdot \frac{Q(Q-1)}{2} - QR\right).$$

By the squeeze theorem, it is enough to only look at the limit of $\rho_n(R_m)$ for n divisible by mM, in which case both r and R are 0. This gives

$$\begin{aligned}
\rho(R_m) &= \lim_{n\to\infty} \rho_n(R_m) = \lim_{n\to\infty} \frac{\sigma_n(R_m)}{(2n+1)^2} \\
&= \lim_{n\to\infty} \frac{2\left(n + m \cdot \frac{q(q-1)}{2} + n^2 - M \cdot \frac{Q(Q-1)}{2}\right)}{4n^2+4n+1} \\
&= \lim_{n\to\infty} \frac{n[(n/m)-1] + 2n^2 - n[(n/M)-1]}{4n^2} \\
&= \lim_{n\to\infty} \frac{n^2/m + 2n^2 - n^2/M}{4n^2} \\
&= \frac{M + 2Mm - m}{4Mm}.
\end{aligned}$$

Recalling that $M = m^2 - m - 1$, we obtain

$$\rho(R_m) = \frac{2m^3 - m^2 - 4m - 1}{4m^3 - 4m^2 - 4m}$$

as required. □

The values of $\rho(R_m)$ for $m = 2, 3, 4, 5$ are $3/8, 8/15, 95/176, 51/95$. The maximum value is the $95/176$ obtained at $m = 4$ and they decrease from this point with $\lim_{m\to\infty} \rho(R_m) = 1/2$. Given that critical sets of the finite squares L_n based on the integers modulo n are conjectured to all have densities less than $1/2$, the existence of infinitely many (indeed, any) critical sets for $L_{\mathbb{Z}}$ with density greater than $1/2$ is perhaps surprising.

6.2 Undefined density

First, recall that it is perfectly reasonable to consider the (asymptotic) density of a subset of $\mathbb{Z}$ (see, for example, [16]). Define $\rho_n(A)$ for $A \subseteq \mathbb{Z}$ to be the number of elements of A between $-n$ and n, i.e., $\rho_n(A) = |A \cap [-n, n]|$. Then the density $\rho(A) = \lim_{n\to\infty} \rho_n(A)$.

One can construct many subsets of $\mathbb{Z}$ with undefined density, i.e., find $A \subseteq \mathbb{Z}$ such that the limit defining its density does not exist. For example, A could have extended gaps with nothing in it, so its lower density is 0, and then periods where you throw everything in so that its upper density is 1.

Using such a subset A of $\mathbb{Z}$ of undefined density we may find a set of undefined density for $L_{\mathbb{Z}}$, for example by gluing part of one critical set on top of another one using indices from the A, as we show below.

It seems useful in the following to view $L_{\mathbb{Z}}$ as a function (the function is addition) from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, because for density it only matters whether the entry is filled, not what the entry in each coordinate is. Thus, we will use $\operatorname{dom} P = \{(x, y) : (x, y, x + y) \in P\}$ for subsquares $P \subseteq L_{\mathbb{Z}}$.

Theorem 6.7. *There is a uniquely completable partial subsquare of $L_{\mathbb{Z}}$ with undefined density.*

Proof. We build the desired subsquare of $L_{\mathbb{Z}}$ as follows. Start with the full bowtie square . On column indices in a set $A \subseteq \mathbb{Z}$ with undefined density, add entries from . Call the resulting subsquare R_A. Clearly R_A is uniquely completable, as it contains the full . It is not critical because all of the entries from could be removed and it would still be uniquely completable.

To see that R_A also has undefined density, we have the following computation:

$$\begin{aligned}\rho(R_A) = \lim_{n\to\infty} \frac{|\operatorname{dom}(R_A) \cap [-n, n]^2|}{(2n + 1)^2} &= \rho() + \lim_{n\to\infty} \frac{|A \cap [-n, n]| \cdot (n + 1)}{(2n + 1)^2} \\ &= \rho() + 1/2 \lim_{n\to\infty} \frac{|A \cap [-n, n]|}{2n + 1}.\end{aligned}$$

Therefore the density of R_A depends on the density of A, and since A's density is undefined, this means that the density of R_A is also undefined. □

7 Partitions into Critical Sets

In [1], it is shown that M_n can be partitioned into four critical sets for all $n > 1$ and into three critical sets when $n \in \{4, 5, 6\}$. Theorem 7.1 shows that the infinite analog $L_{\mathbb{Z}}$ may be partitioned into three critical sets.

Theorem 7.1. *It is possible to partition $L_{\mathbb{Z}}$ into three critical sets that have densities* 1/4, 1/4 *and* 1/2.

Proof. We check that the quartered partial square , the bowtie partial square $_{(0,1)}$ and the transpose of a bowtie partial square $_{(0,1)}$ together contain each entry of $L_{\mathbb{Z}}$ exactly once.

The boundaries of the three sets are the two axes and the line $y = -x$. The set ▞ covers the interiors of the first and third quadrants, the negative x-axis and the positive y-axis. The sets ◥$_{(0,1)}$ and ◢$_{(0,1)}$ cover the interiors of the second and fourth quadrants, with the 0s in the second quadrant (other than the identity) included in ◥$_{(0,1)}$ and the 0s in the fourth quadrant (including the identity) included in ◢$_{(0,1)}$. The positive x-axis is in ◥$_{(0,1)}$ and the negative y-axis is in ◢$_{(0,1)}$.

The density of ▞ is $1/2$ and the densities of ◥$_{(0,1)}$ and ◢$_{(0,1)}$ are each $1/4$, thus these critical sets provide the required partition. □

8 Questions

We have shown that $L_{\mathbb{Z}}$ has a critical set of density ρ for $\rho = 1/4$, for $\rho = 3/8$ and for infinitely many values of ρ in the range $1/2 \le \rho \le 95/176$.

Question 8.1. *What is the full spectrum of possible densities for critical sets of $L_{\mathbb{Z}}$? In particular, what are the smallest and largest possible densities?*

Consider the partial square S given by the union of

$$\{(x, y, x + y) : x > 0 \text{ and } y > x\}\,,\ \{(x, y, x + y) : x < 0 \text{ and } 0 \le y < -x\}$$

and

$$\{(x, y, x + y) : x > 0 \text{ and } -x \le y < -x/2\}\,.$$

It has density $5/16$. If it is a critical set then this is a new value; however, we have been unable to show whether or not it is critical. Similar constructions are possible that give candidates for further new densities close to $1/4$.

More generally, we ask the same question for arbitrary infinite latin squares:

Question 8.2. *What is the full spectrum of possible densities for critical sets of an infinite latin square?*

In particular, the conjectured lower bound is $1/4$ for the finite case; are there critical sets in an infinite square (whether $L_{\mathbb{Z}}$ or a different square) with a lower density?

In Section 6.2 we constructed a uniquely completable set with undefined density. However, it was not critical.

Question 8.3. *Is there a critical set with undefined density?*

In Section 7 we saw that it is possible to partition $L_{\mathbb{Z}}$ into three critical sets with densities $1/4$, $1/4$ and $1/2$.

Question 8.4. *Is it possible to partition $L_{\mathbb{Z}}$ into two or four (or more) critical sets? What about other infinite latin squares?*

More generally, we ask:

Question 8.5. *For what sets* $\{\rho_1, \rho_2, \dots, \rho_k : \sum \rho_i = 1\}$ *is it possible to partition* $L_{\mathbb{Z}}$ *into* k *critical sets with these densities? What possibilities exist for other infinite latin squares?*

Note that a partition with five or more parts requires a critical set with density less than 1/4.

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