

Convex hulls of Grassmannians and combinatorics of symmetric hypermatrices

KAZUMASA NARITA

*Graduate School of Mathematics
Nagoya University, Furo-cho
Chikusa-ku, Nagoya 464-8602
Japan
m19032e@math.nagoya-u.ac.jp*

Abstract

It is known that the complex Grassmannian of k -dimensional subspaces can be identified with the set of projection matrices of rank k . It is also classically known that the convex hull of this set is the set of Hermitian matrices with eigenvalues between 0 and 1 and summing to k . We give a new proof of this fact. We also give an existence theorem for a certain combinatorial class of hypermatrices by a similar argument. This existence theorem can be rewritten into an existence theorem for a uniform weighted hypergraph with given weighted degree sequence.

1 Introduction

The convex hull of the set of projection matrices of constant rank is classically known. In this paper, we give a new approach to identify this convex hull and study combinatorics of symmetric hypermatrices. These subjects appear to be unrelated, but the proofs of both results are similar.

Tai [20] showed that projective spaces can be embedded isometrically into certain Euclidean spaces of matrices. More explicitly, the complex projective space $\mathbf{C}P^n$, for example, can be embedded in a space of Hermitian matrices as follows:

$$\mathbf{C}P^n \xrightarrow{\cong} \{A \in HM(n+1) \mid A^2 = A, \operatorname{tr}A = 1\}, [z] \mapsto \frac{1}{z^*z}zz^*$$

Here z is a column vector in \mathbf{C}^{n+1} . This embedding is well-defined and sends $[z]$, which can be identified with a complex line in \mathbf{C}^{n+1} , to the projection matrix onto the complex line $[z]$. Later, Dimitrić [7] showed that Tai's embedding can be generalized to Grassmannians by identifying points in Grassmannians and the corresponding projection matrices. That is, the complex Grassmannian of k -dimensional subspaces $G_k(\mathbf{C}^n)$ can be embedded into $HM(n)$ as

$$G_k(\mathbf{C}^n) \cong \{A \in HM(n) \mid A^2 = A, \operatorname{tr}A = k\}. \quad (1.1)$$

If a Hermitian matrix A is positive semidefinite, we write $A \succeq 0$. Fillmore and Williams [10] proved the following theorem:

Theorem 1.1. ([10]) *The convex hull of the embedded Grassmannian $G_k(\mathbf{C}^n)$ in $HM(n)$ is*

$$\Omega := \{A \in HM(n) \mid A \succeq 0, I_n - A \succeq 0, \operatorname{tr} A = k\}$$

and the set of extreme points of Ω is $G_k(\mathbf{C}^n)$.

They used this fact to study k -numerical ranges of matrices, the notion introduced by Halmos [14, p. 111]. Overton and Womersley [18] showed that Theorem 1.1 implies Fan's theorem [9], which is an extension of Rayleigh's principle and gives a characterization of the sum of the largest k eigenvalues of a Hermitian matrix.

As Overton and Womersley [18] pointed out, the conventional proof of Fan's theorem relies heavily on the following classical theorem (see [17, p. 785], for example):

Theorem 1.2. *Let Ω_1 be the set of permutation matrices of order n and Ω_2 be the set of doubly stochastic matrices of order n . Then Ω_2 is the convex hull of Ω_1 and Ω_1 is the set of extreme points of Ω_2 .*

In this paper, we give another proof of Theorem 1.1 by using a combinatorial theorem by Rado [19]. This approach should be of interest since it provides a new path deducing Fan's theorem from Theorem 1.2 if one notices that Rado's theorem can be proved by using Theorem 1.2 (see [22, p. 60], for example).

In the second part of the paper, we study combinatorics of symmetric hypermatrices and weighted uniform hypergraphs. A d -uniform hypergraph is a hypergraph each of whose edge has exactly d vertices. A d -uniform hypergraph is called *simple* if it has no repeated edges. A finite sequence of nonnegative integers is called *d -graphic* if it can be realized as the degree sequence of some simple d -uniform hypergraph. In particular, a simple 2-uniform hypergraph is nothing but a simple graph. Havel [15] and Hakimi [13] independently gave a characterization of a 2-graphic sequence. Today there are many other characterizations including the classic Erdős–Gallai theorem [8] (see [16]). Dewdney [6] generalized the Havel–Hakimi algorithm and characterized a d -graphic sequence as follows:

Theorem 1.3. ([6]) *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $D = (d_1, \dots, d_n)$ be a nonincreasing sequence of nonnegative integers. Then D is d -graphic if and only if there exists a nonincreasing sequence $D' = (d'_2, \dots, d'_n)$ of nonnegative integers satisfying all of the following conditions:*

1. D' is $(d - 1)$ -graphic.
2. $\sum_{i=2}^n d'_i = (d - 1)d_1$.
3. $D'' := (d_2 - d'_2, \dots, d_n - d'_n, 0)$ is d -graphic.

In contrast to the case $d = 2$, this is the the only known characterization of a d -graphic sequence for $d \geq 3$. It is known that this characterization does not yield

an efficient algorithm, so conditions for a sequence D to be d -graphic have been investigated for a long time (see [1], [2] and [11], for example).

On the other hand, Hakimi [13] also characterized when a nonincreasing sequence of nonnegative real numbers is realized as the weighted degree sequence of some weighted graph. This characterization can be interpreted as an existence theorem for a symmetric nonnegative matrix with given row sums and with only 0's on the main diagonal (see Section 4 of this paper and also [3, Section 2.3, 7.2]). In this paper, we generalize Hakimi's theorem to symmetric hypermatrices by a similar argument to that for determining the convex hulls of Grassmannians. The notion of alternating sign matrices, one of combinatorial matrix classes, was generalized to hypermatrices by Brualdi and Dahl [4] recently. Still few results are known about combinatorics of hypermatrices. Hence the generalization of Hakimi's theorem to symmetric hypermatrices itself is of interest. In addition, the theorem can be rewritten into the following:

Theorem 1.4. *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $D = (d_1, \dots, d_n)$ be a nonincreasing sequence of nonnegative numbers. Then there exists a d -uniform weighted hypergraph with the weighted degree sequence D if and only if the inequality*

$$dd_1 \leq \sum_{i=1}^n d_i \tag{1.2}$$

holds. Moreover, if this holds, then there exists a d -uniform weighted hypergraph with the weighted degree sequence D that has at most n edges with positive weights.

Letting $d = 2$ recovers the aforementioned theorem due to Hakimi. (See Section 4 for the precise definitions of a d -uniform weighted hypergraph and a weighted degree sequence.) At the end of this paper, we clarify the relationship between Theorem 1.3 and Theorem 1.4.

2 Majorization

In this section, we set up tools from combinatorics that we use later. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, we rearrange the components in decreasing order and label the new subscripts $x = (x_{[1]}, \dots, x_{[n]})$ so that we have $x_{[1]} \geq \dots \geq x_{[n]}$. If $x, y \in \mathbf{R}^n$ satisfy

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \quad \text{for all } 1 \leq k \leq n - 1$$

and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$$

then we say that x is *majorized* by y , which is denoted by $x \preceq y$. If $\tilde{x} \in \mathbf{R}^n$ can be obtained by a rearrangement of components of $x \in \mathbf{R}^n$, then \tilde{x} is called a *permutation* of x (see [17, Chaper 1] or [22, Section 3.2], for example).

Rado [19] showed that for $x, y \in \mathbf{R}^n$, x is majorized by y if and only if x is a convex combination of permutations of y . Although $y \in \mathbf{R}^n$ has $n!$ permutations in general, Zhan [21] pointed out that the number of permutations in the convex combination can be reduced less than or equal to n by using Carathéodory’s theorem. That is, we have the following result, which is a crucial tool in this paper:

Theorem 2.1. *Let $x, y \in \mathbf{R}^n$. Then x is majorized by y if and only if x is a convex combination of at most n permutations of y .*

3 Convex Hulls of Grassmannians

Let $\mathbf{F} = \mathbf{R}$ or \mathbf{C} . The Grassmannian of k -dimensional subspaces of \mathbf{F}^n is denoted by $G_k(\mathbf{F}^n)$. We denote by $FM(n)$ the space of real symmetric matrices of order n if $\mathbf{F} = \mathbf{R}$, or the space of Hermitian matrices of order n if $\mathbf{F} = \mathbf{C}$. If $A \in FM(n)$ is positive semidefinite, we write $A \succeq 0$. It is known that the Grassmannian $G_k(\mathbf{F}^n)$ can be identified with set of projectors in $FM(n)$:

$$G_k(\mathbf{F}^n) \cong \{A \in FM(n) \mid A^2 = A, \operatorname{tr}A = k\}. \tag{3.1}$$

This embedding is isometric with respect to the canonical metrics on Grassmannian and $FM(n)$ (see [7] for details). Hereinafter, we identify the both sides of the equation (3.1). We set

$$\mathcal{H}_k(\mathbf{F}^n) := \{A \in FM(n) \mid A \succeq 0, I_n - A \succeq 0, \operatorname{tr}A = k\}. \tag{3.2}$$

One can immediately see that $\mathcal{H}_k(\mathbf{F}^n)$ is a convex subset of $FM(n)$. A point of a convex body $C \subset \mathbf{R}^N$ is called *extreme* if it is not an interior point of any line segments in C . The set of extreme points of C is denoted by $\operatorname{ext}C$. Fillmore and Williams [10] proved the following proposition:

Proposition 3.1. *The Grassmannian $G_k(\mathbf{F}^n)$ is the set of extreme points of $\mathcal{H}_k(\mathbf{F}^n)$, that is, $G_k(\mathbf{F}^n) = \operatorname{ext}\mathcal{H}_k(\mathbf{F}^n)$.*

The classical Krein–Milman theorem says that the convex hull of the extreme points of a convex set coincides with the convex set itself. Hence by the theorem, the above proposition implies the following theorem:

Theorem 3.2. ([10]) *The convex hull of $G_k(\mathbf{F}^n)$ in $FM(n)$ is $\mathcal{H}_k(\mathbf{F}^n)$.*

We give another proof of this theorem by using Theorem 2.1.

Proof. Each element $A \in G_k(\mathbf{F}^n)$ satisfies $A^2 = A$ and $\operatorname{tr}A = k$. This implies that the eigenvalues of A are 1 and 0 with multiplicity k and $n - k$ respectively. Hence it follows that $G_k(\mathbf{F}^n) \subset \mathcal{H}_k(\mathbf{F}^n)$. Since $\mathcal{H}_k(\mathbf{F}^n)$ is convex, it suffices to show that each element $X \in \mathcal{H}_k(\mathbf{F}^n)$ can be expressed as a convex combination of elements of $G_k(\mathbf{F}^n)$. For any $X \in \mathcal{H}_k(\mathbf{F}^n)$, there exists U such that

$$UXU^{-1} = \operatorname{diag}(r_1, \dots, r_n), \quad 1 \geq r_1 \geq r_2 \geq \dots \geq r_n \geq 0, \quad \sum_{i=1}^n r_i = k, \tag{3.3}$$

where U is an orthogonal matrix if $\mathbf{F} = \mathbf{R}$, or a unitary matrix if $\mathbf{F} = \mathbf{C}$. For any k -tuple (i_1, \dots, i_k) with $1 \leq i_1 < \dots < i_k \leq n$, we denote by $A_{i_1 \dots i_k}$ the diagonal matrix that has units in the i_j -th diagonal entries ($j = 1, 2, \dots, k$) and zeros in the others. Clearly, we have $A_{i_1, \dots, i_k} \in G_k(\mathbf{F}^n)$. Let $r := (r_1, \dots, r_n) \in \mathbf{R}^n$. Consider

$$x := (1, \dots, 1, 0, \dots, 0) \in \mathbf{R}^n,$$

where the number of 1's is k and that of 0's is $n - k$. Then (3.3) implies $r \preceq x$. Hence Theorem 2.1 implies that r can be written as a convex combination of permutations of x . Thus it follows that there exist $\{c_{i_1, \dots, i_k} \geq 0 \mid 1 \leq i_1 < \dots < i_k \leq n\}$ such that

$$UXU^{-1} = \text{diag}(r_1, \dots, r_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} c_{i_1 \dots i_k} A_{i_1 \dots i_k}$$

and

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} c_{i_1 \dots i_k} = 1.$$

Since we have $U^{-1}A_{i_1 \dots i_k}U \in G_k(\mathbf{F}^n)$, X can be written as a convex combination of elements in $G_k(\mathbf{F}^n)$. This completes the proof. \square

4 Symmetric Hypermatrices

Let \mathbf{N} be the set of positive integers. For $n \in \mathbf{N}$, we set $\langle n \rangle := \{1, 2, \dots, n\}$. Let $d \in \mathbf{N}$. For n_1, \dots, n_d , a (real) d -hypermatrix is defined to be a function $A : \langle n_1 \rangle \times \dots \times \langle n_d \rangle \rightarrow \mathbf{R}$. A 2-hypermatrix is just an ordinary matrix. A value $A(i_1, \dots, i_d)$ is called an *entry*. **In this paper, we only consider the case where $n_1 = \dots = n_d = n$.** By fixing one index i_k to be l , one gets a $(d - 1)$ -dimensional subarray $\{A(i_1, \dots, i_{k-1}, l, i_{k+1}, \dots, i_d) \mid i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_d \in \langle n \rangle\}$. This $(d - 1)$ -dimensional subarray is called a *slice* of the d -hypermatrix with respect to $i_k = l$. A slice itself can be regarded as a $(d - 1)$ -hypermatrix. We set

$$I_k := \{(k, i_2, \dots, i_d) \in \langle n \rangle^d \mid i_2 < \dots < i_d, \quad i_2, \dots, i_d \in \langle n \rangle \setminus \{k\}\}$$

for each $k \in \langle n \rangle$.

A d -hypermatrix A is called *nonnegative* if each entry of A is nonnegative. Let \mathfrak{S}_d be the symmetric group of degree d . A d -hypermatrix A is called *symmetric* if the equation

$$A(i_{\pi(1)}, \dots, i_{\pi(d)}) = A(i_1, \dots, i_d)$$

holds for any $(i_1, \dots, i_d) \in \langle n \rangle^d$ and any $\pi \in \mathfrak{S}_d$. A d -hypermatrix A is defined to be *strongly hollow* if every entry $A(i_1, \dots, i_d)$ that has two identical indices $i_k = i_l$ ($k \neq l$) is zero. A strongly hollow 2-hypermatrix is just an ordinary hollow matrix, i.e. a square matrix whose diagonal entries are all equal to zero.

Consider $R = (r_1, \dots, r_n) \in \mathbf{R}^n$ with $r_1 \geq \dots \geq r_n \geq 0$. We denote by $\mathcal{N}_0(d; R)$ the set of strongly hollow symmetric nonnegative d -hypermatrices whose sum of all the entries in each slice with respect to $i_1 = j$ ($j \in \langle n \rangle$) is equal to r_j . Note that i_1

in the above definition can be replaced by any fixed i_k ($k \in \langle n \rangle$) since we consider a symmetric hypermatrix. A strongly hollow symmetric nonnegative d -hypermatrix A belongs to $\mathcal{N}_0(d; R)$ if and only if the equation

$$(d - 1)! \sum_{\mathbf{i} \in I_k} A(\mathbf{i}) = r_k \tag{4.1}$$

holds for each $k \in \langle n \rangle$.

The set $\mathcal{N}_0(d; R)$ might be empty. For example, $\mathcal{N}_0(d; R)$ is obviously empty when $n = 1$ and $R = (1)$. One can easily verify the emptiness also when $n = 2$ and $R = (2, 1)$. Hence it is natural to look for the condition that the set $\mathcal{N}_0(d; R)$ is nonempty. Using a similar argument to that in the proof of Theorem 3.2, one can prove the following theorem.

Theorem 4.1. *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $R = (r_1, \dots, r_n)$ be a vector with $r_1 \geq \dots \geq r_n \geq 0$. Then the set $\mathcal{N}_0(d; R)$ is nonempty if and only if the inequality*

$$dr_1 \leq \sum_{i=1}^n r_i \tag{4.2}$$

holds. Moreover, if this holds, then there exists a d -hypermatrix in $\mathcal{N}_0(d; R)$ the number of whose nonzero entries is at most $n \cdot d!$.

For $d = 2$, the theorem was proved by Hakimi [13] in a graph-theoretic way.

Proof. We assume $A \in \mathcal{N}_0(d; R)$. Then, the equation (4.1) holds for each $k \in \langle n \rangle$. Fix $2 \leq i_2 < i_3 < \dots < i_d \leq n$. The symmetry of A implies

$$\begin{aligned} &(d - 1) \cdot A(1, i_2, \dots, i_d) \\ &= A(i_2, 1, i_3, \dots, i_d) + A(i_3, 1, i_2, i_4, \dots, i_d) + \dots + A(i_d, 1, i_2, \dots, i_{d-1}). \end{aligned} \tag{4.3}$$

Each of the ordered d -tuples $(i_2, 1, i_3, \dots, i_d)$, $(i_3, 1, i_2, i_4, \dots, i_d)$, \dots , $(i_d, 1, i_2, \dots, i_{d-1})$ belongs to one of I_2, \dots, I_n . Since only the elements

$$\{A(i_1, \dots, i_d) \mid (i_1, \dots, i_d) \in \cup_{j=2}^n I_j, i_l = 1 \text{ for some } l\}$$

can appear on the right-hand side of (4.3), we have

$$0 \leq r_2 + \dots + r_n - (d - 1)r_1,$$

and so the inequality (4.2).

We show the converse. Set

$$S = \frac{1}{d} \sum_{i=1}^n r_i.$$

We know $r_1 \leq S$ by the assumption. Consider a vector

$$R' := \left(\frac{r_1}{S}, \dots, \frac{r_n}{S} \right).$$

For $1 \leq i_1 < i_2 < \dots < i_d \leq n$, we denote by $v_{i_1 i_2 \dots i_d}$ the vector in \mathbf{R}^n whose i_1, i_2, \dots, i_d entries are equal to 1 and the other entries are equal to 0. Now we have

$$R' \preceq v_{12\dots d} = (1, \dots, 1, 0, \dots, 0),$$

where the number of 1's is d and that of 0's is $n - d$. Hence Theorem 2.1 implies that there exist nonnegative constants $\{c_{i_1 i_2 \dots i_d} \geq 0 \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}$ such that

$$\sum_{1 \leq i_1 < i_2 < \dots < i_d \leq n} c_{i_1 i_2 \dots i_d} v_{i_1 i_2 \dots i_d} = R' \quad \text{and} \quad \sum_{1 \leq i_1 < i_2 < \dots < i_d \leq n} c_{i_1 i_2 \dots i_d} = 1.$$

Then for arbitrary $k \in \langle n \rangle$, we obtain

$$\sum_{l=1}^d \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{l-1} \\ < i_l = k < i_{l+1} < \dots < i_d \leq n}} c_{i_1 i_2 \dots i_d} = \frac{r_k}{S}.$$

From the set $\{c_{i_1 i_2 \dots i_d} \geq 0 \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}$, one can naturally construct a strongly hollow symmetric d -hypermatrix $(\tilde{c}_{i_1 i_2 \dots i_d})$ satisfying

$$\tilde{c}_{\pi(i_1)\pi(i_2)\dots\pi(i_d)} = c_{i_1 i_2 \dots i_d} \quad \text{for any } \pi \in \mathfrak{S}_d.$$

Since each $c_{i_1 i_2 \dots i_d}$ is nonnegative, the hypermatrix $(\tilde{c}_{i_1 i_2 \dots i_d})$ is nonnegative. Then for each $k \in \langle n \rangle$, we have

$$\sum_{j_1, \dots, j_{d-1} \in \langle n \rangle \setminus k} \tilde{c}_{k j_1 \dots j_{d-1}} = (d-1)! \sum_{l=1}^d \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{l-1} \\ < i_l = k < i_{l+1} < \dots < i_d \leq n}} c_{i_1 i_2 \dots i_d} = \frac{(d-1)! r_k}{S}.$$

Thus we conclude

$$\left(\frac{S}{(d-1)!} \tilde{c}_{i_1 i_2 \dots i_d} \right) \in \mathcal{N}_0(d; R).$$

The second half of the theorem follows from Theorem 2.1 and the symmetry of $\left(\frac{S}{(d-1)!} \tilde{c}_{i_1 i_2 \dots i_d} \right)$. □

In the following lemma, we show that if the set $\mathcal{N}_0(d; R)$ is nonempty for given $d \geq 2$ and $R \in \mathbf{R}^n$, it is a convex polytope, possibly a singleton.

Lemma 4.2. *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $R = (r_1, \dots, r_n)$ be a vector with $r_1 \geq \dots \geq r_n \geq 0$. Assume that the set $\mathcal{N}_0(d; R)$ is nonempty. Then the set $\mathcal{N}_0(d; R)$ is a convex polytope for $n \geq d + 2$, and a singleton for $n = d$ or $d + 1$.*

Proof. First we consider the case where $n = d$. The assumption on R and Theorem 4.1 imply

$$dr_1 \leq \sum_{i=1}^d r_i \leq dr_1$$

and so $r_1 = \dots = r_d$. Then the strongly hollow symmetric d -hypermatrix A given by

$$A(\pi(1), \dots, \pi(d)) = A(1, 2, \dots, d) = \frac{r_1}{(d-1)!} \quad \text{for any } \pi \in \mathcal{S}_d \quad (4.4)$$

is the unique element in $\mathcal{N}_0(d; R)$.

Hereinafter, we consider the case where $n \geq d + 1$. We rewrite the slice sum constraints (4.1) as follows. Let w be a vector in $\mathbf{R}^{\binom{n}{d}}$ obtained by aligning all the elements in $\{A(i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq n\}$ in an arbitrary order. For $1 \leq i_1 < i_2 < \dots < i_d \leq n$, we denote by $v_{i_1 i_2 \dots i_d}$ the vector in \mathbf{R}^n whose i_1, i_2, \dots, i_d entries are equal to 1 and the other entries are equal to 0. We construct an $n \times \binom{n}{d}$ matrix B by arranging the vectors $\{v_{i_1 i_2 \dots i_d} \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}$ in the unique order so that if the j -th entry of w is $A(i_1, \dots, i_d)$, then the j -th column of B is $v_{i_1 i_2 \dots i_d}$. Then the linear system (4.1) is equivalent to

$$Bw = \frac{1}{(d-1)!} R^T. \quad (4.5)$$

Since each column of B is a permutation of $v_{1\dots d}$ and we have $n \geq d + 1$, one can easily check that the rank of B is n .

For $n = d + 1$, the system (4.5) has $(d + 1)$ equations and $(d + 1)$ variables $\{A(1, \dots, k - 1, \check{k}, k + 1, \dots, d + 1) \mid 1 \leq k \leq d + 1\}$, where \check{k} denotes the omission of k . Therefore, the solutions $\{A(1, \dots, k - 1, \check{k}, k + 1, \dots, d + 1) \mid k \in \langle d + 1 \rangle\}$ are uniquely determined. We find the solutions explicitly and show that they are nonnegative. Summing up all the equations in (4.1), one obtains

$$\sum_{k=1}^{d+1} A(1, \dots, k - 1, \check{k}, k + 1, \dots, d + 1) = \frac{1}{d!} \sum_{i=1}^{d+1} r_i. \quad (4.6)$$

For each $k \in \langle d + 1 \rangle$, the left-hand side of the equation (4.1) is the sum of all the variables except for $A(1, \dots, k - 1, \check{k}, k + 1, \dots, d + 1)$. Thus the difference between (4.6) and (4.1) gives

$$A(1, \dots, k - 1, \check{k}, k + 1, \dots, d + 1) = \frac{1}{d!} \left(\sum_{i=1}^{d+1} r_i \right) - \frac{r_k}{(d-1)!}, \quad (4.7)$$

which is nonnegative by the assumption (4.2). Hence the set $\mathcal{N}_0(d; R)$ is a singleton for $n = d + 1$.

Finally we consider the case where $n \geq d + 2$. Then the system (4.5) has n equations and $\binom{n}{d}$ variables. The assumption that $n \geq d + 2 \geq 4$ implies

$$\binom{n}{d} \geq \frac{n(n-1)}{2} > n.$$

Hence the solution space of (4.5) is of dimension $\binom{n}{d} - n > 0$. The assumption that a hypermatrix is nonnegative implies that the set $\mathcal{N}_0(d; R)$ is the intersection of half spaces and the solution space. Hence $\mathcal{N}_0(d; R)$ is a polytope of dimension $\binom{n}{d} - n$. Moreover, one can readily see that $\mathcal{N}_0(d; R)$ is a convex polytope. Thus the claim follows. \square

In the above proof, we have described the unique element in $\mathcal{N}_0(d; R)$ in the cases where $\mathcal{N}_0(d; R)$ is a singleton. For $n = d$, (4.4) implies that the number of positive entries of the unique element in $\mathcal{N}_0(d; R)$ is exactly $d!$ unless that $r_1 = \dots = r_d = 0$. For $n = d + 1$, the number of positive entries of the unique element in $\mathcal{N}_0(d; R)$ is at most $(d + 1)!$ by (4.7). This gives another proof of the second half of Theorem 4.1 when $\mathcal{N}_0(d; R)$ is a singleton.

Assume that $\mathcal{N}_0(d; R)$ is a nonempty convex polytope. In the following proposition, we show that an extreme point in $\mathcal{N}_0(d; R)$ must have at most $n \cdot d!$ positive entries. Since every convex polytope has an extreme point, the following proposition gives another proof of the second half of Theorem 4.1 in case where $\mathcal{N}_0(d; R)$ is a convex polytope. The proposition is a generalization of the lemma in [5] and the proof is similar to the original one.

Proposition 4.3. *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $R = (r_1, \dots, r_n)$ be a vector with $r_1 \geq \dots \geq r_n \geq 0$. Assume that $\mathcal{N}_0(d; R)$ is a nonempty convex polytope. If $A \in \mathcal{N}_0(d; R)$ is an extreme point, then A has at most $n \cdot d!$ positive entries.*

Proof. $A \in \mathcal{N}_0(d; R)$ corresponds to a solution of the linear system

$$(d - 1)! \sum_{1 \leq i_1 < \dots < i_d \leq n} A(i_1, \dots, i_d) v_{i_1, \dots, i_d} = R^T, \quad A(i_1, \dots, i_d) \geq 0, \quad (4.8)$$

where each v_{i_1, \dots, i_d} is the vector in \mathbf{R}^n that has 1's in i_1 -th, i_2 -th, \dots and i_d -th entries and 0's elsewhere. It is known that a solution of (4.8) is extreme if and only if the vectors $\{v_{i_1, \dots, i_d} \mid A(i_1, \dots, i_d) > 0\}$ are linearly independent (see [12, Theorem 5 on p. 55], for example). Hence the cardinality of $\{A(i_1, \dots, i_d) > 0 \mid 1 \leq i_1 < \dots < i_d \leq n\}$ is at most n . Thus the claim follows from the symmetry of A . \square

Finally, we rewrite Theorem 4.1 in terms of hypergraph theory. Let $V = \{v_1, \dots, v_n\}$ be a finite set and $d \in \mathbf{N}$ with $d \geq 2$. Let e_1, \dots, e_m be subsets of V such that $\cup_{i=1}^m e_i = V$ and $|e_i| = d$ for each $i \in \langle m \rangle$. Set $E = \{e_i \mid i \in \langle m \rangle\}$. Let $w : E \rightarrow \mathbf{R}$ be a map and we assume that $w \geq 0$ in what follows. Then the triple (V, E, w) is called a d -uniform weighted hypergraph. The elements of V , E and $\{w(e_i) \in \mathbf{R} \mid i \in \langle m \rangle\}$ are called *vertices*, *edges* and *weights* respectively. For each $i \in \langle n \rangle$, we set

$$d_i := \sum_{v_i \in e_j} w(e_j)$$

and call it the *weighted degree at v_i* . The sequence $D := (d_1, \dots, d_n)$ is called the *weighted degree sequence* of (V, E, w) . For a given strongly hollow symmetric nonnegative d -hypermatrix A , one can associate a d -uniform weighted hypergraph (V, E, w) as follows: Set

$$V = \{v_1, \dots, v_n\},$$

$$E = \{e_{i_1 \dots i_d} := \{v_{i_1}, \dots, v_{i_d}\} \mid 1 \leq i_1 < \dots < i_d \leq n\},$$

and

$$w(e_{i_1 \dots i_d}) = A(i_1, \dots, i_d).$$

Then the triple (V, E, w) is a d -uniform weighted hypergraph. Conversely, for a given d -uniform weighted hypergraph (V, E, w) , one can associate a strongly hollow symmetric nonnegative d -hypermatrix. Consider a strongly hollow symmetric nonnegative d -hypermatrix A whose j -th slice sum is $r_j \geq 0$. Then the associated d -uniform weighted hypergraph (V, E, w) satisfies

$$d_j = \frac{r_j}{(d - 1)!}$$

for each $j \in \langle n \rangle$. Thus Theorem 4.1 can be interpreted as a result about realizability of a d -uniform weighted hypergraph for a given weighted degree sequence.

Theorem 4.4. *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $D = (d_1, \dots, d_n)$ be a sequence with $d_1 \geq \dots \geq d_n \geq 0$. Then there exists a d -uniform weighted hypergraph with the weighted degree sequence D if and only if the inequality*

$$dd_1 \leq \sum_{i=1}^n d_i \tag{4.9}$$

holds. Moreover, if this holds, then there exists a d -uniform weighted hypergraph with the weighted degree sequence D that has at most n edges with positive weights.

For $n = d$ or $d + 1$, Lemma 4.2 and the above theorem imply the following corollary:

Corollary 4.5. *Let $d \in \mathbf{N}$ with $d \geq 2$ and $n = d$ or $d + 1$. Let $D = (d_1, \dots, d_n)$ be a sequence with $d_1 \geq \dots \geq d_n \geq 0$. If the inequality (4.9) holds, then there exists a unique d -uniform weighted hypergraph with the weighted degree sequence D .*

A d -uniform hypergraph is called *simple* if it has no repeated edges. The degree sequence of a simple d -uniform hypergraph (V, E) can be seen as the weighted degree sequence of a d -uniform weighted hypermatrix (V, E, w) with $w(e) = 1$ for any $e \in E$. A nonincreasing sequence of nonnegative integers is called *d -graphic* if it can be realized as the degree sequence of some simple d -uniform hypergraph. Dewdney [6] proved the following:

Theorem 4.6. ([6]) *Let $d, n \in \mathbf{N}$ with $d \geq 2$ and $n \geq d$. Let $D = (d_1, \dots, d_n)$ be a nonincreasing sequence of nonnegative integers. Then D is d -graphic if and only if there exists a nonincreasing sequence $D' = (d'_2, \dots, d'_n)$ of nonnegative integers satisfying all of the following conditions:*

1. D' is $(d - 1)$ -graphic.
2. $\sum_{i=2}^n d'_i = (d - 1)d_1$.
3. $D'' := (d_2 - d'_2, \dots, d_n - d'_n, 0)$ is d -graphic.

Remark 4.7. If $D = (d_1, \dots, d_n)$, a nonincreasing sequence of nonnegative integers, is d -graphic, then D satisfies the condition (4.9). In fact, if D is d -graphic, there exists D' as in Theorem 4.6. Then we have

$$dd_1 = d_1 + \sum_{i=2}^n d'_i \leq d_1 + \sum_{i=2}^n d_i = \sum_{i=1}^n d_i,$$

where the condition (2) is used at the left equality and the middle inequality follows from the condition (3). This remark is reasonable since the degree sequence of a simple d -uniform hypergraph (V, E) is nothing but the weighted degree sequence of a d -uniform weighted hypergraph (V, E, w) with $w(e) = 1$ for any $e \in E$.

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