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Group divisible designs $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$

Atthakorn Sakda

Faculty of Science and Technology
Surat Thani Rajabhat University
Surat Thani 84100, Thailand
`atthakorn.sak@sru.ac.th`

Chariya Uiyyasathian*

Department of Mathematics and Computer Science
Faculty of Science, Chulalongkorn University
Bangkok 10330, Thailand
`chariya.u@chula.ac.th`

Abstract

In 2017 the current authors gave a complete solution for the existence problem of group divisible designs (or PBIBDs) with block size $k = 3$ with four groups of sizes $(n, n, n, 1)$ and any two indices (λ_1, λ_2). In this paper, we solve the problem for four groups of sizes $(n, n, n, 2)$ with any two indices (λ_1, λ_2). The key elements in our construction of these group divisible designs are quasigroups with holes, while the construction in the original paper used orthogonal latin squares and maximal packings.

1 Introduction

Group divisible designs are pairwise balanced incomplete block designs (PBIBDs) where the set of symbols is partitioned into groups with two different associates. Formally, a *group divisible design* $\mathsf{GDD}(g = g_1 + g_2 + \cdots + g_s, s, k; \lambda_1, \lambda_2)$ is an ordered pair $(G, \mathcal{B})$ where G is a g-set of symbols that is partitioned into s sets, called *groups*, of sizes $g_1, g_2, \ldots, g_s$, and $\mathcal{B}$ is a collection of k-subsets of G, called *blocks*, such that each pair of symbols from the same group appears together in exactly λ_1 blocks, and each pair of symbols from distinct groups appears together in exactly λ_2 blocks. Such pairs of elements in the same group are known to statisticians as *first associates*, and pairs of elements from the different groups are called *second associates*. (See [4, 5].) The parameters λ_1 and λ_2 are referred to as the *first index* and *second index*,

* Corresponding author

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respectively. The interest in the existence of group divisible designs has been strong over the years, leading Colbourn et al. in [3] to state that "Group divisible designs are one of the most basic ingredients in the construction of combinatorial designs of many types; consequently much effort has gone into the construction of large classes of GDDs".

Many papers in the literature have focused on the designs with $k = 3$. When all groups are of the same size, the existence of $\mathsf{GDD}(g = n + n + \cdots + n, s, 3; \lambda_1, \lambda_2)$ was completely solved for any pair of λ_1 and λ_2 by Fu, Rodger and Sarvate [4, 5]. The problem becomes more complicated when not all groups have the same size. We mention some notable results in this particular case. Colbourn, Hoffman, and Rees [3] provided a necessary and sufficient condition for the existence of $\mathsf{GDD}(g = n+n+\cdots+n+t, s+1, 3; 0, 1)$. When $\lambda_1 \geq \lambda_2$ the existence of $\mathsf{GDD}(g = n+m, 2, 3; \lambda_1, \lambda_2)$ had been solved for $m \neq 2$, $n \neq 2$, as seen in the work by Pabhapote et al. [2, 11, 12, 14, 15] in 2009–2012 using Steiner triple systems and related designs. Generally, the problem becomes difficult if there is a group of size less than the size of the blocks. Therefore, when $k = 3$, it is often challenging to construct designs with some groups of size 1 or 2. In 2013, Chaffee and Rodger [1] provided a complete solution of the design with two groups $\mathsf{GDD}(g = n + m, 2, 3; \lambda_1, \lambda_2)$ for $m = 2$ or $n = 2$, using a classic result of Colbourn and Rosa on quadratic leaves. For designs with three groups, Hurd and Sarvate [7] made some progress concerning the existence of $\mathsf{GDD}(g = n + 2 + 1, 3, 3; \lambda_1, \lambda_2)$ when $\lambda_1 \geq \lambda_2$. Subsequently, Lapchinda et al. provided a solution for the existence of $\mathsf{GDD}(g = n+n+1, 3, 3; \lambda_1, \lambda_2)$ for all (λ_1, λ_2), as seen in [8, 9]. Designs with four groups, where one group has a different size from the others, are also worth discussing, as one can construct these designs using Latin squares, which generally does not work for other cases. $\mathsf{GDD}(g = n + n + n + 1, 4, 3; \lambda_1, \lambda_2)$s are constructed using orthogonal Latin squares and maximal packings, as presented in [13]. In this paper, we solve the existence problem of $\mathsf{GDD}(g = n + n + n + 2, 4, 3; \lambda_1, \lambda_2)$ for all (λ_1, λ_2). Unlike in [13], orthogonal Latin squares do not work; here, we present a new construction technique that uses idempotent quasigroups with holes. These quasigroups with holes can also be viewed as Latin squares with holes.

Since we are dealing only with GDDs having four groups and block size three, for the sake of brevity, $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ is used to present $\mathsf{GDD}(g = n + n + n + 2, 4, 3; \lambda_1, \lambda_2)$ from this point forward, and we refer to blocks as *triples*. Our necessary conditions for the existence problem of a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ can be easily obtained from a graph model by describing it graphically as follows. Let λK_v denote the graph with v vertices, where each pair of vertices is joined by λ edges. Consider two graphs, G_1 and G_2. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition of a graph H* is a partition of the edges of H such that each element of the partition induces a copy of G. The existence of a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ is equivalent to the existence of a K_3-decomposition of $\lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} \lambda_1 K_2))$, with the blocks or triples of this design being the triangles K_3 in the graph decomposition. We provide necessary conditions for the existence of a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ in Lemma 3.1, obtained straightforwardly from a graphical point of view. Furthermore, the following main theorem establishes that these necessary conditions are also sufficient, providing a complete solution to the existence problem of our designs.

Theorem 1.1 (Main Theorem). *Let $n \geq 1$ and $\lambda_1, \lambda_2 \geq 0$ be integers. There exists a* $\mathsf{GDD}(n, n,\ n,\ 2;\ \lambda_1, \lambda_2)$ *if and only if*

(*i*) $\lambda_1 = \lambda_2 = 0$, *or*
(*ii*) $\lambda_2 \neq 0, n = 2, 6 \mid \lambda_1$ *and* $\lambda_1 \leq 3\lambda_2$, *or*
(*iii*) $\lambda_2 \neq 0, n \neq 2, 2 \mid (\lambda_1 + n\lambda_2), 3 \mid \lambda_1$ *and* $\lambda_1 \leq 3n\lambda_2$.

Using our main result, we can similarly apply the construction of a $\mathsf{GDD}(g = n + \cdots + n + 1, t + 1, 3; \lambda_1, \lambda_2)$ as outlined in the last section of [13] to construct a $\mathsf{GDD}(g = n + \cdots + n + 2, t + 1, 3; \lambda_1, \lambda_2)$ where t is divisible by 3. Furthermore, if a $\mathsf{GDD}(g = n + n + 2, 3, 3; \lambda_1, \lambda_2)$ also exists, we obtain the GDDs where $t \equiv 1, 2 \pmod 3$ as well.

2 Preliminary Background

In this section, we describe the two main tools used in our construction of GDDs, namely quasigroups with holes and triple systems. A *triple system* $\mathsf{TS}(n, \lambda)$ of index λ and order n is an ordered pair $(S, \mathcal{T})$, where S is an n-set, and $\mathcal{T}$ is a collection of 3-subsets of S called triples or blocks, such that each pair of distinct elements of S appears together in λ triples. A triple system $\mathsf{TS}(n, \lambda)$ can be also considered as a $\mathsf{GDD}(g = n, 1, 3; \lambda, \lambda_2)$ or a $\mathsf{GDD}(g = 1 + 1 + 1 + \cdots + 1, n, 3; \lambda_1, \lambda)$ when λ_1 and λ_2 are nonnegative integers. The existence of the triple systems is assured by Theorem 2.1. See more details in [6, 10].

Theorem 2.1. [6] *Let n be any positive integer and λ a nonnegative integer. Then a* $\mathsf{TS}(n, \lambda)$ *exists if and only if λ and n satisfy either one of the followings:*

(*i*) $\lambda \equiv 0 \pmod 6$ *for all positive integers* $n \neq 2$,
(*ii*) $\lambda \equiv 1$ *or* $5 \pmod 6$ *for all n with* $n \equiv 1$ *or* $3 \pmod 6$,
(*iii*) $\lambda \equiv 2$ *or* $4 \pmod 6$ *for all n with* $n \equiv 0$ *or* $1 \pmod 3$, *and*
(*iv*) $\lambda \equiv 3 \pmod 6$ *for all odd integers.*

A *quasigroup* of order n is a pair $(Q, \circ)$, where Q is a set of size n and $\circ$ is a binary operation on Q such that for each pair of elements $a, b \in Q$, both equations $a \circ x = b$ and $y \circ a = b$ have a unique solution. Let $P \subseteq Q$; then $(P, \circ)$ is a *subquasigroup* of $(Q, \circ)$ provided $(P, \circ)$ is also a quasigroup. If $i \circ i = i$ for all $i \in Q$, then $(Q, \circ)$ is called an *idempotent quasigroup.* A quasigroup is just a Latin square with a headline and a sideline. ([10], p. 4.) It is well known that for every positive integer n there exists a quasigroup of order n, and there exists an idempotent quasigroup of order n for all n, $n \neq 2, 6$. i (See Chapter 8 in [16].) Let $Q = \{1, 2, \ldots, 2k\}$ and let $\mathcal{H} = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k - 1, 2k\}\}$. In what follows, the two-element subsets $\{2i - 1, 2i\} \in \mathcal{H}$ are called *holes.* A *quasigroup with holes* $\mathcal{H}$ is a quasigroup $(Q, \circ)$ of order $2k$ in which for each $h \in \mathcal{H}$, the pair $(h, \circ)$ is a subquasigroup of $(Q, \circ)$. We list two relevant results used in this paper. For further details, see [10] and [16].

Theorem 2.2. [16] *If $(P, \circ)$ is a quasigroup of order s and $n \geq 2s$, then there exists a quasigroup $(Q, \circ)$ of order n with $(P, \circ)$ as a subquasigroup.*

Theorem 2.3. [10] *For all $k \geq 3$ there exists a quasigroup of order $2k$ with the holes $\mathcal{H} = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\}$.*

Theorems 2.2 and 2.3 can also be interpreted as statements about Latin squares.

3 Proof of Theorem 1.1

We begin by establishing necessary conditions for the existence of the designs. We remark that edges joining vertices from the same group are called *pure* edges, while edges joining vertices from different group are called *cross* edges.

Lemma 3.1 (Necessary Conditions)**.** *Let $n \geq 1$ and $\lambda_1, \lambda_2 \geq 0$ be integers. If there exists a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ then*

(i) $2 \mid (\lambda_1 + n\lambda_2)$,
(ii) $3 \mid \lambda_1$,
(iii) $\lambda_1 \leq 3n\lambda_2$, *and*
(iv) *if* $n = 2$ *then* $\lambda_1 \leq 3\lambda_2$.

Proof. Let $G = \lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} (\lambda_1 K_n \vee_{\lambda_2} \lambda_1 K_2))$. Since G has a K_3-decomposition, each vertex has even degree, and the number of edges is divisible by three. It is worth noting that G contains vertices of degree $(n-1)\lambda_1 + (2n+2)\lambda_2$ and $\lambda_1 + 3n\lambda_2$. The number of edges is $(3\frac{n(n-1)}{2} + 1)\lambda_1 + 3(n^2 + 2n)\lambda_2$. These observations imply that $2 \mid (\lambda_1 + n\lambda_2)$ and $3 \mid \lambda_1$, which correspond to conditions (i) and (ii).

Now, let W be the group of size two in the given design. Each pure edge in W must be contained in a triangle with two other cross edges. Thus, the number of pure edges in W must be at most half of the number of cross edges, satisfying condition (iii). Similarly, if all four groups have size two, condition (iv) holds due to the condition on the pure edges in all groups. $\square$

We devote the rest of the paper to show that the conditions in Lemma 3.1 are sufficient. Unless stated otherwise, for any positive integer n, we let $X_n = \{x_1, x_2, \ldots, x_n\}$, $Y_n = \{y_1, y_2, \ldots, y_n\}$, $Z_n = \{z_1, z_2, \ldots, z_n\}$, and $W = \{w_1, w_2\}$ be disjoint sets, and define $V_n = X_n \cup Y_n \cup Z_n \cup W$. Note that a collection of blocks may have repeated blocks; thus, the union symbol "$\cup$" in our construction will be used for the union of multi-sets.

If $\lambda_2 = 0$, then the two elements in any group of size 2 cannot be in any triple. Therefore, there is no $\mathsf{GDD}(n, n, n, 2; \lambda_1, 0)$ for any n if $\lambda_1 > 0$. Thus, Theorem 1.1(i) holds; so, we consider only a construction of a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ where $\lambda_2 > 0$. Moreover, it can be noted from Lemma 3.1 that we have an extra necessary condition (iv) when $n = 2$. Therefore, we will consider the case $n = 2$ separately. Lemma 3.2 implies that the condition in Theorem 1.1(ii) is sufficient.

Lemma 3.2. *Let $\lambda_1 \geq 0$ and $\lambda_2 > 0$ be integers such that $\lambda_1 \leq 3\lambda_2$ and $6|\lambda_1$. Then there exists a* $\mathsf{GDD}(2, 2, 2, 2; \lambda_1, \lambda_2)$.

Proof. Let λ_1 and λ_2 be such that $6|\lambda_1$ and $\lambda_1 \leq 3\lambda_2$. Then there exist nonnegative integers x, y such that $\lambda_1 = 6x$ and $\lambda_2 = 2x+y$, and hence, we can construct a desired $\mathsf{GDD}(2, 2, 2, 2; 6x, 2x + y)$ by using x copies of all triples of a $\mathsf{GDD}(2, 2, 2, 2; 6, 2)$ and y copies of all triples of a $\mathsf{GDD}(2, 2, 2, 2; 0, 1)$. Thus it suffices to construct a $\mathsf{GDD}(2, 2, 2, 2; 6, 2)$ and a $\mathsf{GDD}(2, 2, 2, 2; 0, 1)$.

Let $\mathcal{B}_1$ be the collection of all triples of V_2 containing two elements from the one group and one element from the other, and let $\mathcal{B}_2 = \{\{w_1, x_1, y_2\}, \{w_1, y_1, z_2\}, \{w_1, z_1, x_2\}, \{w_2, x_2, y_1\}, \{w_2, y_2, z_1\}, \{w_2, z_2, x_1\}, \{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$. It can be verified that $(V_2, \mathcal{B}_1)$ is a $\mathsf{GDD}(2, 2, 2, 2; 6, 2)$ and $(V_2, \mathcal{B}_2)$ is a $\mathsf{GDD}(2, 2, 2, 2; 0, 1)$. □

Now we consider the construction of a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$ wherever $n \neq 2$ and $(n, \lambda_1, \lambda_2)$ satisfies all necessary conditions in Lemma 3.1. Lemmas 3.8–3.9 and Corollary 3.10 are used to prove Theorem 1.1(*iii*). We first construct several special cases for small GDDs in Lemmas 3.3 through 3.7, where the technique used here differs slightly from the construction of larger GDDs in general cases.

Let $n \neq 2$ and $(n, \lambda_1, \lambda_2)$ satisfy our necessary conditions in Lemma 3.1. We begin by considering the case $\lambda_2 = 1$. It follows that $\lambda_1 \leq 3n$ by condition (*iii*). Moreover, the necessary conditions (*i*) and (*ii*) imply the following two statements.

If n is even, then $\lambda_1 \equiv 0 \pmod 6$, and thus $\lambda_1 \equiv 3n \pmod 6$.
If n is odd, then λ_1 is odd and $\lambda_1 \equiv 0 \pmod 3$, and thus $\lambda_1 \equiv 3n \pmod 6$.

Hence for $\lambda_2 = 1$, we can always write $\lambda_1 = 3n - 6t$ for some $t \in \{0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In particular, if $\lambda_2 = 1$ and $n \leq 5$, then (n, λ_1) must be one of the following:

$$\mathcal{S} = \{(1, 3), (3, 3), (3, 9), (4, 0), (4, 6), (4, 12), (5, 3), (5, 9), (5, 15)\}.$$

Lemma 3.3 will construct the desired GDDs where $(n, \lambda_1) \in \{(1, 3), (3, 9), (4, 12), (5, 15)\}$, while GDDs with the rest of (n, λ_1) in $\mathcal{S}$ will be constructed in Lemmas 3.4–3.7.

Lemma 3.3. *There exists a* $\mathsf{GDD}(n, n, n, 2; 3n, 1)$ *for all positive integer $n \neq 2$.*

Proof. Let $(Q, \circ)$ be a quasigroup of order n where $Q = \{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, n\}$, let

$$\mathcal{B}_i = \{\{w_1, w_2, x_i\}, \{w_1, w_2, y_i\}, \{w_1, w_2, z_i\}\}, \text{ and}$$
$$\mathcal{C}_i = \{\{x_i, y_1, z_{i\circ 1}\}, \{x_i, y_2, z_{i\circ 2}\}, \ldots, \{x_i, y_n, z_{i\circ n}\}\}.$$

By Theorem 2.1(*i*) and (*iv*), there exists a $\mathsf{TS}(n, 3n)$. Let $(X_n, \mathcal{T}_1)$, $(Y_n, \mathcal{T}_2)$ and $(Z_n, \mathcal{T}_3)$ be those $\mathsf{TS}(n, 3n)$'s obtained from the theorem. Then $(V_n, \mathcal{B} \cup \mathcal{C} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3)$, where $\mathcal{B}$ is the union of all $\mathcal{B}_i$ and $\mathcal{C}$ is the union of all $\mathcal{C}_i$, is the desired GDD. □

Lemma 3.4. *There exist a* $\mathsf{GDD}(3,3,3,2;3,1)$ *and a* $\mathsf{GDD}(4,4,4,2;0,1)$.

Proof. (i) $(V_3,\mathcal{B})$ is a $\mathsf{GDD}(3,3,3,2;3,1)$ where

$$\begin{aligned}\mathcal{B}=\{&\{w_1,w_2,x_3\},\{w_1,w_2,y_3\},\{w_1,w_2,z_3\},\{w_1,x_1,y_2\},\{w_1,y_1,z_2\},\{w_1,z_1,x_2\},\\&\{w_2,x_2,y_1\},\{w_2,y_2,z_1\},\{w_2,z_2,x_1\},\{x_1,x_2,x_3\},\{x_1,x_2,x_3\},\{x_1,x_2,x_3\},\\&\{x_3,y_1,z_1\},\{x_3,y_2,z_2\},\{y_3,x_1,z_1\},\{y_3,x_2,z_2\},\{z_3,x_1,y_1\},\{z_3,x_2,y_2\},\\&\{y_1,y_2,y_3\},\{y_1,y_2,y_3\},\{y_1,y_2,y_3\},\{z_1,z_2,z_3\},\{z_1,z_2,z_3\},\{z_1,z_2,z_3\},\\&\{x_3,y_3,z_3\}\}.\end{aligned}$$

(ii) $(V_4,\mathcal{C}_1\cup\mathcal{C}_2)$ is a $\mathsf{GDD}(4,4,4,2;0,1)$ where

$$\begin{aligned}\mathcal{C}_1=\{&\{w_1,x_1,y_4\},\{w_1,y_1,z_4\},\{w_1,z_1,x_4\},\{w_1,x_2,y_3\},\{w_1,y_2,z_3\},\{w_1,z_2,x_3\},\\&\{w_2,x_3,y_2\},\{w_2,y_3,z_2\},\{w_2,z_3,x_2\},\{w_2,x_4,y_1\},\{w_2,y_4,z_1\},\{w_2,z_4,x_1\}\};\\\mathcal{C}_2=\{&\{x_i,y_j,z_{(5-i-j)\bmod 5}\}:i,j\in\{1,2,3,4\}\text{ and }i+j\neq 5\}.\end{aligned}$$

□

Lemma 3.5. *There exists a* $\mathsf{GDD}(4,4,4,2;6,1)$.

Proof. Let

$$\begin{aligned}\mathcal{B}=\{&\{w_1,w_2,x_3\},\{w_1,w_2,x_4\},\{w_1,w_2,y_3\},\{w_1,w_2,y_4\},\{w_1,w_2,z_3\},\{w_1,w_2,z_4\},\\&\{w_1,x_1,y_2\},\{w_1,y_1,z_2\},\{w_1,z_1,x_2\},\{w_2,x_2,y_1\},\{w_2,y_2,z_1\},\{w_2,z_2,x_1\},\\&\{x_1,y_1,z_1\},\{x_2,y_2,z_2\}\}.\end{aligned}$$

For $Q=\{1,2,3,4\}$ and by Theorem 2.2, there exists a quasigroup $(Q,\circ)$ with subquasigroup $(P,\circ)$ where $P=\{1,2\}$. Now let

$$\mathcal{C}=\{\{x_i,y_j,z_{i\circ j}\}:i,j\in\{1,2,3,4\}\text{ and }(i,j)\notin\{(1,1),(1,2),(2,1),(2,2)\}\}.$$

By Theorem 2.1(i), there exists a $\mathsf{TS}(4,6)$. Let $(X_3,\mathcal{T}_1)$, $(Y_3,\mathcal{T}_2)$ and $(Z_3,\mathcal{T}_3)$ be those $\mathsf{TS}(4,6)$'s obtained from the theorem. Then $(V_4,\mathcal{B}\cup\mathcal{C}\cup\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3)$ is a $\mathsf{GDD}(4,4,4,2;6,1)$ as desired. □

Lemma 3.6. *There exists a* $\mathsf{GDD}(5,5,5,2;3,1)$.

Proof. Let

$$\begin{aligned}\mathcal{B}_1=\{&\{w_1,w_2,x_5\},\{w_1,w_2,y_5\},\{w_1,w_2,z_5\},\{w_1,x_3,y_1\},\{w_1,x_4,y_2\},\{w_1,y_3,z_1\},\\&\{w_1,y_4,z_2\},\{w_1,z_3,x_1\},\{w_1,z_4,x_2\},\{w_2,x_1,y_3\},\{w_2,x_2,y_4\},\{w_2,y_1,z_3\},\\&\{w_2,y_2,z_4\},\{w_2,z_1,x_3\},\{w_2,z_2,x_4\}\};\\\mathcal{B}_2=\{&\{x_5,y_1,z_4\},\{x_5,y_2,z_3\},\{x_5,y_3,z_2\},\{x_5,y_4,z_1\},\\&\{y_5,x_1,z_4\},\{y_5,x_2,z_3\},\{y_5,x_3,z_2\},\{y_5,x_4,z_1\},\\&\{z_5,x_1,y_4\},\{z_5,x_2,y_3\},\{z_5,x_3,y_2\},\{z_5,x_4,y_1\},\{x_5,y_5,z_5\}\};\\\mathcal{B}_3=\{&\{x_1,y_1,z_1\},\{x_1,y_2,z_2\},\{x_2,y_1,z_2\},\{x_2,y_2,z_1\},\\&\{x_3,y_3,z_3\},\{x_3,y_4,z_4\},\{x_4,y_3,z_4\},\{x_4,y_4,z_3\}\}.\end{aligned}$$

By Theorem 2.1(iv), there exists a $\mathsf{TS}(5,3)$. Let $(X_5,\mathcal{T}_1)$, $(Y_5,\mathcal{T}_2)$ and $(Z_5,\mathcal{T}_3)$ be those $\mathsf{TS}(5,3)$'s obtained from the theorem. Then $(V_5,\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3)$ is a $\mathsf{GDD}(5,5,5,2;3,1)$. □

Lemma 3.7. *There exists a* $\mathsf{GDD}(5,5,5,2;9,1)$.

Proof. Let

$$\begin{aligned}\mathcal{B}=\{&\{w_1,w_2,x_3\},\{w_1,w_2,y_3\},\{w_1,w_2,z_3\},\{w_1,w_2,x_4\},\{w_1,w_2,y_4\},\{w_1,w_2,z_4\},\\&\{w_1,w_2,x_5\},\{w_1,w_2,y_5\},\{w_1,w_2,z_5\},\{w_1,x_1,y_2\},\{w_1,y_1,z_2\},\{w_1,z_1,x_2\},\\&\{w_2,x_2,y_1\},\{w_2,y_2,z_1\},\{w_2,z_2,x_1\},\{x_1,y_1,z_1\},\{x_2,y_2,z_2\}\}.\end{aligned}$$

For $Q=\{1,2,3,4,5\}$ and by Theorem 2.2, there exists a quasigroup $(Q,\circ)$ of order 5 with subquasigroup $(P,\circ)$ where $P=\{1,2\}$ is of order 2. Let

$$\mathcal{C}=\{\{x_i,y_j,z_{i\circ j}\}:i,j\in\{1,2,\ldots,5\}\text{ and }(i,j)\notin\{(1,1),(1,2),(2,1),(2,2)\}\}.$$

By Theorem 2.1(iv), there exists a $\mathsf{TS}(5,9)$. Let $(X_5,\mathcal{T}_1)$, $(Y_5,\mathcal{T}_2)$ and $(Z_5,\mathcal{T}_3)$ be those $\mathsf{TS}(5,9)$'s obtained from the theorem. Then $(V_5,\mathcal{B}\cup\mathcal{C}\cup\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3)$ is a $\mathsf{GDD}(5,5,5,2;9,1)$ as desired. □

Lemma 3.8. *There exists a* $\mathsf{GDD}(n,n,n,2;\,3n-6t,1)$ *for every positive integer* $n\neq 2$ *and all nonnegative integers* $t\leq\frac{n}{2}$.

Proof. Let $n\neq 2$ be a positive integer, and $t\leq\frac{n}{2}$ a nonnegative integer. The existence of the GDDs when $n\leq 5$ is already covered by Lemmas 3.3 through 3.7. Now, assuming that $n\geq 6$, the construction is carried out separately depending on the parity of n.

Case 1. *n is even.* Let

$$\begin{aligned}\mathcal{B}_1&=\{\{w_1,w_2,x_i\},\{w_1,w_2,y_i\},\{w_1,w_2,z_i\}:i\in\{2t+1,2t+2,\ldots,n\}\};\\\mathcal{B}_2&=\{\{w_1,x_{2i-1},y_{2i}\},\{w_1,y_{2i-1},z_{2i}\},\{w_1,z_{2i-1},x_{2i}\},\\&\qquad\{w_2,x_{2i},y_{2i-1}\},\{w_2,y_{2i},z_{2i-1}\},\{w_2,z_{2i},x_{2i-1}\},\\&\qquad\{x_{2i-1},y_{2i-1},z_{2i-1}\},\{x_{2i},y_{2i},z_{2i}\}:i\in\{1,2,\ldots,t\}\}.\end{aligned}$$

According to Theorem 2.3, for $n\geq 6$, there exists a quasigroup with holes. Let $Q_1=\{1,2,\ldots,n\}$, $\mathcal{H}_1=\{\{1,2\},\{3,4\},\ldots,\{2t-1,2t\}\}$, and $(Q_1,\circ)$ be such a quasigroup of order n with holes $\mathcal{H}_1$. Define

$$\mathcal{B}_3=\{\{x_i,y_j,z_{i\circ j}\}:i,j\in Q_1,\text{ and }i\text{ and }j\text{ are not in the same hole in }\mathcal{H}_1\}.$$

By Theorem 2.1(i), there exists a $\mathsf{TS}(n,3n-6t)$. Let $(X_n,\mathcal{T}_1)$, $(Y_n,\mathcal{T}_2)$, and $(Z_n,\mathcal{T}_3)$ be those $\mathsf{TS}(n,3n-6t)$'s obtained from the theorem. Then $(V_3,\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3)$ is a $\mathsf{GDD}(n,n,n,2;\,3n-6t,1)$.

Case 2. *n is odd.* Let

$$\begin{aligned}\mathcal{C}_1 = \{&\{w_1, w_2, x_i\}, \{w_1, w_2, y_i\}, \{w_1, w_2, z_i\} : i \in \{2t+1, 2t+2, \dots, n\}\}\\ &\cup \{\{x_n, y_n, z_n\}\};\\ \mathcal{C}_2 = \{&\{w_1, x_{2i-1}, y_{2i-1}\}, \{w_1, y_{2i}, z_{2i}\}, \{w_1, x_{2i}, z_{2i-1}\},\\ &\{w_2, x_{2i}, y_{2i}\}, \{w_2, y_{2i-1}, z_{2i-1}\}, \{w_2, x_{2i-1}, z_{2i}\},\\ &\{x_n, y_{2i}, z_{2i-1}\}, \{x_n, y_{2i-1}, z_{2i}\}, \{y_n, x_{2i-1}, z_{2i-1}\},\\ &\{y_n, x_{2i}, z_{2i}\}, \{z_n, x_{2i-1}, y_{2i}\}, \{z_n, x_{2i}, y_{2i-1}\} : i \in \{1, 2, \dots, t\}\};\\ \mathcal{C}_3 = \{&\{x_n, y_{2i}, z_{2i-1}\}, \{x_n, y_{2i-1}, z_{2i}\}, \{y_n, x_{2i-1}, z_{2i}\},\\ &\{y_n, x_{2i}, z_{2i-1}\}, \{z_n, x_{2i-1}, y_{2i}\}, \{z_n, x_{2i}, y_{2i-1}\},\\ &\{x_{2i-1}, y_{2i-1}, z_{2i-1}\}, \{x_{2i}, y_{2i}, z_{2i}\} : i \in \{t+1, t+2, \dots, \tfrac{n-1}{2}\}\}.\end{aligned}$$

Let $Q_2 = \{1, 2, 3, \dots, n-1\}$, $\mathcal{H}_2 = \{\{1, 2\}, \{3, 4\}, \dots, \{n-2, n-1\}\}$, and $(Q_2, \circ)$ be a quasigroup of order $n-1$ with hole $\mathcal{H}_2$. Define

$$\mathcal{C}_4 = \{\{x_i, y_j, z_{i\circ j}\} : i, j \in Q_2, \text{ and } \{i, j\} \notin \mathcal{H}_2\}.$$

By Theorem 2.1(*iv*), there exists a $\mathsf{TS}(n, 3n-6t)$. Then let $(X_n, \mathcal{T}_4)$, $(Y_n, \mathcal{T}_5)$ and $(Z_n, \mathcal{T}_6)$ be those $\mathsf{TS}(n, 3n-6t)$'s obtained from the theorem. Thus $(V_3, \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{T}_4 \cup \mathcal{T}_5 \cup \mathcal{T}_6)$ is a $\mathsf{GDD}(n, n, n, 2; 3n-6t, 1)$. □

Lemma 3.9. *There exists a* $\mathsf{GDD}(n, n, n, 2;\ 0, \lambda)$ *for every positive integer* $n \neq 2$ *and every even integer* λ.

Proof. It suffices to construct a $\mathsf{GDD}(n, n, n, 2; 0, 2)$ because a $\mathsf{GDD}(n, n, n, 2; 0, \lambda)$ can be constructed by using $\frac{\lambda}{2}$ copies of all blocks of a $\mathsf{GDD}(n, n, n, 2; 0, 2)$.
For $i \in \{1, 2, \dots, n\}$, let

$$\mathcal{B}_i = \{\{w_1, x_i, y_i\}, \{w_1, y_i, z_i\}, \{w_1, z_i, x_i\}, \{w_2, x_i, y_i\}, \{w_2, y_i, z_i\}, \{w_2, z_i, x_i\}\};$$

$Q = \{1, 2, \dots, n\}$, and $(Q, \circ)$ be an idempotent quasigroup of order n. Define

$$\mathcal{C} = \{\{x_i, y_j, z_{i\circ j}\} : i, j \in Q \text{ and } i \neq j\}\}.$$

Then $(V_n, \mathcal{B} \cup \mathcal{C} \cup \mathcal{C})$, where $\mathcal{B}$ is the union of all $\mathcal{B}_i$, is a $\mathsf{GDD}(n, n, n, 2; 0, 2)$ as desired. □

Lemmas 3.8 and 3.9 lead to the following corollary, which is crucial for the proof of the main result.

Corollary 3.10. *Let* $n \neq 2$ *be a positive integer.*
(*i*) *If n is even, then there exists a* $\mathsf{GDD}(n, n, n, 2;\ 6s, 1)$ *where* $s \in \{0, 1, 2, \dots, \frac{n}{2}\}$. *There exists a* $\mathsf{GDD}(n, n, n, 2;\ 0, \lambda)$ *for any even n and nonnegative integer* λ.
(*ii*) *If n is odd, then there exist a* $\mathsf{GDD}(n, n, n, 2; 6s+3, 1)$ *where* $s \in \{0, 1, 2, \dots, \frac{n-1}{2}\}$ *and a* $\mathsf{GDD}(n, n, n, 2;\ 6s, 2)$ *where* $s \in \{0, 1, 2, \dots, n\}$.

Proof. (i) Since n is even and $6s \leq 3n$, we have $3n - 6s \equiv 0 \pmod 6$. Write $3n - 6s = 6t$, and so $6s = 3n - 6t$. By Lemma 3.8 when $t \leq \frac{n}{2}$, there exists a $\mathsf{GDD}(n, n, n, 2; 6s, 1)$. The second statement follows immediately by setting $s = 0$.

(ii) Since n is odd and $6s + 3 \leq 3n$, we have $3n - (6s + 3) \equiv 0 \pmod 6$. Write $3n - (6s + 3) = 6t$, and so $6s + 3 = 3n - 6t$. Again by Lemma 3.8 when $t \leq \frac{n-1}{2}$, there exists a $\mathsf{GDD}(n, n, n, 2; 6s + 3, 1)$. Moreover, by choosing any s_1 and s_2 in $\{0, 1, 2, \ldots, \frac{n-1}{2}\}$ such that $s = s_1 + s_2$, we can construct a $\mathsf{GDD}(n, n, n, 2; 6s, 2)$, where $1 \leq s \leq n-1$, from a $\mathsf{GDD}(n, n, n, 2; 6s_1+3, 1)$ and a $\mathsf{GDD}(n, n, n, 2; 6s_2+3, 1)$. Together with Lemma 3.3 and Lemma 3.9, there exists a $\mathsf{GDD}(n, n, n, 2; 6s, 2)$ where $0 \leq s \leq n$ as desired. □

Finally, we are ready for the construction of our designs when $n \neq 2$. The construction in Lemma 3.11 guarantees that the condition in Theorem 1.1(iii) is sufficient, which completes the proof of Theorem 1.1. This yields a complete solution for the existence of a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$.

Lemma 3.11. *Let $n \neq 2$ be a positive integer, and $\lambda_1 \geq 0$ and $\lambda_2 > 0$ be nonnegative integers such that $2 \mid (\lambda_1 + n\lambda_2)$, $3 \mid \lambda_1$ and $\lambda_1 \leq 3n\lambda_2$. Then there exists a $\mathsf{GDD}(n, n, n, 2; \lambda_1, \lambda_2)$.*

Proof. Write $\lambda_1 = q(3n) + r$, where $0 \leq q$ and $0 \leq r < 3n$. By Lemma 3.1(iii), we have $q(3n) + r \leq 3n\lambda_2$, and so $\lambda_2 \geq q + \frac{r}{3n}$. Note that we can construct a $\mathsf{GDD}(n, n, n, 2; q(3n), q)$ by using q copies of a $\mathsf{GDD}(n, n, n, 2; 3n, 1)$, which exists by Lemma 3.3. Thus, it remains to construct a $\mathsf{GDD}(n, n, n, 2; r, \lambda_2 - q)$.

Case $r = 0$. The construction depends on the parity of n. If n is even, by Corollary 3.10(i), there exists a $\mathsf{GDD}(n, n, n, 2;\ 0, \lambda_2 - q)$. Now, assume that n is odd. Since $\lambda_1 = q(3n)$, we have $\lambda_1 \equiv q \pmod 2$. According to Lemma 3.1$(i)$, $\lambda_1 \equiv \lambda_2 \pmod 2$, therefore we have $\lambda_2 \equiv q \pmod 2$. Hence, $\lambda_2 - q$ is even. By Lemma 3.9, there exists a $\mathsf{GDD}(n, n, n, 2; 0, \lambda_2 - q)$ as desired.

Case $r > 0$. Since $\lambda_2 \geq q + \frac{r}{3n}$, we have $\lambda_2 \geq q + 1$. For an even n, we have $\lambda_1 \equiv 0 \pmod 6$ by Lemma 3.1(i) and (ii). It follows that $r = \lambda_1 - q(3n) \equiv 0 \pmod 6$. By Corollary 3.10$(i)$, there exist a $\mathsf{GDD}(n, n, n, 2; r, 1)$ and a $\mathsf{GDD}(n, n, n, 2; 0, \lambda_2 - (q+1))$ as desired. Now, assume that n is odd. Since $\lambda_1 = q(3n) + r \equiv 0 \pmod 3$ by Lemma 3.1(ii), we have $r \equiv 0, 3 \pmod 6$. If $r \equiv 3 \pmod 6$, then $\lambda_1 \equiv \lambda_2 = q(3n) + r \equiv q + r \equiv q + 1 \pmod 2$. Hence, by Corollary 3.10(ii), there exist a $\mathsf{GDD}(n, n, n, 2; r, 1)$ and a $\mathsf{GDD}(n, n, n, 2; 0, \lambda_2 - (q + 1))$. If $r \equiv 0 \pmod 6$, then $\lambda_2 \equiv \lambda_1 \equiv q(3n) \equiv q \pmod 2$. Furthermore, since $\lambda_2 \geq q + 1$, we have $\lambda_2 \geq q + 2$, and $\lambda_2 - (q + 2) \equiv 0 \pmod 2$. Hence, by Corollary 3.10(ii), there exist a $\mathsf{GDD}(n, n, n, 2; r, 2)$ and a $\mathsf{GDD}(n, n, n, 2; 0, \lambda_2 - (q + 2))$. Therefore, for any cases, there exists a $\mathsf{GDD}(n, n, n, 2; r, \lambda_2 - q)$ as desired. □

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