

List-edge-coloring triangulations with maximum degree at most five

JOSHUA HARRELSON

Department of Mathematics and Statistics
Middle Georgia State University
Macon, GA 31206, U.S.A.
joshua.harrelson@mga.edu

JESSICA McDONALD*

Department of Mathematics and Statistics
Auburn University
Auburn, AL 36849, U.S.A.
mcdonald@auburn.edu

Abstract

We prove that triangulations with maximum degree at most 5 satisfy the List-Edge-Coloring Conjecture.

1 Introduction

An *edge list assignment* for a graph G is a function L that assigns to each edge $e \in E(G)$ a list of colors $L(e)$. Given such an L , an *L -edge-coloring* of G is a (proper) edge-coloring of G such that every edge e is given a color from $L(e)$. Note that the classical notion of a k -edge-coloring of G can be viewed as an L -edge-coloring for the list assignment L defined by $L(e) = \{1, \dots, k\}$ for all $e \in E(G)$. We say a graph G is *k -list-edge-colorable* if it is L -edge-colorable for every edge list assignment L such that $|L(e)| \geq k$ for all $e \in E(G)$. The *list-chromatic index* of G , denoted $\chi'_\ell(G)$, is the minimum k such that G has a k -list-edge coloring. We immediately get that $\chi'_\ell(G) \geq \chi'(G) \geq \Delta$ for every graph G , where $\chi'(G)$ is the chromatic index of G (the minimum k such that G is k -edge-colorable), and $\Delta := \Delta(G)$ is the maximum degree of G .

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In this paper we consider every graph to be simple, hence Vizing’s Theorem [25] says that $\chi'(G) \leq \Delta + 1$ for all graphs G . Vizing [24] conjectured that this upper bound also holds for list-edge colorings.

Conjecture 1.1 (Vizing [24]). *If G is a graph, then $\chi'_\ell(G) \leq \Delta + 1$.*

Conjecture 1.1 has been verified for all graphs with $\Delta \leq 4$. The $\Delta = 3$ case was proved by Vizing [24] in 1976 and independently by Erdős, Rubin, and Taylor [11] in 1979. The $\Delta = 4$ case of Conjecture 1.1 was proved in 1998 by Juvan, Mohar, Škrekovski [17]. Since there are graphs with $\Delta = 3$ and $\Delta = 4$ having $\chi'(G) = \Delta + 1$, these results are tight. However, in general, we may hope for more than Conjecture 1.1. The famous List-Edge Coloring Conjecture (LECC), which follows, has been attributed to many sources, some as early as 1975 (see e.g. [16]).

Conjecture 1.2 (LECC). *If G is a graph, then $\chi'(G) = \chi'_\ell(G)$.*

The LECC is true for a number of special families, most notably bipartite graphs due to Galvin [12] in 1995 (see also [18] for an extension). As an example of how far away this conjecture still is however, consider that it has not yet been established for all even cliques (odd cliques were established by Häggkvist and Janssen [13], and cliques of order equal to a prime plus one were established by Schauz [22]).

More is known about Conjectures 1.1 and 1.2 for planar graphs, where both edge-coloring and list-edge-coloring are somewhat simpler. While it is NP-complete to decide whether a graph has chromatic index Δ or $\Delta + 1$ (Holyer [15]), all planar graphs with $\Delta \geq 7$ have $\chi'(G) = \Delta$ (Sanders and Zhao [20], Zhang [26]). We may therefore expect $\chi'_\ell(G) = \Delta$ for all planar graphs G with $\Delta \geq 7$, but this has only been established for $\Delta \geq 12$ (by Borodin, Kostochka and Woodall [4]). Conjecture 1.1 has been pushed further, and we now know it holds for planar graphs when $\Delta \geq 8$ (Bonamy [2]). This leaves the planar case of Conjecture 1.1 open for graphs with $5 \leq \Delta \leq 7$. In this paper we show that the stronger LECC holds for $\Delta \leq 5$ when G is a triangulation (i.e. when all faces are triangles).

Theorem 1.3. *If G is a triangulation with $\Delta \leq 5$, then $\chi'_\ell(G) = \chi'(G)$.*

Most of the above-mentioned papers utilize the discharging method in their proofs. A particularly insightful proof by Cohen and Havet [8] highlights the trouble that triangular faces cause in such arguments (their proof shows Conjecture 1.1 for $\Delta \geq 9$, which was previously established by Borodin [3]). Theorem 1.3 deals with these troublesome cases for $\Delta \leq 5$. In fact, there are not so many such cases: in Section 2 of this paper we will show that there are exactly twelve triangulations with $\Delta \leq 5$, eight of which have $\Delta = 5$. In Section 3 we note that some of these graphs are known to satisfy the LECC and we explain how the recent work of Dvořák [9] can be used to computationally show the remaining graphs also satisfy the List-Edge Coloring Conjecture.

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$ V_3 $	4	3	3	2	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0
$ V_4 $	0	0	1	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5	6
$ V_5 $	0	3	1	6	4	2	0	9	7	5	3	1	12	10	8	6	4	2	0

Table 1: degree sequences of $|V_3|, |V_4|, |V_5|$ satisfying $12 = 3|V_3| + 2|V_4| + |V_5|$.

2 Twelve triangulations with $\Delta \leq 5$

Triangulations with $\Delta \leq 5$ are special in that only finitely many degree sequences are possible, via Euler’s formula. For what follows we let $V_i = \{v \in V(G) : \deg(v) = i\}$.

Lemma 2.1. *If G is a triangulation with $\Delta \leq 5$, then either $G = K_3$ or $V(G) = V_3 \cup V_4 \cup V_5$ and $12 = 3|V_3| + 2|V_4| + |V_5|$.*

Proof. The only triangulation with a vertex of degree 2 or less is the triangle K_3 so, we may assume that $V(G) = V_3 \cup V_4 \cup V_5$. Since G is a triangulation, Euler’s formula gives us $|E(G)| = 3|V(G)| - 6$. Applying the degree-sum formula this yields the following:

$$12 = 6|V(G)| - \sum_{v \in V(G)} \deg(v)$$

Substituting $|V(G)| = |V_3| + |V_4| + |V_5|$ and $\sum_{v \in V(G)} \deg(v) = 3|V_3| + 4|V_4| + 5|V_5|$ gives our desired result. \square

Note that if we apply the argument of Lemma 2.1 to triangulations with vertices of degree 6, then the $|V_6|$ term will cancel, meaning that $|V_6|$ would be unrestricted.

There are only 19 linear combinations of $|V_3|, |V_4|,$ and $|V_5|$ that satisfy the equation in Lemma 2.1, and they are listed in Table 1. We will hereafter refer to each of these potential degree sequences by their case number, as listed in this table. Note that cases 8 and 14 are impossible since they each contain a set of 5 vertices of degree 5 all of which must be adjacent to one another, forcing a copy of K_5 . In the remainder of this section, we will provide arguments to eliminate all but 11 of the remaining cases, and then show that each of these cases corresponds to a unique triangulation, with case i corresponding to graph G_i in Figure 1. We define \mathcal{G} as the set of these 11 graphs together with K_3 . Specifically, given the pictures in Figure 1, we define $\mathcal{G} = \{G_i : i \in \{0, 1, 4, 6, 7, 11, 13, 15, 16, 17, 18, 19\}\}$.

Theorem 2.2. \mathcal{G} is the set of all triangulations with $\Delta(G) \leq 5$.

It is important to note that there is an alternative to the argument we are about to present for Theorem 2.2: it can also be derived by computer, using the results of Brinkmann and McKay on *plantri* (see [6]).

We start our work towards Theorem 2.2 with the following observation and lemma.

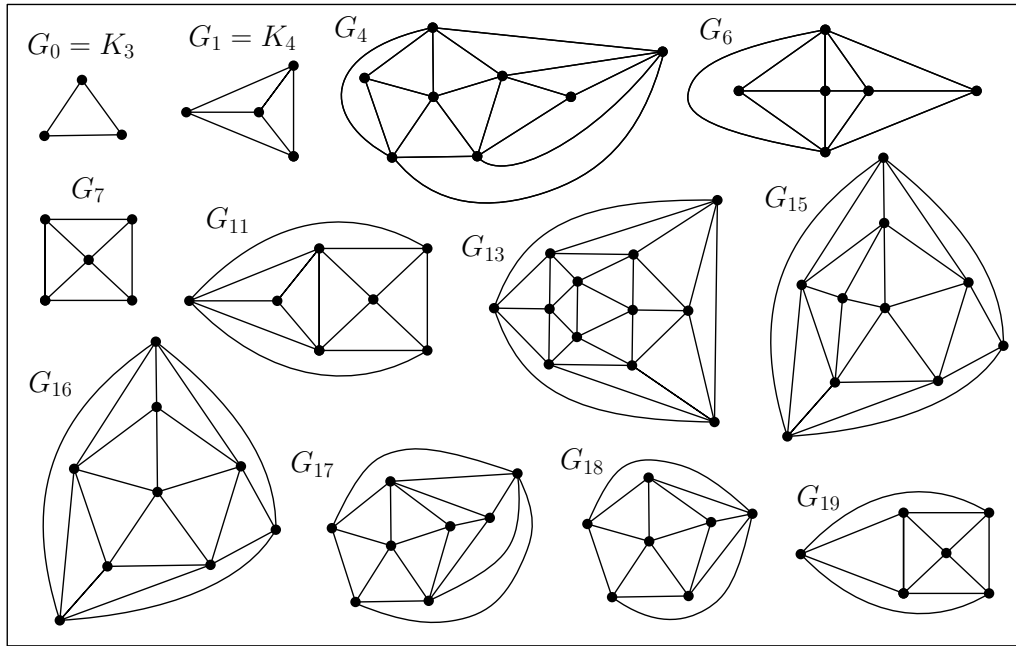


Figure 1: The set \mathcal{G} of all triangulations with $\Delta \leq 5$.

Observation 2.3. Let G be a triangulation with path $P = \{x, y, z\}$ for $x, y, z \in V(G)$. If xy and yz lay in the same face, then $x \sim z$.

Lemma 2.4. Let G be a triangulation with $\Delta(G) = 5$ and let $v \in V_3$. Then $N(v)$ contains no vertices of degree 3 and at most one vertex of degree 4. Moreover, if $u \in N(v) \cap V_4$, then u, v have two common neighbours $a, b \in V_5$, and u has one other neighbour c , such that a, b, c induces a triangle in G . See Figure 2.

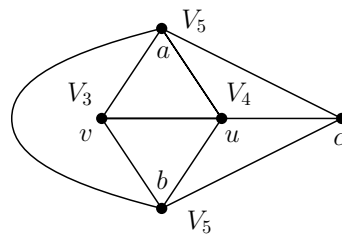


Figure 2: Adjacent vertices $v \in V_3$ and $u \in V_4$, as described in Lemma 2.4.

Proof. Let $v \in V_3$ and let $G' = G - v$. Since G is a triangulation we know $N(v)$ induces a triangle, so G' is also a triangulation.

Suppose first that v has some neighbor $u \in V_3$ in G . This means $deg_{G'}(u) = 2$. Since G' is a triangulation it must be that $G' = K_3$. However, this implies that $G = K_4$, which does not have maximum degree 5.

Suppose now that v has some neighbor $u \in V_4$ in G . Let $N_G(v) = \{u, a, b\}$. Since G is a triangulation, $N_G(v)$ induces a triangle, and hence $N_G(u) = \{v, a, b, c\}$ for

some other vertex c . Moreover, since u has degree 4, edges bu, uc (and au, uc) are on the same face, meaning that $ca, cb \in E(G)$ by Observation 2.3. Also, since v has degree 3, edges av, vb are on the same face, and hence $ab \in E(G)$ by Observation 2.3. So we have precisely the situation pictured in Figure 2, save that we have not yet argued that $a, b \in V_5$.

Let $G'' = G' - u = G - \{v, u\}$. We know G' is a triangulation and that $\text{deg}_{G'}(u) = 3$. This means G'' is also a triangulation. If either of a or b , has degree 4 in G , then they have degree 2 in G'' . This would imply that $G'' = K_3$, and hence that G has no vertex of degree 5, contradiction. \square

In any triangulation, the neighbourhood of a vertex x contains a cycle on which its neighbours appear, say in clockwise order, in the embedding around x . The following lemma says something about this ordering.

Lemma 2.5. *Let G be a triangulation with $\Delta(G) = 5$, and let u, v be non-adjacent vertices in G . If u, v have a common neighbour y , then they must have a second common neighbour z , and moreover y, z must be consecutive neighbours in both the clockwise embedding of $N(u)$ and of $N(v)$.*

Proof. If u, v have precisely one common neighbour y , then y has degree at least 6, owing to its adjacencies to u, v , to its two neighbours on the cycle around $N(u)$, and similarly in $N(v)$. See, for example, the left picture in Figure 3. In fact, even if u, v have a second common neighbour z , y will still have degree at least 6, unless z is consecutive to y on both the cycles around $N(u)$ and $N(v)$. See the right picture in Figure 3. \square

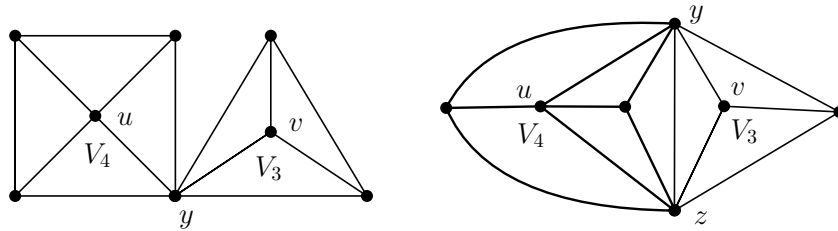


Figure 3: Two situations considered in Lemma 2.5.

Lemma 2.6. *Let G be a triangulation with $\Delta(G) = 5$ and let $v \in V_3, u \in V_4$ with $v \not\sim u$. Then either $N(u) \cap N(v) = \emptyset$, or G is one of the triangulations G_6, G_{11} pictured in Figure 1.*

Proof. If $N(u) \cap N(v) = \emptyset$, then we are done so assume $N(u) \cap N(v) \neq \emptyset$. By Lemma 2.5, $|N(u) \cap N(v)| \in \{2, 3\}$. Suppose first that $|N(u) \cap N(v)| = 2$, say with common neighbors a, b and with c the third neighbor of v . We may assume by Lemma 2.5 that $a \sim b, a \sim c$, and $b \sim c$ with a, b, c the clockwise order of the embedding of $N(v)$. Let x, y be the other neighbors of u ; we may assume by Lemma

2.5 that $a \sim b, a \sim x, x \sim y, y \sim b$ with a, b, y, x the counter-clockwise order of the embedding of $N(u)$. See the left picture in Figure 4. Both a, b have five neighbors among $N[v] \cup N[u]$, and hence edges ac, ax are on the same face, and edges bc, cy are on the same face. So by Observation 2.3, $cx, cy \in E(G)$. However now c also has 5 neighbours among $N[v] \cup N[u]$. Since c, x, y is a triangle and G is a triangulation, this means there are no other vertices or edges in the graph, and indeed the only possible triangulation is G_{11} pictured in Figure 1.

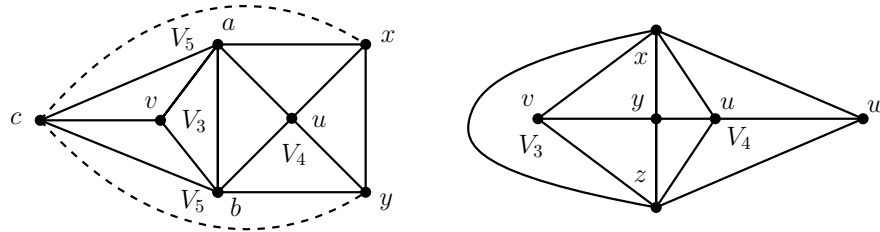


Figure 4: Two situations considered in the proof of Lemma 4.

We may now assume that $|N(u) \cap N(v)| = 3$, say with common neighbors x, y, z . See the right picture in Figure 4. Then one of these vertices, say y , is forced to have degree 4 in G , and the other two, say x, z , have five neighbors among $N(v) \cap N(u)$. Moreover, if w is the other neighbor of u , then x, z, w forms a triangle. Since G is a triangulation, this again means there are no other vertices or edges in the graph, and indeed the only possible triangulation is G_6 pictured in Figure 1. \square

Proposition 2.7. *If G is a triangulation with $\Delta(G) = 5$ then G has a degree sequence corresponding to one of the following eight cases: 4, 6, 11, 13, 15, 16, 17, 18.*

Proof. Given our discussion following Table 1 (eliminating cases 8 and 14), and since cases 1, 7, 19 do not involve vertices of degree 5, it remains only for us to eliminate cases 2, 3, 5, 9, 10 and 12 as possible degree sequences.

Lemma 2.4 says that every 3-vertex is adjacent to at least two 5-vertices. Both case 3 and case 12 have a 3-vertex, but only one 5-vertex, which makes this impossible. Case 2 can also be eliminated, since there $|V_3| = |V_5| = 3$ (and $|V_4| = 0$), so Lemma 2.4 implies that G contains a copy of $K_{3,3}$.

We may now assume, for a contradiction, that G has a degree sequence corresponding to case 5, 9, or 10. In each case we can choose a pair of vertices $v \in V_3$ and $u \in V_4$.

Suppose first that $v \sim u$. Then we get the situation described by Lemma 2.4 and depicted in Figure 2 with $a, b, c \in V(G)$. Let $S = \{u, v, a, b, c\}$. Between S and $V(G) \setminus S$ we will have one edge each from a and b , and up to two edges from c , for a total of at most four edges.

In case 5, $V(G) \setminus S$ consists of exactly two vertices (either two vertices of degree 5 or one vertex of degree 5 and one vertex of degree 3), and their degree requires at least $4 + 2 = 6$ edges from S , contradiction. In case 9, $V(G) \setminus S$ consists of 4 vertices,

all of which have degree 5, so they need a total of at least $4(5 - 3) = 8$ edges from S , contradiction. In case 10, $V(G) \setminus S$ consists of exactly three vertices (either all of degree 5, or two of degree 5 and one of degree 4), and their degree requires at least $3 + 3 + 2 = 8$ edges from S , contradiction.

We may now assume that $v \not\sim u$. Since neither of G_6 nor G_{11} has a degree sequence corresponding to case 5, 9, or 10, we know by Lemma 2.6 that $N(v) \cap N(u) = \emptyset$. This means that $N[v] \cup N[u]$ contains 9 different vertices in G , immediately disqualifying cases 5 or 10 (which have a total of 7 and 8 vertices, respectively). Case 9 has exactly 9 vertices, meaning that $N[v] \cup N[u] = V(G)$. Moreover, all vertices besides u, v have degree 5 in this case. We know that $N(u)$ contains a cycle, since G is a triangulation. Each of the four vertices in $N(u)$ must have two additional neighbors, in addition to u and its two neighbors on this cycle. By planarity, this means that the cycle in $N(u)$ is induced. However then, in order to satisfy the degree requirements of vertices in $N(u)$, there must be exactly $4(2) = 8$ edges between $N(u)$ and $N(v)$. On the other hand, the vertices of $N(v)$ require exactly $3(2) = 6$ edges, contradiction. \square

We can now establish Theorem 2.2.

Proof. (Theorem 2.2) First suppose that $\Delta(G) < 5$. By Lemma 2.1 this means that either $G = K_3 \in \mathcal{G}$ or the degree sequence of G corresponds to one of the cases $\{1, 7, 19\}$. If G has the degree sequence of case 1, then it must be K_4 . If G has the degree sequence of case 7 or case 19, then there is a $v \in V(G)$ such that $\deg(v) = 4$. We know v forms a 4-wheel with its neighbors since G is a triangulation. If G has the degree sequence of case 7, then there are no vertices outside this 4-wheel and joining two nonconsecutive neighbors of v will yield $G_7 \in \mathcal{G}$. If G has the degree sequence of case 19, then there is 1 vertex outside the 4-wheel which must be joined to all neighbors of v to yield $G_{19} \in \mathcal{G}$.

We may now assume that G is a triangulation with $\Delta(G) = 5$; by Proposition 2.7 this means the degree sequence of G corresponds to one of the following eight cases: 4, 6, 11, 13, 15, 16, 17, 18. It is already known that the icosahedron (which is $G_{13} \in \mathcal{G}$) is the unique embedding of a 5-regular triangulation, and we won't repeat this argument here.

Suppose that G has the degree sequence of case 6 or case 11. Then we have $|V_3| \geq 1$, $|V_4| \geq 2$, and $v(G) \in \{6, 7\}$. Let $v \in V_3$; by Lemma 2.4 we know there is a $u \in V_4$ such that $u \not\sim v$. In particular, this means that we cannot have $N(v) \cap N(u) = \emptyset$ (since that would require at least $4+5=9$ vertices in G). So by Lemma 2.6, $G \in \{G_6, G_{11}\} \subseteq \mathcal{G}$.

Now suppose that G has the degree sequence of case 4, 15, 16, 17, or 18. Note that $|V(G)| \in \{7, 8, 9, 10\}$. Choose $w \in V_5$ and let w_1, \dots, w_5 be the clockwise neighbours of w , appearing along a cycle C . Since $|V(G)| > 6$ there exists $u \in V(G)$ with $u \not\sim w$. If u, w have no common neighbours then $N[u]$ accounts for an extra 4 to 6 vertices in the graph. However, this is only possible with $|V(G)| = 10$ and $\deg(u) = 3$. These two values cannot happen simultaneously since the only case with $V_3 \neq 0$

case 4, and it has only 8 total vertices. So u and w must have common neighbors. By Lemma 2.5 we may assume that, without loss of generality, $u \sim w_1, w_2$, and moreover, $N(w) \cap N(u) = \{w_1, w_2, \dots, w_i\}$ for some $2 \leq i \leq 5$.

Suppose first that $i = 5$. Then u, w both have degree 5, w_1, \dots, w_5 have degree 4, and G must look like the graph $G_{18} \in \mathcal{G}$.

We may now assume that $2 \leq i \leq 4$. Then w_1 and w_i have degree 5 in $N[w] \cap N[u]$ due to their adjacencies to u, w and to the two vertices consecutive to them around both $N(u)$ and $N(w)$. We also know w_2, \dots, w_{i-1} (which is possibly an empty list), all have degree 4. See, for example, Figure 5.

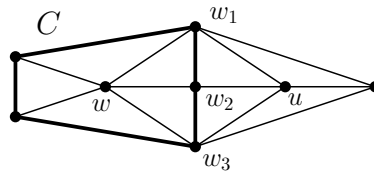


Figure 5: An example where $i = 3$.

Suppose now that we are in case 4. Since $V_4 = \emptyset$, we know that $i = 2$. We claim that we can choose our afore-mentioned $u \not\sim w$ so that $u \in V_3$. The only issue with this would be if C contains both V_3 vertices, which we claim cannot happen. Otherwise, let U be the two vertices of G not in $N[w]$. We would have at most 6 edges between C and U (at most two from each of the three 5-vertices on C). On the other hand, U consists of two 5-vertices in this case, meaning that there are at least 8 edges between U and C , contradiction. So, we may indeed choose $u \not\sim w$ with $u \in V_3$. Let $N(u) = \{w_1, w_2, x\}$ for some $x \in V(G)$. See Figure 6(a). Since w_1 has degree 5, w_5w_1 and w_1x lie on the same face, so by Observation 2.3, $w_5x \in E(G)$. Similarly, $w_3x \in E(G)$. Since now x has degree 5 and w_5, w_3 have degree at least 4, it must be that w_4 is the other vertex of degree 3 in G . But then w_3w_4, w_4w_5 lie on the same face, so $w_3w_5 \in E(G)$, and G must look like the graph $G_4 \in \mathcal{G}$.

We may now assume we are in case 15, 16, 17, or 18. We claim that either we may choose our previously discussed $u \not\sim w$ so that $u \in V_4$, or that we have $G = G_{18}$. Since each vertex of degree 4 can be adjacent to at most four vertices of degree 5, having $(|V_4|, |V_5|)$ be $(2, 8)$ or $(3, 6)$ (as in Cases 15 and 16) means this choice of u is certainly possible. Having this value be $(4, 4)$ as in case 17 also means this is possible, since while two vertices of degree 4 may share an identical neighborhood of 4 vertices of degree 5, planarity will not allow for our vertices of degree 4 to have such identical neighborhoods. In Case 18 it could be possible for the two 5-vertices to have identical neighborhoods, consisting of five 4-vertices, but then we get exactly G_{18} . So we indeed have our claim.

We may now choose the vertex $u \not\sim w$ with $u \in V_4$. So $i \in \{2, 3, 4\}$. Suppose first that $i = 4$. Then w_1, w_4 have degree 5, so w_4w_5, w_5w_1 are on the same face, and hence w_5 has degree 3. See Figure 6(b). But this contradicts the fact that we are in one of cases 15, 16, 17, or 18, where $V_3 = \emptyset$. Suppose next that $i = 3$. Let $N(u) = \{w_1, w_2, w_3, x\}$ for some $x \in V(G)$. See Figure 6(c). Then w_1 has degree 5,

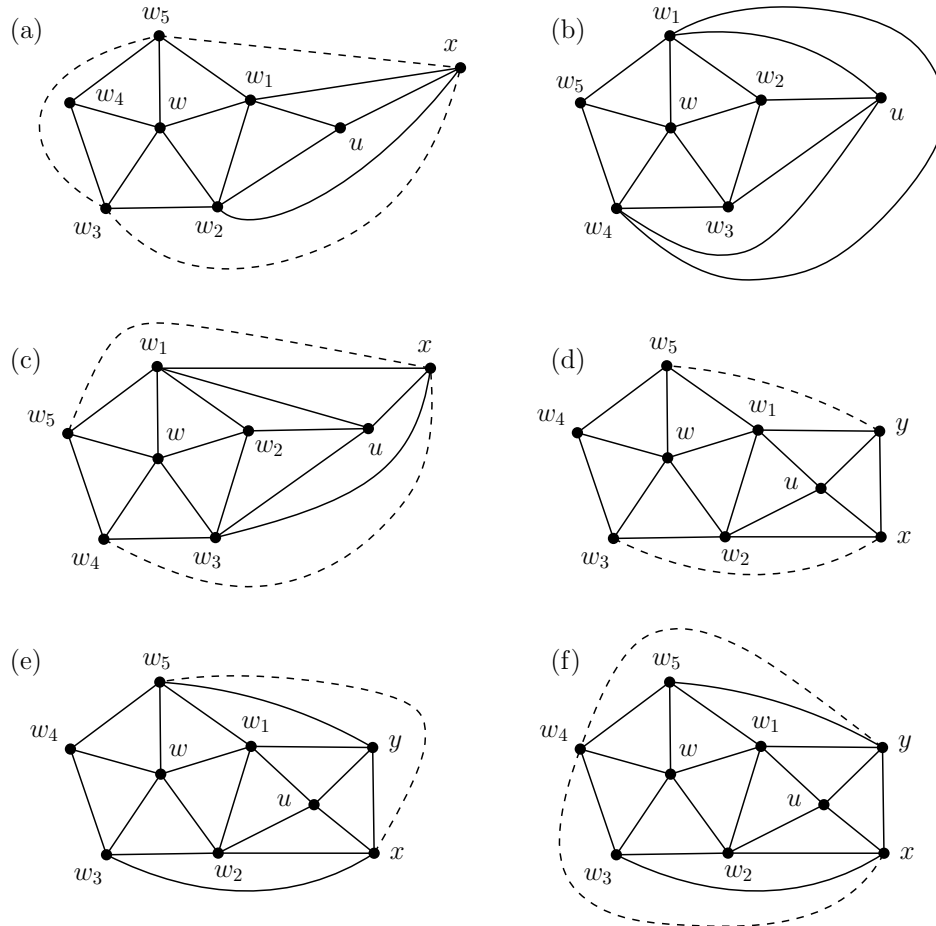


Figure 6: Various situations considered in the proof of Theorem 2.2.

so w_5w_1, w_1x are on the same face, meaning $w_5x \in E(G)$. Similarly, since w_4 has degree 5, $w_4x \in E(G)$. We now have x of degree 5, and x, w_5, w_4 forming a triangle in G . Hence $V(G) = N[u] \cup N[v]$ and G looks exactly like $G_{17} \in \mathcal{G}$.

We may now assume that $i = 2$. Let $N(u) = \{w_1, w_2, x, y\}$ with this being the counter-clockwise order around u . Then w_1, w_2 have degree 5, so $yw_5, xw_3 \in E(G)$. See Figure 6(d). Note that $|N[u] \cup N[w]| = 9$, meaning that we must be in either case 15 (10 vertices) or case 16 (9 vertices).

Suppose first that we are in case 15, meaning there exists one more vertex $z \in V(G) \setminus N[u] \cup N[w]$. Note that, by our above arguments, we may in fact assume that all non-adjacent 4-vertices and 5-vertices in G share exactly two common neighbors. This means that if z has degree 4, it must be adjacent to two neighbours of w_1 and two neighbours of w_2 , meaning its neighbour set is precisely w_5, y, x, w_3 . But then this forces w_4 to be degree 3 in G , contradiction. Hence z must have degree 5, with

neighbors w_5, y, x, w_3, w_4 . But then G must be exactly the graph $G_{15} \in \mathcal{G}$.

Suppose finally that we are in case 16, meaning that $V(G) = N[u] \cup N[w]$. Since there are no degree 3 vertices in case 16, we may assume wlog that $x \sim w_4$. See Figure 6(f). Since there are exactly three 4-vertices in case 16, exactly one of w_4, w_5, y must have degree 4. Since w_5 is already adjacent to w_4, y , it must be w_5 that has degree 4, and w_4, y must be adjacent. Hence G looks exactly like $G_{16} \in \mathcal{G}$. \square

3 List-edge-coloring the triangulations

Let G be a graph with $E(G) = \{e_1, \dots, e_m\}$. Define a polynomial p_G on the variables x_i for $1 \leq i \leq m$ as

$$p_G = \prod_{e_i \sim e_j, i < j} (x_i - x_j),$$

where by $e_i \sim e_j$ we mean that these edges are adjacent in G . Given some p_G and some $f : E(G) \rightarrow \mathbb{N}$, by $[x^f]p_G$ we mean the coefficient of the monomial in p_G where x_i has exponent $f(e_i)$, for $1 \leq i \leq m$. Given some $e \in E(G)$, by $[x^{f+1_e}]p_G$ we mean the same thing as $[x^f]p_G$ except with the exponent of the variable corresponding to e increased by one. These quantities are significant when considering the list-edge-colorability of a graph, as evidenced by the following theorems.

Theorem 3.1 (Alon and Tarsi [1]). *Let G be a graph and let $f : E(G) \rightarrow \{0, 1, \dots, k - 1\}$. If $[x^f]p_G \neq 0$, then G is k -list-edge-colorable.*

Theorem 3.2 (Dvořák [9]). *Let G be a graph, let L be an edge-list-assignment with $|L_e| = k$ for all $e \in E(G)$, and let $f : E(G) \rightarrow \{0, 1, \dots, k - 1\}$ be such that $\sum_{e \in E(G)} f(e) = \sum_{v \in V(G)} \binom{\deg(v)}{2} - 1$. If G is not L -edge-colorable, then for every color c ,*

$$\sum_{e \in E(G)} [x^{f+1_e}]p_G \cdot \chi_{L,c}(e) = 0,$$

where $\chi_{L,c}(e)$ is 1 if $c \in L_e$ and 0 otherwise.

It is worth noting that Theorems 3.1 and 3.2 are simplified versions of what appeared in [1] and [9], tailored to suit our needs here. In particular, these are the line-graph versions of more general theorems about vertex-coloring, and in the more general versions the lists do not need to have a constant size. Despite this simplification, there are two fundamental challenges in applying Theorem 3.1 successfully: one must actually compute the relevant coefficients of the polynomial in question and, if all such coefficients are zero, we do not get any information on whether the graph is k -list-edge-colorable or not. In [9], Dvořák helps on both of these fronts by providing an algorithm to find the coefficients of Theorem 3.1 that is more efficient in practice than direct computation which also incorporates Theorem 3.2 when needed.

Note that in order to get $[x^f]p_G \neq 0$ in Theorem 3.1, we must have $\sum_{e \in E(G)} f(e) = \sum_{v \in V(G)} \binom{\deg(v)}{2}$, since every term in p_G will have this same total degree. However, if

all such functions f result in $[x^f]p_G = 0$ (i.e. Theorem 3.1 gives no information), then Theorem 3.2 tells us to look at f with total sum one less to potentially get valuable information. In this situation each monomial considered by $[x^{f+1e}]p_G$ will have the same total degree as before, but one exponent may be as large as k . Theorem 3.2 uses the coefficients of these monomials to produce a system of linear equations in the variables $\chi_{L,c}(e)$ for all $e \in E(G)$ (with one equation for each appropriate f), which can be used to determine information about a potentially “bad” list-assignment L . Such potentially bad L would then be output by Dvořák’s algorithm. Of course, if there are only a small number of possibly bad L output by the algorithm, then it may be possible to directly check these to determine whether the graph is k -list-edge-colorable or not. In the end, we will see that this outcome does not happen when we apply Dvořák’s algorithm to the particular graphs we are interested in.

Theorem 3.3. *If G is a triangulation with $\Delta \leq 5$, then $\chi'_l(G) = \chi'(G)$.*

Proof. By Theorem 2.2, \mathcal{G} is the set of all triangulations with $\Delta \leq 5$. Some of these graphs are already known to satisfy the LECC, in particular $G_0 = K_3$, $G_1 = K_4$, and the icosahedron, G_{13} (Ellingham and Goddyn [10]). Since $\chi'_l(K_6) = 5$ is known (Cariolaro, Cariolaro, Schauz, and Sun [7]) and $G_6 \subset K_6$ with $\chi'(G_6) \geq \Delta(G_6) = 5$, we immediately get that G_6 satisfies the LECC.

It remains now for us to verify the LECC for the graphs in $\mathcal{G} \setminus \{G_0, G_1, G_6, G_{13}\}$. Of these, only two (G_7 and G_{19}) have maximum degree 4 (see Figure 1), and we will show they are 4-list-edge-colorable. The rest of the graphs have maximum degree 5, and we will show they are all 5-list-edge-colorable, except for G_{16} . Since G_{16} has 9 vertices and 21 edges, $\chi(G_{16}) \geq \frac{21 \cdot 2}{8} > 5$; we will show that G_{16} is 6-list-edge-colorable.

The two graphs we plan to show are 4-list-edge-colorable, G_7 and G_{19} , have the property that $G_7 \subset G_{19}$ (see Figure 1). Therefore it suffices to just verify that G_{19} is 4-list-edge-colorable. Similarly, we claim that two of the graphs we plan to show are 5-list-edge-colorable, G_4 and G_{11} , satisfy $G_{11} \subset G_4$. To see this, first note that G_4 has $|V_3| = 2$ and $|V_5| = 6$ (see Table 1) and its vertices of degree 3 are non-adjacent (see Figure 1). By deleting either vertex of degree 3 in G_4 , we get another triangulation with $|V_3| = 1, |V_4| = 3, |V_5| = 3$, which by Theorem 2.2 must be G_{11} . Therefore in order to show that G_4 and G_{11} are 5-list-edge-colorable, it suffices to just check G_4 .

We now apply Dvořák’s afore-mentioned algorithm to G_i for each $i \in \{4, 15, 16, 17, 18, 19\}$. This algorithm is available at

<https://gitlab.mff.cuni.cz/dvorz9am/alon-tarsi-method>,

with a link also included in the abstract of [9]. The algorithm can be compiled using g++ on Linux. Since the algorithm is intended to consider list-vertex-coloring, we must take the line graph of each of G_i for each $i \in \{4, 15, 16, 17, 18, 19\}$ before inputting them into the algorithm. For this preprocessing step, we label the vertices of our graphs as shown in Figure 7 and use Algorithm 1 printed in Appendix A. This

algorithm is run in Sage [19] by taking each of our labeled graphs and our desired list-size for each graph as an input. The output of Algorithm 1 is the required line graph input for Dvořák’s algorithm, in the correct format, so we can apply it at that point. It is worth noting that we used Dvořák’s code verbatim except for changing his parameter MAX_VERTICES from 21 to 24 to accommodate our larger graphs.

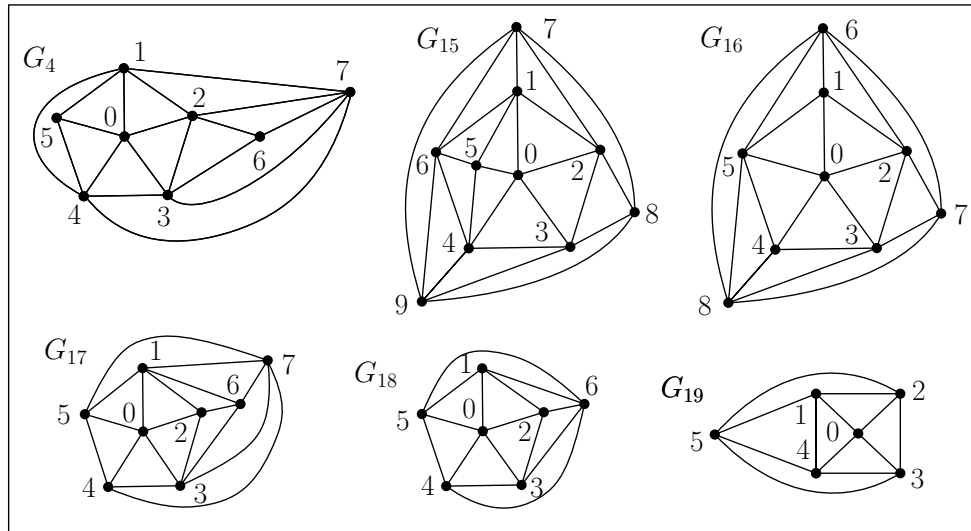


Figure 7: Some labelled triangulations from \mathcal{G} .

For each G_i with $i \in \{4, 15, 16, 17, 18, 19\}$, the input and output of Algorithm 1, as well as the printed output of Dvořák’s algorithm, are listed a Computations B.1 through B.6 in Appendix B. In the output of Algorithm 1 (i.e. input to Dvořák’s algorithm), note that we have used the endl ine symbol to denote an actual line break, just for space efficiency in this paper. In each of the computations, the output of Dvořák’s algorithm is: “Always colorable (using standard Alon–Tarsi)”. \square

As a final note, we remark that Dvořák’s algorithm has succeeded where two other approaches failed. When trying to determine whether the graphs \mathcal{G} satisfied the list-edge-coloring conjecture, the present authors first employed the so-called “kernel method,” initiated by Galvin [12] and generalized by Borodin, Kostochka, and Woodall ([4] and [5]). Included in the first author’s PhD thesis [14] are the details of using this method to show that G_{11} , G_{17} , and G_{18} satisfy Vizing’s conjecture, but this method did not work for G_4 , G_{15} , G_{16} , and G_{19} even to verify this weaker version of LECC. Also included in this dissertation are computations that show that some of these graphs satisfy the LECC. These computations use an implementation by Schauz [21], based on his extension [23] of Theorem 3.1. Schauz’s algorithm is for regular graphs only, but by embedding our graphs in regular graphs, one can show that the LECC holds for G_4 , G_{11} , G_{16} , G_{17} , and G_{19} . Schauz’s algorithm was also able to show that G_{15} and G_{18} satisfy Vizing’s conjecture, but it fell short to show they satisfy LECC.

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A Algorithm 1

Algorithm 1

```

edges = [(0,1),(0,2),(0,3),(0,4),(1,2),(2,3),(3,4),(1,4),(1,5),(2,5),
         (3,5),(4,5)]
listsize = 5

G = Graph(edges)
edges_dict = {}
counter = 0
for edge in G.edges(labels=False):
    edges_dict[edge] = counter
    counter += 1

H=G.line_graph(labels=False)
H.edges(labels=False)

h_prime_edges = []
for edge in H.edges(labels=False):
    v_1 = edge[0]
    v_2 = edge[1]
    n_1 = edges_dict[v_1]
    n_2 = edges_dict[v_2]
    h_prime_edges.append((n_1,n_2))

H_prime = Graph(h_prime_edges)
print(len(H_prime.vertices()), len(H_prime.edges()))
M=[listsize for n in range(0,len(H_prime.vertices()))]
print(*M)
for edge in H_prime.edges(labels=False):
    print('{} {}'.format(edge[0],edge[1]))

```

B Computations

Computation B.1 (G_4).

Algorithm 1 edges: $[(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(2,3),(3,4),(4,5),(1,5),(2,6),(3,6),(6,7),(2,7),(3,7),(1,7),(4,7)]$

Algorithm 1 listsize: 5

Algorithm 1 output:17 58 \\5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 \\0 1 \\0 2 \\0 3 \\0 4 \\0 5 \\0 6 \\0 7 \\1 2 \\1 3 \\1 4 \\1 5 \\1 8 \\1 9 \\1 10 \\2 3 \\2 4 \\2 8 \\2 11 \\2 12 \\2 13 \\3 4 \\3 11 \\3 14 \\3 15 \\4 6 \\4 14 \\5 6 \\5 7 \\5 8 \\5 9 \\5 10 \\6 7 \\6 14 \\7 10 \\7 13 \\7 15 \\7 16 \\8 9 \\8 10 \\8 11 \\8 12 \\8 13 \\9 10 \\9 12 \\9 16 \\10 13 \\10 15 \\10 16 \\11 12 \\11 13 \\11 14 \\11 15 \\12 13 \\12 16 \\13 15 \\13 16 \\14 15 \\15 16 \\

Dvořák output:Always colorable (using standard Alon–Tarsi)

Computation B.2 (G_{15}).

Algorithm 1 edges: $[(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(2,3),(3,4),(4,5),(1,5),(1,6),(4,6),(5,6),(6,7),(6,9),(1,7),(2,7),(7,8),(7,9),(2,8),(3,8),(8,9),(3,9),(4,9)]$

Algorithm 1 listsize: 5

Algorithm 1 output:24 92 \\5 \\0 1 \\0 2 \\0 3 \\0 4 \\0 5 \\0 6 \\0 7 \\0 8 \\1 2 \\1 3 \\1 4 \\1 5 \\1 9 \\1 10 \\1 11 \\2 3 \\2 4 \\2 9 \\2 12 \\2 13 \\2 14 \\3 4 \\3 12 \\3 15 \\3 16 \\3 17 \\4 6 \\4 15 \\4 18 \\5 6 \\5 7 \\5 8 \\5 9 \\5 10 \\5 11 \\6 7 \\6 8 \\6 15 \\6 18 \\7 8 \\7 16 \\7 18 \\7 19 \\7 20 \\8 10 \\8 19 \\8 21 \\8 22 \\9 10 \\9 11 \\9 12 \\9 13 \\9 14 \\10 11 \\10 19 \\10 21 \\10 22 \\11 13 \\11 21 \\11 23 \\12 13 \\12 14 \\12 15 \\12 16 \\12 17 \\13 14 \\13 21 \\13 23 \\14 17 \\14 20 \\14 22 \\14 23 \\15 16 \\15 17 \\15 18 \\16 17 \\16 18 \\16 19 \\16 20 \\17 20 \\17 22 \\17 23 \\18 19 \\18 20 \\19 20 \\19 21 \\19 22 \\20 22 \\20 23 \\21 22 \\21 23 \\22 23 \\

Dvořák output:Always colorable (using standard Alon–Tarsi)

Computation B.3 (G_{16}).

Algorithm 1 edges: $[(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(2,3),(3,4),(4,5),(1,5),(5,6),(1,6),(2,6),(2,7),(3,7),(3,8),(4,8),(5,8),(6,7),(7,8),(6,8)]$

Algorithm 1 listsize: 6

Algorithm 1 output:21 78 \\6 \\0 1 \\0 2 \\0 3 \\0 4 \\0 5 \\0 6 \\0 7 \\1 2 \\1 3 \\1 4 \\1 5 \\1 8 \\1 9 \\1 10 \\2 3 \\2 4 \\2 8 \\2 11 \\2 12 \\2 13 \\3 4 \\3 11 \\3 14 \\3 15 \\4 6 \\4 14 \\4 16 \\4 17 \\5 6 \\5 7 \\5 8 \\5 9 \\5 10 \\6 7 \\6 14 \\6 16 \\6 17 \\7 9 \\7 16 \\7 18 \\7 19 \\8 9 \\8 10 \\8 11 \\8 12 \\8 13 \\9 10 \\9 16 \\9 18 \\9 19 \\10 12 \\10 18 \\10 20 \\11 12 \\11 13 \\11 14 \\11 15 \\12 13 \\12 18 \\12 20 \\13 15 \\13 17 \\13 19 \\13 20 \\14 15 \\14 16 \\14 17 \\15 17 \\15 19 \\15 20 \\16 17 \\16 18 \\16 19 \\17 19 \\17 20 \\18 19 \\18 20 \\19 20 \\

Dvořák output:Always colorable (using standard Alon–Tarsi)

Computation B.4 (G_{17}).

Algorithm 1 edges: $[(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(2,3),(3,4),(4,5),(1,5),(1,6),(2,6),$
 $(3,6),(1,7),(3,7),(4,7),(5,7),(6,7)]$

Algorithm 1 listsize: 5

Algorithm 1 output: 18 64 \\5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 \\0 1 \\0 2 \\0 3
 \\0 4 \\0 5 \\0 6 \\0 7 \\0 8 \\1 2 \\1 3 \\1 4 \\1 5 \\1 9 \\1 10 \\2 3 \\2 4 \\2
 9 \\2 11 \\2 12 \\2 13 \\3 4 \\3 11 \\3 14 \\3 15 \\4 6 \\4 14 \\4 16 \\5 6 \\5
 7 \\5 8 \\5 9 \\5 10 \\6 7 \\6 8 \\6 14 \\6 16 \\7 8 \\7 10 \\7 12 \\7 17 \\8 13
 \\8 15 \\8 16 \\8 17 \\9 10 \\9 11 \\9 12 \\9 13 \\10 12 \\10 17 \\11 12 \\11 13
 \\11 14 \\11 15 \\12 13 \\12 17 \\13 15 \\13 16 \\13 17 \\14 15 \\14 16 \\15 16
 \\15 17 \\16 17 \\

Dvořák output: Always colorable (using standard Alon–Tarsi)

Computation B.5 (G_{18}).

Algorithm 1 edges: $[(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(2,3),(3,4),(4,5),(1,5),(1,6),(2,6),$
 $(3,6),(4,6),(5,6)]$

Algorithm 1 listsize: 5

Algorithm 1 output: 15 50 \\5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 \\0 1 \\0 2 \\0 3 \\0 4
 \\0 5 \\0 6 \\0 7 \\1 2 \\1 3 \\1 4 \\1 5 \\1 8 \\1 9 \\2 3 \\2 4 \\2 8 \\2 10 \\2
 11 \\3 4 \\3 10 \\3 12 \\3 13 \\4 6 \\4 12 \\4 14 \\5 6 \\5 7 \\5 8 \\5 9 \\6 7
 \\6 12 \\6 14 \\7 9 \\7 11 \\7 13 \\7 14 \\8 9 \\8 10 \\8 11 \\9 11 \\9 13 \\9 14
 \\10 11 \\10 12 \\10 13 \\11 13 \\11 14 \\12 13 \\12 14 \\13 14 \\

Dvořák output: Always colorable (using standard Alon–Tarsi)

Computation B.6 (G_{19}).

Algorithm 1 edges: $[(0,1),(0,2),(0,3),(0,4),(1,2),(2,3),(3,4),(1,4),(1,5),(2,5),(3,5),(4,5)]$

Algorithm 1 listsize: 4

Algorithm 1 output: 12 36 \\4 4 4 4 4 4 4 4 4 4 4 4 4 4 \\0 1 \\0 2 \\0 3 \\0 4 \\0 5
 \\0 6 \\1 2 \\1 3 \\1 4 \\1 7 \\1 8 \\2 3 \\2 7 \\2 9 \\2 10 \\3 5 \\3 9 \\3 11
 \\4 5 \\4 6 \\4 7 \\4 8 \\5 6 \\5 9 \\5 11 \\6 8 \\6 10 \\6 11 \\7 8 \\7 9 \\7 10
 \\8 10 \\8 11 \\9 10 \\9 11 \\10 11 \\

Dvořák output: Always colorable (using standard Alon–Tarsi)

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