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Convex subgraphs and spanning trees of the square cycles

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Abstract

We classify connected spanning convex subgraphs of C_n^2, the square of the n-vertex cycle. We then show that every spanning tree of C_n^2 is contained in a unique nontrivial connected spanning convex subgraph of C_n^2. As a result, we obtain a purely combinatorial derivation of the formula for the number of spanning trees of C_n^2.

1 Introduction

It is well known that the number $t(G)$ of spanning trees of a connected graph G can be computed using the matrix-tree theorem (see e.g., [2, Section 13.2]). More precisely, $t(G)$ is the product of nonzero eigenvalues of the Laplacian of G, divided by the number of vertices of G. For families of graphs whose Laplacian eigenvalues can be computed, this method is very useful in computing $t(G)$, except that the results sometimes need to be simplified since eigenvalues may not be rational integers. Extensive work has been done to simplify the formula for $t(G)$ for circulant graphs (see [5, 6, 8]). For example, the derivation of the number $t(C_n^2)$ of spanning trees of C_n^2, the square the n-vertex cycle, using the matrix-tree theorem was done first by Baron et al. [1]. Kleitman and Golden [3] used a different approach to compute

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$t(C_n^2)$. Namely, they used topological properties of a planar embedding of C_n^2 to derive a formula for $t(C_n^2)$ when n is even, and mentioned that a similar method can be used to derive the same formula for odd n, without giving details. If n is even, C_n^2 is isomorphic to the rose window graph $R_{n/2}(1,1)$ [7]. The graph C_n^2 is also denoted by $C_n(1,2)$ [6] and $C_n^{1,2}$ [8].

In this paper, we transform the topological argument given by Kleitman and Golden [3] to a purely combinatorial one, using the theory of graph homotopy [4]. This allows us to give a uniform proof of the formula for $t(C_n^2)$ independent of the parity of n. The key idea in our proof is the fact that every spanning tree of C_n^2 is contained in a unique nontrivial connected spanning convex subgraph. Although this fact appeared implicitly in [3] when n is even, the classification of connected convex subgraphs of C_n^2 is new.

The organization of this paper is as follows. In Section 2, we fix notation for the square of a cycle as a circulant graph, and give some properties of the Fibonacci sequence. We give a classification of connected spanning convex subgraphs of C_n^2 in Section 3. In Section 4, we show that the set of the spanning trees of C_n^2 coincides with the disjoint union of the set of the spanning trees of strip graphs with tails $S_{n,k,j}$, defined in Section 2. As a consequence, we deduce a combinatorial proof of the formula for $t(C_n^2)$ which does not depend on the parity of n.

2 Preliminaries

Definition 2.1. A graph that is connected and has no closed paths is called a *tree*. For a graph G, we say that G' satisfying

$$E(G') \subseteq E(G), V(G) = V(G')$$

is a *spanning subgraph* of G. If a spanning subgraph G' in a connected graph G is a tree, then G' is called a *spanning tree* of the graph G.

Definition 2.2. Let n be an integer with $n \geq 5$. The *square of the n-vertex cycle*, or the *square cycle* for short, denoted C_n^2, is defined by $V(C_n^2) = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, $E(C_n^2) = \{\{v_i, v_j\} \mid v_i, v_j \in V(C_n^2),\ i, j \in \mathbb{Z},\ i - j = 1, 2\}$, where $v_i = i + n\mathbb{Z} \in \mathbb{Z}_n$.

Let n be an integer with $n \geq 5$. Then, $E(C_n^2) = \{e_i \mid i \in \mathbb{Z}\} \cup \{f_i \mid i \in \mathbb{Z}\}$, where we define *frame* e_i and *window* f_i as follows.

$$e_i = \{v_i, v_{i+1}\},\ f_i = \{v_i, v_{i+2}\} \quad (i \in \mathbb{Z}).$$

We denote by $\mathcal{W}(n)$ and $\mathcal{F}(n)$ the set of frames and windows, respectively as follows.

$$\mathcal{W}(n) = \{f_i \mid 0 \leq i \leq n-1\},$$
$$\mathcal{F}(n) = \{e_i \mid 1 \leq i \leq n\}.$$

By a *triangle* of C_n^2 we mean a set

$$T_i = \{e_i, e_{i+1}, f_i\} \quad (i \in \mathbb{Z}).$$

Then,

$$E(C_n^2) = \bigcup_{i=0}^{n-1} T_i.$$

Definition 2.3. Given i ($i \in \mathbb{Z}$), if a subgraph G of C_n^2 satisfies $|T_i \cap E(G)| \leq 1$ or $T_i \subseteq E(G)$, then G is said to be *convex* with respect to the triangle T_i. A subgraph G of C_n^2 is said to be *convex* if G is convex with respected to T_i for all i ($i \in \mathbb{Z}$).

Definition 2.4. The graph S_k defined by $V(S_k) = \{1, 2, \ldots, k\}$, $E(S_k) = \{\{i, j\} \mid i, j \in V(S_k), 1 \leq |i - j| \leq 2\}$ is called a *strip graph*.

The sequence of numbers F_n defined by the recurrence relation $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ ($n = 0, 1, 2, \ldots$) is called the *Fibonacci sequence*. The following two lemmas are due to Kleitman and Golden [3].

Lemma 2.5. *For $n \geq 2$, $t(S_n) = F_{2n-2}$.*

Lemma 2.6. *For $n \geq 2$,*

$$F_n^2 = \begin{cases} \sum_{k=0}^{(n-2)/2} F_{4k+2} & \text{if } n \text{ is even}, \\ 1 + \sum_{k=1}^{(n-1)/2} F_{4k} & \text{if } n \text{ is odd}. \end{cases}$$

The following substructures appeared implicitly in [3]. In fact, an escape route is the set of edges crossed by a path from the interior to the outside region, in the planar drawing of C_n^2 (see [3, Fig. 4]). The removal of an escape route gives a strip graph with tails (see [3, Fig. 5]).

Definition 2.7. Let $n \geq 5$. For integers j and k with $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$, we define the graph $S_{n,k,j}$ as follows:

$$V(S_{n,k,j}) = V(C_n^2),$$
$$E(S_{n,k,j}) = E(C_n^2) \setminus ES(n, k, j), \quad (j, k \in \mathbb{Z},\ 0 \leq k \leq \lceil \frac{n-2}{2} \rceil),$$

where

$$ES(n, k, j) = \{f_j, f_{j+2k+1}\} \cup \{e_{j+1}, \ldots, e_{j+2k+1}\} \quad (j, k \in \mathbb{Z},\ 0 \leq k \leq \lceil \frac{n-2}{2} \rceil).$$

The graph $S_{n,k,j}$ is called a *strip graph with tails*, and $ES(n, k, j)$ is called the *escape route*.

The graphs $S_{n,k,j}$ are connected spanning convex subgraphs of C_n^2. Clearly, C_n^2 and $(\mathbb{Z}_n, \mathcal{W}(n))$ for n odd are also connected spanning subgraphs of C_n^2, and we call these subgraphs *trivial* connected spanning subgraphs.

For a graph G, let T_G be the set of all spanning trees of G. Then $t(G) = |T_G|$. Since $S_{n,k,j}$ can be obtained from the strip graph S_{n-2k} by attaching two tails of length k, the following lemma holds.

Lemma 2.8. *For $j, k \in \mathbb{Z}$, $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$, $t(S_{n,k,j}) = t(S_{n-2k})$.*

3 Spanning convex subgraphs

In this section, we prove our first main result which gives a classification of connected spanning convex subgraphs of C_n^2.

Lemma 3.1. *Let G be a connected spanning convex subgraph of C_n^2. If k and p are integers with $0 \le p < n$ and*

$$\{e_{k-1}, f_k, f_{k+2}, \dots, f_{k+2p-2}, e_{k+2p}\} \subseteq E(G),$$

then $\{e_k, e_{k+1}, \dots, e_{k+2p-1}\} \subseteq E(G)$.

Proof. We prove the assertion by induction on p. If $p = 0$, then it is trivial. Therefore, we may assume that $p \ge 1$.

Suppose that there exists an integer i with $0 \le i \le 2p-1$ such that $e_{k+i} \in E(G)$. If i is even, then since G is convex with respected to T_{k+i}, $e_{k+i+1} \in E(G)$. Therefore, we can apply the induction to $\{e_{k-1}, f_k, f_{k+2}, \dots, f_{k+i-2}, e_{k+i}\}$ and $\{e_{k+i+1}, f_{k+i+2}, f_{k+i+4}, \dots, f_{k+2p-2}, e_{k+2p}\}$. Similarly, if i is odd, then we can apply the induction.

It remains to derive a contradiction by assuming

$$e_k, e_{k+1}, \dots, e_{k+2p-1} \notin E(G). \tag{1}$$

Since G is convex with respect to T_{k-1},

$$f_{k-1} \notin E(G). \tag{2}$$

Similarly, since G is convex with respect to T_{k+2p-1}

$$f_{k+2p-1} \notin E(G). \tag{3}$$

From (1), (2), and (3), we see that the set $\{v_{k+1}, v_{k+3}, \dots, v_{k+2p-1}\}$ is separated from its complement in the connected spanning subgraph G. This is a contradiction. $\square$

Lemma 3.2. *Let G be a nontrivial connected spanning convex subgraph of C_n^2. If $E(G)$ contains no frame, then n is odd, and $G = S_{n,\frac{n-1}{2},j}$ for some integer j with $0 \le j \le n-1$.*

Proof. By the assumption, $E(G)$ consists only of windows. Since G is connected, n is odd. Since G is nontrivial, $|E(G)| \le n-1$. Since G is connected, $|E(G)| \ge n-1$. Therefore, $|E(G)| = n-1$. Then there exists j such that $E(G) = \mathcal{W}(n) \setminus \{f_j\} = E(S_{n,\frac{n-1}{2},j})$. This proves $G = S_{n,\frac{n-1}{2},j}$. $\square$

Lemma 3.3. *Let G be a nontrivial connected spanning convex subgraph of C_n^2. If $E(G)$ contains a frame, then $G = S_{n,k,j}$ for some integers j, k with $0 \le j \le n-1$, $0 \le k \le \lfloor \frac{n-2}{2} \rfloor$.*

Proof. If $\mathcal{F}(n) \subset E(G)$, then it is easy to see that $G = C_n^2$, contradicting the assumption that G is nontrivial. Since $\mathcal{F}(n) \cap E(G) \neq \emptyset$, there exists i, l with $0 \leq i \leq n-1$, $1 \leq l \leq n-1$ satisfying $\{e_i, e_{i+1}, \dots, e_{i+l-1}\} \subseteq E(G)$ and $e_{i-1}, e_{i+l} \notin E(G)$. Without loss of generality, we may assume that $i = 0$. In this case, we have

$$\{e_0, e_1, \dots, e_{l-1}\} \subseteq E(G), \tag{4}$$

$$e_{-1} \notin E(G), \tag{5}$$

$$e_l \notin E(G). \tag{6}$$

Since G is convex with respected to T_j $(0 \leq j \leq l-2)$, (4) implies

$$f_0, f_1, \dots, f_{l-2} \in E(G). \tag{7}$$

Since G is convex with respected to T_{-1}, (4) and (5) imply

$$f_{-1} \notin E(G). \tag{8}$$

Since G is convex with respect to T_{l-1}, (4) and (6) imply

$$f_{l-1} \notin E(G). \tag{9}$$

Let s and t be the largest non-negative integers such that

$$f_{-2}, f_{-4}, \dots, f_{-2s} \in E(G), \tag{10}$$

and

$$f_l, f_{l+2}, \dots, f_{l+2t-2} \in E(G), \tag{11}$$

respectively. Then, $f_{-2s-2} \notin E(G)$ and $f_{l+2t} \notin E(G)$.

We show that

$$e_l, e_{l+1}, \dots, e_{l+2t} \notin E(G) \text{ and } t < \frac{n-l}{2}, \tag{12}$$

$$e_{-1}, e_{-2}, \dots, e_{-2s-1} \notin E(G) \text{ and } s < \frac{n-l}{2}. \tag{13}$$

Assume that there exists an integer m with $0 \leq m \leq 2t$ and that $e_{l+m} \in E(G)$. We may choose minimal such m. By (6), we have $m > 0$. If m is odd, then by (11) and by the convexity of G, $T_{l+m-1} \subseteq E(G)$. Therefore, $e_{l+m-1} \in E(G)$. This contradicts the minimality of m. If m is even, then by (4) and (11), we have $\{e_{l-1}, f_l, f_{l+2}, \dots, f_{l+m-1}, e_{l+m}\} \subseteq E(G)$. Then, by Lemma 3.1, we have $e_{l+m-1} \in E(G)$, again contradicting the minimality of m. Therefore, (12) holds. Similarly, we can prove (13).

Let $K = \{v_{-2s}, v_{-2s+2}, \dots, v_0, v_1, \dots, v_l, v_{l+2}, \dots, v_{l+2t}\}$. If $K \neq \mathbb{Z}_n$, then by (8), (9), (12) and (13), G is disconnected. This is a contradiction. Therefore, $K = \mathbb{Z}_n$, and in particular, $s + l + 1 + t = |K| = n$. From (12) and (13), $s = t = \frac{n-l-1}{2}$. Then, from (10), (11) and (12), we have

$$\{f_{l+1}, f_{l+3}, \dots, f_{n-2}\} \subseteq E(G),$$
$$\{f_l, f_{l+2}, \dots, f_{n-3}\} \subseteq E(G),$$
$$e_l, e_{l+1}, \dots, e_{n-1} \notin E(G),$$

respectively. Together with (4), (7), (8) and (9), these imply $E(G) = E(S_{n,\frac{n-l-1}{2},l-1})$. This proves $G = S_{n,\frac{n-l-1}{2},l-1}$. $\square$

Theorem 3.4. *Let G be a nontrivial connected spanning convex subgraph of C_n^2. Then there exists integers j, k with $0 \leq j \leq n-1$, $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$ such that $G = S_{n,k,j}$.*

Proof. This is immediate from Lemmas 3.2 and 3.3. $\square$

4 Enumerating spanning trees of the square cycles

In this section, we prove our second main result which states that every spanning tree of C_n^2 is contained in a unique connected spanning convex subgraph. As a consequence, we obtain an alternative proof of the formula for the number of spanning trees of C_n^2. Our method is a combinatorial formulation of the topological proof given in [3]. The tool we use is the theory of graph homotopy. We refer the reader to [4] for the precise definition of the homotopy group. Roughly speaking, the homotopy group $\pi(G, v_0)$ of the graph G with respect to a vertex v_0 is the group formed by equivalence classes of circuits through v_0. It contains the subgroup $\pi(G, v_0, 3)$ which is "generated" by triangles. It is clear that $\pi(G, v_0) = \pi(G, v_0, 3)$ if G is a tree, strip graph, or strip graph with tails, while $\pi(G, v_0) \neq \pi(G, v_0, 3)$ if G is a cycle of length at least 4 or $G = C_n^2$ with $n \geq 7$.

Theorem 4.1. *Let n be an integer with $n \geq 5$. For every spanning tree G of C_n^2, there exists a unique nontrivial connected spanning convex subgraph H of C_n^2 such that $E(G) \subseteq E(H)$. Each such graph H has a form $S_{n,k,j}$ for $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$, $0 \leq j \leq n-1$. More precisely,*

$$T_{C_n^2} = \bigcup_{k=0}^{\lceil \frac{n-2}{2} \rceil} \bigcup_{j=0}^{n-1} T_{S_{n,k,j}} \quad \textit{(disjoint)}. \tag{14}$$

Proof. Since the assertion can be verified directly for $n = 5$ and 6, we assume $n \geq 7$. According to Lewis [4], for a graph G we can define its homotopy group $\pi(G, v_0)$ and the normal subgroup $\pi(G, v_0, 3)$ of $\pi(G, v_0)$ generated by the triangles. Clearly $\pi(G, v_0)$ is the trivial group for the spanning tree G of C_n^2, so in particular $\pi(G, v_0) = \pi(G, v_0, 3)$ holds. For a spanning tree G of C_n^2 which is not convex with respect to some triangle T_i, $\pi(G', v_0) = \pi(G', v_0, 3)$ also holds for the graph G' obtained from G by adding the unique missing edge of T_i. This process can be iterated until we reach a convex subgraph containing G. The resulting graph H is a connected spanning convex subgraph H of C_n^2, and hence it is one of the graphs classified in Theorem 3.4, or one of the trivial connected spanning convex subgraph. Since $\pi(H, v_0) = \pi(H, v_0, 3)$ holds only for nontrivial connected spanning convex subgraph H, there exist j, k with $0 \leq j \leq n-1$, $0 \leq k \leq \lceil \frac{n-2}{2} \rceil$ such that $E(G) \subseteq E(S_{n,k,j})$.

It remains to show that the union in (14) is disjoint. Suppose $E(G) \subseteq E(S_{n,k',j'})$ for some j', k' with $0 \leq k' \leq \lceil \frac{n-2}{2} \rceil$, $0 \leq j' \leq n-1$. Then the subgraph with edge set

$E(S_{n,k,j}) \cap E(S_{n,k',j'})$ is a nontrivial connected spanning convex subgraph of C_n^2, and hence coincides with $S_{n,k',j'}$ for some j'', k'' with $0 \leq k'' \leq \lceil \frac{n-2}{2} \rceil$, $0 \leq j'' \leq n-1$. This implies $E(S_{n,k'',j''}) \subseteq E(S_{n,k,j})$ which is possible only when $(j,k) = (j'',k'')$. Then we have $(j,k) = (j',k')$. Therefore, the union in (14) is disjoint. $\square$

Corollary 4.2 (Kleitman and Golden [3])**.**

$$t(C_n^2) = nF_n^2.$$

Proof.

$$\begin{aligned}
t(C_n^2) &= \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} \sum_{j=0}^{n-1} t(S_{n,k,j}) && \text{(by Theorem 4.1)} \\
&= n \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} t(S_{n-2k}) && \text{(by Lemma 2.8)} \\
&= \begin{cases} n \sum_{k=0}^{(n-2)/2} t(S_{2k+2}) & \text{if } n \text{ is even,} \\ n + n \sum_{k=1}^{(n-1)/2} t(S_{2k+1}) & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} n \sum_{k=0}^{(n-2)/2} F_{4k+2} & \text{if } n \text{ is even,} \\ n(1 + \sum_{k=1}^{(n-1)/2} F_{4k}) & \text{if } n \text{ is odd} \end{cases} && \text{(by Lemma 2.5)} \\
&= nF_n^2 && \text{(by Lemma 2.6).}
\end{aligned}$$

$\square$

Remark. We have verified by computer that the assertion of Theorem 4.1 holds for the graphs C_{10}^3 and C_{11}^3 (see [8] for a definition), if we modify the definition of trivial connected spanning subgraphs to be the ones whose homotopy group is nontrivial. There are exactly 63 and 96 nontrivial connected convex subgraphs up to automorphism of C_{10}^3 and C_{11}^3, respectively, and every spanning tree is contained in a unique nontrivial connected convex subgraph.

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