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Faces of directed edge polytopes

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Abstract

Given a finite quiver (directed graph) without loops and multiedges, the convex hull of the column vector of the incidence matrix is called the directed edge polytope and is an interesting example of a lattice polytope. In this paper, we give a complete characterization of facets of the directed edge polytope of an arbitrary finite quiver without loops and multiedges in terms of the connectivity and the existence of a rank function. Our result can be regarded as an extension of the result on facets of symmetric edge polytopes to directed edge polytopes, shown by Higashitani, Jochemko and Michałek. When the quiver in question has a rank function, we obtain a characterization of faces of arbitrary dimensions.

1 Introduction

Motivated by optimal transportation problems, Vershik [14] proposed to study the convex polytope $\mathrm{KR}(X,d)$ constructed from a finite metric space (X,d). When $X=\{1,\ldots,n\}$, it is defined by

$$\mathrm{KR}(X,d)=\mathrm{conv}\left(\frac{\boldsymbol{e}_i-\boldsymbol{e}_j}{d(i,j)}\;\middle|\;1\le i,j\le n\right),$$

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where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. This is called the *fundamental polytope* in Vershik's paper. It is also called the *Kantorovich–Rubinstein polytope* [8, 9]. When X is a tree-like metric space, Delucchi and Hoessley [4] proved a nice formula of the f-vector by using the relation between tree-like metric spaces and hyperplane arrangements. The starting point of this work was the second author's attempt to extend their work to graphs with cycles.

Given a finite simple graph G, define a metric d_{graph} on the vertex set G_0 by the shortest length of paths, where each edge is equipped with length one. It turns out that the Kantorovich–Rubinstein polytope of the metric space (G_0, d_{graph}) has already been studied under different names. It coincides with the *symmetric edge polytope* $\mathrm{SE}(G)$ introduced by Matsui et al. [10]. When G is the complete graph K_n, it is called the *root polytope* of the root system A_n and its faces are completely determined by Cho [2].

We may also generalize the definition of a symmetric edge polytope to a finite quiver (directed graph) $Q = (Q_0, Q_1)$ without loops and multiedges by

$$\mathrm{DE}(Q) = \operatorname{conv}\left(\varepsilon_{(i,j)} = \boldsymbol{e}_i - \boldsymbol{e}_j \;\middle|\; (i,j) \in Q_1\right).$$

Here Q_0 is the set of vertices and the set of edges Q_1 is regarded as a subset of $Q_0 \times Q_0$. The polytope $\mathrm{DE}(Q)$ is called the *directed edge polytope* of Q in [6]. The symmetric edge polytope $\mathrm{SE}(G)$ is nothing but $\mathrm{DE}(D(G))$, where $D(G)$ is the double of G, i.e. the quiver obtained from G by replacing each edge $v - w$ by two directed edges $v \to w$ and $v \leftarrow w$.

The aim of this paper is to find an explicit combinatorial description of all facets of $\mathrm{DE}(Q)$ and thus obtain combinatorial descriptions of facets of $\mathrm{SE}(G) = \mathrm{KR}(G_0, d_{\text{graph}})$ for a finite simple graph.

In general, the directed edge polytope $\mathrm{DE}(Q)$ is defined as a convex polytope in the vector space $\mathbb{R}^{Q_0} = \{\rho : Q_0 \to \mathbb{R}\}$. Since the vertex set of $\mathrm{DE}(Q)$ is given by $\left\{ \varepsilon_{(v,w)} \;\middle|\; (v,w) \in Q_1 \right\}$, any face of $\mathrm{DE}(Q)$ can be written in the form $\mathrm{DE}(R)$ for a subquiver R with $R_0 = Q_0$. Let us call such a subquiver a *lluf subquiver*[1].

Given a lluf subquiver R of Q, our problem is thus to determine when $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, i.e. $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ and $\dim \mathrm{DE}(R) = \dim \mathrm{DE}(Q) - 1$. It turns out that the existence of a *rank function*, i.e. a function $\rho : Q_0 \to \mathbb{R}$ satisfying $\rho(v) + 1 = \rho(w)$ for any edge $v \to w$, plays a key role in both problems. As we see in Proposition 3.8, such a function makes the vertex set Q_0 into a graded poset.

Theorem 1.1. *For a finite quiver Q without loops and multiedges, we have*

$$\dim(\mathrm{DE}(Q)) = \begin{cases} |Q_0| - |\pi_0(Q)| - 1 & (\textit{if } Q \textit{ has a rank function}) \\ |Q_0| - |\pi_0(Q)| & (\textit{otherwise}), \end{cases}$$

where $\pi_0(Q)$ is the set of connected components of Q.

[1]In category theory, a subcategory containing all possible morphisms of a parent category is called a full subcategory. On the other hand, a subcategory containing all objects of a parent category is called a wide subcategory. It is also called a lluf subcategory, which is an anagram of 'full'. Since the term 'wide' in graph theory is sometimes used as another meaning, we call a subquiver containing all vertices a lluf quiver in this paper.

Remark 1.2. In [12], Ohsugi and Hibi obtained essentially the same notion as the directed edge polytope for a tournament obtained from an orientation of a complete graph. A tournament is equvalent to an asymmetric quiver whose underlying graph is a complete graph. They calculate the dimension of them.

It should be noted that, even if Q is connected, a subquiver R representing a facet of $\mathrm{DE}(Q)$ as $\mathrm{DE}(R)$ might not be connected. In fact, the number of connected components is another key player in our work.

Theorem 1.3. *Let Q be a finite quiver without loops and multiedges. For a lluf subquiver R of Q with $\dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1$, $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$ if and only if one of the following conditions holds:*

1. *$|\pi_0(R)| = |\pi_0(Q)| + 1$, R is a component-wise full subquiver of Q (Definition 2.6), and the contraction of R in Q (Definition 4.2) Q/R is acyclic.*
2. *$|\pi_0(R)| = |\pi_0(Q)|$ and there exists a rank function ρ of R such that*
$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0$$
for any $(v, w), (v', w') \in Q_1 \setminus R_1$.

Note that lower dimensional faces can be obtained from facets by iterating the process of taking facets. Thus we obtain the following characterization of all faces of $\mathrm{DE}(Q)$ for any quiver Q with a rank function.

Theorem 1.4. *Suppose Q has a rank function. For a proper subquiver R of Q, the polytope $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$, if and only if R is a component-wise full subquiver of Q and Q/R is acyclic.*

When $Q = D(G)$ for a simple graph G, it does not have a rank function and the condition (2) in Theorem 1.3 applies. It is immediate to translate the condition (2) into the following form.

Corollary 1.5 (Corollary 5.4)**.** *For a connected lluf subquiver R of $D(G)$ with $\dim(\mathrm{DE}(R)) = \dim(\mathrm{SE}(G)) - 1$, $\mathrm{DE}(R)$ is a facet of $\mathrm{SE}(G)$ if and only if $|\pi_0(R)| = |\pi_0(D(G))|$ and there exists a function $\rho \in \mathbb{R}^{G_0}$ such that*
$$\rho(v) - \rho(w) = \begin{cases} 1 & \text{if } (v, w) \in R_1, \\ -1 & \text{if } (w, v) \in R_1, \\ 0 & \text{otherwise} \end{cases}$$
for $(v, w) \in D(G)_1$.

This is essentially equivalent to a characterization of facets of symmetric edge polytopes in [7] when G is connected.

After fixing notation and terminology in Section 2, Theorem 1.1 is proved in Section 3, and Theorems 1.3 and 1.4 are proved in Section 4. We end this paper with sample computations in Section 5. In particular, a complete characterization of all faces of the symmetric edge polytope of a cyclic graph is obtained, which was previously done by the second author without using the characterization in this paper and became the starting point of this work. This is a full version of the results announced as an extended abstract [11].

2 Notation and terminology

First we fix notation and terminology for quivers.

Definition 2.1. We call a pair $Q = (Q_0, Q_1)$ a *quiver* if Q_0 is a finite set and Q_1 is a subset of $Q_0 \times Q_0 \setminus \{ (v, v) \mid v \in Q_0 \}$. An element of Q_0 is called a *vertex* of Q, and an element (v, w) in Q_1 is called an *edge* of Q from v to w.

The following classes of quivers play essential roles in this paper.

Definition 2.2. A quiver Q is said to be

1. *acyclic* if there does not exist $v_0, \ldots, v_n \in Q_0$ such that $n > 1$, $v_n = v_0$, and $(v_t, v_{t+1}) \in Q_1$ for $t = 0, \ldots, n-1$,
2. *asymmetric* if
$$(v, w) \in Q_1 \implies (w, v) \notin Q_1,$$
and
3. *symmetric* if
$$(v, w) \in Q_1 \iff (w, v) \in Q_1.$$

Note that a quiver may be neither symmetric nor asymmetric.

Definition 2.3. We define the *underlying graph* of a quiver Q to be the (undirected) graph obtained from Q by using all vertices of Q and by replacing all directed edges of Q with undirected edges.

Underlying graphs may have multiple edges. The underlying graph of Q is simple if and only if Q is asymmetric.

In order to describe faces of directed edge polytopes, we need subquivers.

Definition 2.4. A quiver R is called a *subquiver* of Q if $R_0 \subset Q_0$ and $R_1 \subset Q_1$. We say that a subquiver R of Q is

1. *proper* if R_1 is a proper subset of Q_1,
2. *full* if $R_1 = \{ (v, w) \in Q_1 \mid v, w \in R_0 \}$, and
3. *lluf* if $R_0 = Q_0$.

We make use of (undirected) walks to define connectivity of quivers.

Definition 2.5. Let Q be a quiver. An *undirected walk* from v_0 to v_n in Q is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices in Q such that $(v_t, v_{t+1}) \in Q_1$ or $(v_{t+1}, v_t) \in Q_1$ for all t.

An undirected walk $(v_0, v_1, \ldots, v_n)$ is called

1. *closed* if $v_0 = v_n$, and
2. an *undirected cycle* if it is closed and $v_i \neq v_j$ for any pair (i, j) with $0 \leq i < j < n$.

Definition 2.6. We say a quiver Q is *connected* if, for any pair (v, w) of vertices of Q, there exists an undirected walk from v to w. A connected maximal subquiver of Q is called a *connected component* of Q. The set of all connected components of Q is denoted by $\pi_0(Q)$. The number $|Q_0| - |\pi_0(Q)|$ is denoted by $c(Q)$ and is called the *coconnectivity* of Q. We say that a luff subquiver R of Q is *component-wise full* if $C_1 = \{ (v, w) \in Q_1 \mid v, w \in C_0 \}$ for each connected component $C \in \pi_0(R)$ of R. In the other words, a luff subquiver R of Q is said to be component-wise full if each connected component of R is a full subquiver of Q.

Note that a quiver is connected if and only if the underlying graph is connected.

Definition 2.7. For a connected quiver Q, we call a lluf asymmetric subquiver R of Q a *spanning polytree* in Q if the underlying graph of R is a tree, i.e. an acyclic connected simple undirected graph. For a quiver Q, we call a lluf subquiver R of Q a *spanning polyforest* in Q if each connected component of R is a spanning polytree in some connected component of Q.

We also need directed walks and cycles.

Definition 2.8. A *directed walk* from v_0 to v_n in a quiver Q is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices in Q such that $(v_t, v_{t+1}) \in Q_1$ for all t. A directed walk $(v_0, v_1, \ldots, v_n)$ is called a *directed cycle* if $v_0 = v_n$ and $v_i \neq v_j$ for any pair (i, j) with $0 \leq i < j < n$.

By definition, a quiver is acyclic if and only if it does not contain a directed cycle.

3 Dimension

Here is our main object of study.

Definition 3.1. Let Q be a quiver. The vector space of maps from Q_0 to $\mathbb{R}$ is denoted by $\mathbb{R}^{Q_0}$. It is equipped with an inner product $\langle\ ,\ \rangle$ defined by

$$\langle \rho, \delta \rangle = \sum_{v \in Q_0} \rho(v)\delta(v)$$

for $\rho, \delta \in \mathbb{R}^{Q_0}$. For a subset $V \subset Q_0$, we define an element $\kappa_V \in \mathbb{R}^{Q_0}$ by

$$\kappa_V(v) = \begin{cases} 1 & (v \in V), \\ 0 & (v \notin V). \end{cases}$$

The set $\{ \kappa_{\{v\}} \mid v \in Q_0 \}$ is a standard basis for the vector space $\mathbb{R}^{Q_0}$. For $(v, w) \in Q_0 \times Q_0$, we define the vector $\varepsilon_{(v,w)}$ by

$$\varepsilon_{(v,w)} = \kappa_{\{v\}} - \kappa_{\{w\}}.$$

Define a convex polytope in $\mathbb{R}^{Q_0}$ by

$$\mathrm{DE}(Q) = \mathrm{conv}\left\{ \varepsilon_{(v,w)} \mid (v,w) \in Q_1 \right\}.$$

This is called the *directed edge polytope* of Q.

Remark 3.2. For a quiver Q, define a map

$$\delta : Q_0 \times Q_1 \longrightarrow \{-1, 0, 1\}$$

by

$$\delta(v,e) = \begin{cases} 1 & \text{if } e = (v,w) \text{ for some } w \in Q_0, \\ -1 & \text{if } e = (w,v) \text{ for some } w \in Q_0, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix obtained from this map by choosing appropriate total orders of Q_0 and Q_1 is called the *incidence matrix* of Q and is denoted by $I(Q)$. Note that the vectors $\varepsilon_{(v,w)}$ are column vectors of $I(Q)$ and the directed edge polytope $\mathrm{DE}(Q)$ is the convex hull of these column vectors.

It is easy to determine the vertices of $\mathrm{DE}(Q)$. For a convex polytope P, let us denote the set of vertices of P by $\mathrm{vert}(P)$.

Lemma 3.3. *For a quiver Q, we have*

$$\mathrm{vert}(\mathrm{DE}(Q)) = \left\{ \varepsilon_{(v,w)} \;\middle|\; (v,w) \in Q_1 \right\}.$$

In general, any face of a convex polytope is the convex hull of a collection of vertices of the original polytope. In particular, any face of $\mathrm{DE}(Q)$ is of the form $\mathrm{DE}(R)$ for a subquiver R. The main purpose of this article is to give a characterization of subquivers corresponding to faces.

Note that even if Q is connected, a subquiver R representing a face of $\mathrm{DE}(Q)$ may not be connected. It turns out that the number of connected components is closely related to the dimension of $\mathrm{DE}(R)$. In fact, an upper bound is given by the coconnectivity $c(R)$.

Lemma 3.4. *Define a vector subspace V_Q of $\mathbb{R}^{Q_0}$ by*

$$V_Q = \bigcap_{R \in \pi_0(Q)} \kappa_{R_0}^{\perp},$$

where $\kappa_{R_0}^{\perp}$ is the orthogonal complement of κ_{R_0} in $\mathbb{R}^{Q_0}$. Then $\mathrm{DE}(Q) \subset V_Q$ and we have $\dim \mathrm{DE}(Q) \leq |Q_0| - |\pi_0(Q)| = c(Q)$.

Proof. Let R be a connected component of Q. We show that $\mathrm{DE}(Q)$ is contained in the hyperplane $\kappa_{R_0}^{\perp}$. For each edge (v,w) in Q, either $\{v,w\} \subset R_0$ or $\{v,w\} \cap R_0 = \emptyset$. In the former case, $\langle \kappa_{R_0}, \varepsilon_{(v,w)} \rangle = 1 - 1$ and in the latter case, $\langle \kappa_{R_0}, \varepsilon_{(v,w)} \rangle = 0 - 0$. Thus $\mathrm{DE}(Q) \subset \kappa_{R_0}^{\perp}$.

Since the hyperplanes defined by the vertex sets of connected components intersect transversally, we have

$$\dim \mathrm{DE}(Q) \leq \dim \left(\bigcap_{R\in\pi_0(Q)} \kappa_{R_0}^{\perp} \right) = \dim \mathbb{R}^{Q_0} - |\pi_0(Q)| = c(Q).$$

□

It turns out that $\dim \mathrm{DE}(Q)$ varies depending on the existence of a rank function, since such a function defines another hyperplane that contains the directed edge polytope.

Definition 3.5. For a quiver Q, a function $\rho \in \mathbb{R}^{Q_0}$ is called a *rank function* of Q if it satisfies $\rho(v) + 1 = \rho(w)$ for each edge $(v, w) \in Q_1$.

Lemma 3.6. *When Q has a rank function ρ, choose an edge $(v, w) \in Q_1$ and define a hyperplane H_ρ in $\mathbb{R}^{Q_0}$ by*

$$H_\rho = \rho^\perp + \varepsilon_{(v,w)} = \left\{ \delta + \varepsilon_{(v,w)} \;\middle|\; \delta \in \rho^\perp \right\}.$$

Then this is independent of the choice of an edge (v, w) and contains $\mathrm{DE}(Q)$*.*

Proof. For edges (v, w) and (v', w') in Q, we have

$$\langle \rho, \varepsilon_{(v,w)} \rangle = \rho(v) - \rho(w) = -1 = \rho(v') - \rho(w') = \langle \rho, \varepsilon_{(v',w')} \rangle,$$

which implies that $\varepsilon_{(v,w)} - \varepsilon_{(v',w')} \in \rho^\perp$ and $\rho^\perp + \varepsilon_{(v,w)} = \rho^\perp + \varepsilon_{(v',w')}$. It also implies that all vertices of $\mathrm{DE}(Q)$ are contained in H_ρ and thus $\mathrm{DE}(Q) \subset H_\rho$. □

The hyperplane H_ρ is transversal to the hyperplanes defined by connected components of Q. And we have the following upper bound of $\dim \mathrm{DE}(Q)$.

Corollary 3.7. *If Q has a rank function, then* $\dim(\mathrm{DE}(Q)) \leq c(Q) - 1$.

The choice of the term "rank function" is justified by the following fact.

Proposition 3.8. *The following are equivalent for a quiver Q:*

1. *Q has a rank function ρ.*
2. *Q is asymmetric and satisfies*

$$|\{ t \mid (v_t, v_{t+1}) \in Q_1 \}| = |\{ t \mid (v_{t+1}, v_t) \in Q_1 \}|$$

for each undirected closed walk $(v_0, v_1, \ldots, v_n)$ in Q.

3. *Q is asymmetric and satisfies*

$$|\{ t \mid (v_t, v_{t+1}) \in Q_1 \}| = |\{ t \mid (v_{t+1}, v_t) \in Q_1 \}|$$

for each undirected cycle $(v_0, v_1, \ldots, v_n)$ in Q.

4. *Q is the Hasse diagram of a graded poset $(Q_0, \leq)$ with rank function ρ.*

Proof. Suppose Q has a rank function ρ. Then Q cannot have a pair (v, w) of vertices with $v \to w$ and $v \leftarrow w$ and hence is asymmetric. Let $(v_0, \ldots, v_n)$ be an undirected closed walk or cycle in Q. Then

$$0 = \rho(v_0) - \rho(v_n) = \sum_{t \in \{\, t \mid (v_t, v_{t+1}) \in Q_1 \,\}} 1 + \sum_{t \in \{\, t \mid (v_{t+1}, v_t) \in Q_1 \,\}} (-1)$$

and we have

$$|\{\, t \mid (v_t, v_{t+1}) \in Q_1 \,\}| = |\{\, t \mid (v_{t+1}, v_t) \in Q_1 \,\}| .$$

Conversely, suppose that the second condition is satisfied. Choose a vertex v_0. For a vertex $w \in Q_0$, choose an undirected walk $(v_0, \ldots, v_n = w)$ from v_0 to w and define

$$\rho(w) = |\{\, t \mid (v_t, v_{t+1}) \in Q_1 \,\}| - |\{\, t \mid (v_{t+1}, v_t) \in Q_1 \,\}| .$$

The second condition guarantees that this is independent of the choice of a walk.

Since any undirected closed walk can be decomposed into undirected cycles, the second and the third conditions are equivalent. Finally the first and the fourth conditions are equivalent by definition. □

In order to obtain lower bounds of $\dim \mathrm{DE}(Q)$, we consider the case of acyclic quivers.

Lemma 3.9. *Let F be a quiver whose underlying graph is acyclic. Then the dimension of the vector space spanned by $\{\, \varepsilon_{(v,w)} \mid (v, w) \in F_1 \,\}$ is given by $c(F)$. Thus we obtain*

$$\dim \mathrm{DE}(F) = \dim \operatorname{aff} \left(\varepsilon_{(v,w)} \;\middle|\; (v, w) \in F_1 \right) = c(F) - 1,$$

where aff *denotes the affine hull.*

Proof. Recall that $I(F)$ is the incidence matrix of F. Then the dimension of the vector space spanned by $\left\{\, \varepsilon_{(v,w)} \;\middle|\; (v, w) \in F_1 \,\right\}$ is $\operatorname{rank} I(F)$. By the additivity of the rank of the incidence matrix with respect to disjoint unions, it suffices to prove that $\operatorname{rank} I(F) = |F_0| - 1$, when F is connected.

By the acyclicity assumption, the underlying graph of F is a tree. Let $n = |F_0|$. We may choose an ordering $F_0 = \{v_1, \ldots, v_n\}$ in such a way that the underlying graph of the full subquiver $F^{(i)}$ with vertices $\{v_i, \ldots, v_n\}$ is a tree for each $i = 1, \ldots, n-1$. In other words, v_i is connected to a vertex in $\{v_{i+1}, \ldots, v_n\}$ by a unique edge for each $i = 1, \ldots, n-1$. With this ordering, $I(F^{(i)})$ is of the form

$$I(F^{(i)}) = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & \\ \mp 1 & & I(F^{(i+1)}) & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

and we obtain $\operatorname{rank} I(F) = n - 1 = |F_0| - 1$ by induction.

Since the origin is not contained in the affine hull, we have

$$\begin{aligned}\dim \operatorname{conv}\left(\varepsilon_{(v,w)} \mid (v,w) \in F_1\right) &= \dim \operatorname{aff}\left(\varepsilon_{(v,w)} \mid (v,w) \in F_1\right)\\ &= c(F) - 1.\end{aligned}$$

□

Corollary 3.10. *For any quiver Q, we have* $\dim(\mathrm{DE}(Q)) \geq c(Q) - 1$.

Proof. Choose a spanning polyforest F in Q. Then we have

$$\dim(\mathrm{DE}(Q)) \geq \dim(\mathrm{DE}(F)) = c(F) - 1 \geq c(Q) - 1,$$

since $R_0 = Q_0$. □

For those quivers that do not have rank functions, we have the following lower bound.

Lemma 3.11. *If Q does not have a rank function, then* $\dim(\mathrm{DE}(Q)) \geq c(Q)$.

Proof. Let F be spanning polyforest of Q. Then $F_0 = Q_0$ and $|\pi_0(F)| = |\pi_0(Q)|$. By Lemma 3.9, we have $\dim(\mathrm{DE}(F)) = c(F) - 1 = c(Q) - 1$.

Since the underlying graph of F is acyclic, F has a rank function ρ. Let H_ρ be the hyperplane in Lemma 3.6. It is given by

$$H_\rho = \rho^\perp + \varepsilon_{(v_0,w_0)}$$

for an edge (v_0, w_0) in F. Since Q does not have a rank function, there exists an edge (v, w) in Q such that $\rho(v) + 1 \neq \rho(w)$. Then we have

$$\begin{aligned}\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \rangle &= \rho(v) - \rho(w) - (\rho(v_0) - \rho(w_0))\\ &= \rho(v) - \rho(w) + 1 \neq 0,\end{aligned}$$

which implies that the vertex $\varepsilon_{(v,w)}$ is not contained in the hyperplane H_ρ. It is not contained in any one of hyperplanes of the form κ_{G_0} for a connected component G of F, either. In other words, $\varepsilon_{(v,w)} \notin H_\rho \cap V_F$. Since $\mathrm{DE}(F) \subset H_\rho \cap V_F$, we have $\dim(\mathrm{DE}(Q)) \geq \dim(\mathrm{DE}(F)) + 1 = c(Q)$. □

Now Theorem 1.1 follows from Corollary 3.7, Corollary 3.10, Lemma 3.4 and Lemma 3.11.

4 Facets

Let R be a lluf subquiver of Q so that both $\mathrm{DE}(R)$ and $\mathrm{DE}(Q)$ are contained in $\mathbb{R}^{Q_0}$. In order to prove Theorem 1.3, we would like to know when $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ and $\dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1$.

We first obtain the following relation between the coconnectivities of Q and R by the dimension condition.

Lemma 4.1. *Let R be a lluf subquiver of Q. If* $\mathrm{DE}(R)$ *is a facet of* $\mathrm{DE}(Q)$*, then* $c(R) = c(Q)$ *or* $c(R) = c(Q) - 1$*. Thus* $|\pi_0(R)| = |\pi_0(Q)|$ *or* $|\pi_0(R)| = |\pi_0(Q)| + 1$*.*

Proof. When Q has a rank function, so does R. And we have

$$c(R) - 1 = \dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1 = c(Q) - 2$$

by Theorem 1.1, or $c(R) = c(Q)-1$. If Q does not have a rank function, $\dim(\mathrm{DE}(Q)) = c(Q)$, and we have $c(R) = c(Q)$ or $c(R) = c(Q) - 1$, depending on the existence of a rank function on R. □

One of the sufficient conditions for $\mathrm{DE}(R)$ being a facet is the acyclicity of the quiver Q/R obtained from Q by “contracting” R.

Definition 4.2. Let R be a component-wise full subquiver of a quiver Q. Define an equivalence relation $\sim$ on Q_0 by

$$v \sim w \iff v \text{ and } w \text{ are connected by an undirected walk in } R.$$

The equivalence class of $v \in Q_0$ is denoted by $[v]$. Define a quiver Q/R by

$$\begin{aligned}(Q/R)_0 &= Q_0/_\sim \\ (Q/R)_1 &= \{\, ([v],[w]) \mid (v,w) \in Q_1 \setminus R_1 \,\}.\end{aligned}$$

Roughly speaking, Q/R is the quiver obtained from Q by collapsing each connected component of R to a point.

Lemma 4.3. *Let R be a component-wise full proper subquiver of a quiver Q. If Q/R is acyclic, then* $\mathrm{DE}(R)$ *is a face of* $\mathrm{DE}(Q)$*.*

Proof. Let us denote the connected components of R by

$$\pi_0(R) = \{R^{(1)}, \dots, R^{(n)}\}.$$

Since Q/R is acyclic, we may assume that, if $v \in R_0^{(i)}$ and $w \in R_0^{(j)}$ are connected by an edge in Q, then $i < j$.

Denote $C_k = \bigcup_{i=1}^{k} R_0^{(i)}$. Then, for $v, w \in Q_0$,

$$\begin{aligned}\langle \kappa_{C_k}, \varepsilon_{(v,w)} \rangle &= \kappa_{C_k}(v) - \kappa_{C_k}(w) \\ &= \begin{cases} 1 & (v \in C_k \text{ and } w \notin C_k) \\ -1 & (v \notin C_k \text{ and } w \in C_k) \\ 0 & (\text{otherwise}). \end{cases}\end{aligned}$$

By our choice, the second case does not occur and we have $\langle \kappa_{C_k}, \varepsilon_{(v,w)} \rangle \geq 0$. In other words, the orthogonal complement $\kappa_{C_k}^{\perp}$ is a supporting hyperplane of $\mathrm{DE}(Q)$ and thus $\mathrm{DE}(Q) \cap \kappa_{C_k}^{\perp}$ is a face of $\mathrm{DE}(Q)$ for each k.

We claim that

$$\mathrm{DE}(Q) \cap \bigcap_{i=1}^{n} \kappa_{C_k}^{\perp} = \mathrm{DE}(R)$$

or

$$\left\{ \varepsilon_{(v,w)} \;\middle|\; (v,w) \in Q_1 \right\} \cap \bigcap_{i=1}^{n} \kappa_{C_k}^{\perp} = \left\{ \varepsilon_{(v,w)} \;\middle|\; (v,w) \in R \right\}.$$

If $(v, w) \in R_1$, $\langle \kappa_{C_k}, \varepsilon_{(v,w)} \rangle = 0$ for all k by the previous calculation. Conversely, suppose that $(v, w) \in Q_1 \setminus R_1$ with $v \in R_0^{(i)}$ and $w \in R_0^{(j)}$. By assumption, $i < j$, which implies that

$$\langle \kappa_{C_i}, \varepsilon_{(v,w)} \rangle = 1 - 0 = 1 \neq 0$$

and we have $\varepsilon_{(v,w)} \notin \kappa_{C_i}^{\perp}$. □

Another sufficient condition for being a face is the following.

Lemma 4.4. *Let R be a lluf proper subquiver of a quiver Q. If R has a rank function $\rho \in \mathbb{R}^{Q_0}$ such that*

$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0$$

for any $(v, w), (v', w') \in Q_1 \setminus R_1$, then $\mathrm{DE}(R)$ *is a face of* $\mathrm{DE}(Q)$.

Proof. Suppose a lluf subquiver R of Q has such a rank function ρ. Fix $(v_0, w_0) \in R_1$ and consider the hyperplane

$$H_\rho = \rho^{\perp} + \varepsilon_{(v_0,w_0)}.$$

We claim that H_ρ is a supporting hyperplane of $\mathrm{DE}(Q)$.

For $(v, w) \in Q_1$, we have

$$\left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle = \rho(v) - \rho(w) - \rho(v_0) + \rho(w_0) = \rho(v) - \rho(w) + 1.$$

Note that $\left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle = 0$ for $(v, w) \in R_1$. By our assumption on ρ we have $\left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle \geq 0$ for any $(v, w) \in Q_1$ or $\left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle \leq 0$ for any $(v, w) \in Q_1$ and H_ρ is a supporting hyperplane of $\mathrm{DE}(Q)$.

It remains to show that $\mathrm{DE}(Q) \cap H_\rho = \mathrm{DE}(R)$. Again by the assumption on ρ,

$$\left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle = \rho(v) - \rho(w) + 1 = 0 \iff (v, w) \in R_1$$

for $(v, w) \in Q_1$. Hence the hyperplane H_ρ contains $\mathrm{DE}(R)$, but the hyperplane does not contain $\varepsilon_{(v,w)}$ for any $(v, w) \in Q_1 \setminus R_1$. Hence the face $\mathrm{DE}(Q) \cap H_\rho$ coincides with $\mathrm{DE}(R)$. □

We next consider necessary conditions for being facets.

Lemma 4.5. *Let R be a subquiver of Q with $|\pi_0(R)| = |\pi_0(Q)| + 1$. If* $\mathrm{DE}(R)$ *is a facet of* $\mathrm{DE}(Q)$*, then R is a component-wise full subquiver of R.*

Proof. Let $n = |\pi_0(R)| = |\pi_0(Q)| + 1$ and $R^{(1)}, \dots, R^{(n)}$ be the complete list of connected components of R so that

$$V_R = \bigcap_{k=1}^{n} \kappa_{R^{(k)}}^{\perp}.$$

Let $(v, w) \in Q_1$ satisfy $v, w \in R_0^{(k)}$ for some k. It suffices to show that $\varepsilon_{(v,w)} \in \mathrm{DE}(R)$, since it is equivalent to $(v, w) \in R_1$ by Lemma 3.3.

By Lemma 3.4, V_R is a $c(R)$-dimensional affine space, which is a hyperplane in V_Q by the assumption $c(R) = c(Q) - 1$. We have $\varepsilon_{(v,w)} \in V_R$, since

$$\begin{aligned}\kappa_{R_0^{(i)}}(\varepsilon_{(v,w)}) &= \begin{cases} 1-1 & (i = k) \\ 0-0 & (i \neq k) \end{cases} \\ &= 0\end{aligned}$$

for any $i = 1, \dots, n$.

If R does not have a rank function, then $\mathrm{DE}(R)$ is a $c(R)$-dimensional polytope contained in V_R. In other words, V_R is a supporting hyperplane of $\mathrm{DE}(R)$ in V_Q, which implies that $\varepsilon_{(v,w)} \in \mathrm{DE}(R)$.

Suppose that R has a rank function. $\mathrm{DE}(R)$ is of dimension $c(R) - 1$ by Theorem 1.1. Since $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$,

$$\dim(\mathrm{DE}(Q)) = \dim(\mathrm{DE}(R)) + 1 = c(R) = c(Q) - 1.$$

By Theorem 1.1, Q also has a rank function, which is denoted by ρ. It defines a hyperplane

$$H_\rho = \rho^{\perp} + \varepsilon_{(v_0,w_0)}$$

in $\mathbb{R}^{Q_0}$ for some $(v_0, w_0) \in Q_1$. We may choose $(v_0, w_0) \in R_1$. By Lemma 3.6, $\mathrm{DE}(R)$ is contained in H_ρ, hence in $H_\rho \cap V_R$.

Since ρ is a rank function on Q, we have $\varepsilon_{(v,w)} \in H_\rho$ for $(v, w) \in Q_1$. If $v, w \in R_1^{(k)}$ for some k, we also have $\varepsilon_{(v,w)} \in V_R$, and hence $\varepsilon_{(v,w)} \in H_\rho \cap V_R$. Note that H_ρ and V_R intersect transversally, and we have

$$\dim(H_\rho \cap V_R) = \dim(V_R) - 1 = c(R) - 1 = \dim(\mathrm{DE}(R)).$$

In other words, $H_\rho \cap V_R$ is the affine hull of $\mathrm{DE}(R)$ in H_ρ and it should be the supporting hyperplane of $\mathrm{DE}(R)$ in H_ρ, since $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$. It implies that $\varepsilon_{(v,w)} \in \mathrm{DE}(R)$. □

Lemma 4.6. *Let R be a subquiver of Q with $|\pi_0(R)| = |\pi_0(Q)| + 1$. If $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, then Q/R is acyclic.*

Proof. Denote

$$\begin{aligned}\pi_0(Q) &= \{Q^{(1)}, \dots, Q^{(n-1)}\} \\ \pi_0(R) &= \{R^{(1)}, \dots, R^{(n)}\}.\end{aligned}$$

Without loss of generality, we may assume that $R^{(i)} \subset Q^{(i)}$ for $i = 1, \dots, n-2$ and $R^{(n-1)} \cup R^{(n)} \subset Q^{(n-1)}$. Let us denote $R' = R^{(n-1)} \cup R^{(n)}$ and $Q' = Q^{(n-1)}$. We should have $R^{(i)} = Q^{(i)}$ for $i = 1, \dots, n-2$ and $\mathrm{DE}(R')$ is a facet of $\mathrm{DE}(Q')$, since $\dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1$. It implies that Q/R is a union of $(n-2)$ quivers consisting of a single vertex and Q'/R'.

If Q/R were to have a directed cycle, it should be contained in Q'/R'. By Lemma 4.5, both $R^{(n-1)}$ and $R^{(n)}$ are full subquivers of Q'. Since R is lluf, such a directed cycle contains edges $(v,w),(v',w') \in Q'_1$ with $v, w' \in R_0^{(n-1)}$ and $w, v' \in R_0^{(n)}$. Then we have

$$\left\langle \kappa_{R_0^{(n-1)}}, \varepsilon_{(v,w)} \right\rangle = 1 \tag{1}$$

$$\left\langle \kappa_{R_0^{(n-1)}}, \varepsilon_{(v',w')} \right\rangle = -1. \tag{2}$$

If R' does not have a rank function, this contracts to the fact that $V_{R'} = \kappa^{\perp}_{R_0^{(n-1)}} \cap \kappa^{\perp}_{R_0^{(n)}}$ is a supporting hyperplane of $\mathrm{DE}(R')$ in $V_{Q'}$, as we have seen in the proof of Lemma 4.5.

If R' has a rank function, so does Q' as is shown in the proof of Lemma 4.5. Let ρ be a rank function of Q'. Then, again by the proof of Lemma 4.5, $H_\rho \cap V_{R'}$ is a supporting hyperplane of $\mathrm{DE}(R')$, which contradicts to (1) and (2).

Hence Q/R is acyclic. □

Lemma 4.7. *Let R be a lluf subquiver of Q with $|\pi_0(R)| = |\pi_0(Q)|$. If $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, then the subquiver R has a rank function $\rho \in \mathbb{R}^{Q_0}$ such that*

$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0.$$

for $(v,w),(v',w') \in Q_1 \setminus R_1$ and Q does not have a rank function.

Proof. Since $c(R) = c(Q)$ and $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$,

$$c(Q) \geq \dim(\mathrm{DE}(Q)) = \dim(\mathrm{DE}(R)) + 1 \geq c(R) = c(Q),$$

Q does not have a rank function and R has a rank function by Theorem 1.1. We also have $\dim(\mathrm{DE}(Q)) = c(R)$ and $\dim(\mathrm{DE}(R)) = c(R) - 1$. Since $|\pi_0(Q)| = |\pi_0(R)|$, v and w are in the same connected component of R for each $(v,w) \in Q_1$. Hence V_R contains $\varepsilon_{(v,w)}$ for all $(v,w) \in Q_1$. In other words, $\mathrm{DE}(Q)$ is a convex polytope in V_R.

Let ρ be a rank function of R. Fix $(v_0, w_0) \in R_1$ and consider the hyperplane $H_\rho = \rho^\perp + \varepsilon_{(v_0,w_0)}$. Then $H_\rho \cap V_R$ is an affine space of dimension $c(R) - 1$ which contains $\mathrm{DE}(R)$. It means that $H_\rho \cap V_R$ is a supporting hyperplane of $\mathrm{DE}(R)$ in V_R. Thus we have

$$(v,w) \in Q_1 \implies \left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle = \rho(v) - \rho(w) + 1 \geq 0$$

or

$$(v,w) \in Q_1 \implies \left\langle \rho, \varepsilon_{(v,w)} - \varepsilon_{(v_0,w_0)} \right\rangle = \rho(v) - \rho(w) + 1 \leq 0.$$

It remains to show that these values are nonzero if $(v, w) \in Q_1 \setminus R_1$. If $\rho(v) - \rho(w) = 1$, then $H_\rho \cap V_R$ contains $\varepsilon_{(v,w)}$. Since $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, it follows that $\mathrm{DE}(R)$ contains $\varepsilon_{(v,w)}$, which implies $(v, w) \in R_1$. $\square$

Now we are ready to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let R be a lluf subquiver of Q with $\dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1$.

Suppose that $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$. By Lemma 4.1, $c(R) = c(Q)$ or $c(R) = c(Q) - 1$. When $c(R) = c(Q)$, Lemma 4.7 implies that the condition (2) in Theorem 1.3 holds. If $c(R) = c(Q) - 1$, the condition (1) in Theorem 1.3 follows from Lemmas 4.5 and 4.6.

Conversely, if a lluf subquiver R satisfies (1) in Theorem 1.3, then $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ by Lemma 4.3. If R satisfies (2) in Theorem 1.3, $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ by Lemma 4.4. Hence, in both cases, $\mathrm{DE}(R)$ is a facet. $\square$

Proof of Theorem 1.4. Suppose that $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$. Then there exists a descending sequence of subquivers

$$Q = Q^{(0)} \supset Q^{(1)} \supset \cdots \supset Q^{(n-1)} \supset Q^{(n)} = R$$

such that $\mathrm{DE}(Q^{(i+1)})$ is a facet of $\mathrm{DE}(Q^{(i)})$ for each $i = 0, \ldots, n-1$. The rank function of Q serves as a rank function of $Q^{(i)}$ and we have

$$c(Q^{(i+1)}) = \dim(\mathrm{DE}(Q^{(i+1)})) + 1 = \dim(\mathrm{DE}(Q^{(i)})) - 1 + 1 = c(Q^{(i)}) - 1.$$

by Theorem 1.1. It implies by Theorem 1.3 that each connected component of $Q^{(i+1)}$ is a full subquiver of $Q^{(i)}$ and that $Q^{(i)}/Q^{(i+1)}$ is an acyclic quiver for all i. Hence each connected component of $Q^{(n)} = R$ is a full subquiver of Q and $Q^{(0)}/Q^{(n)} = Q/R$ is acyclic.

The converse follows from Lemma 4.3. $\square$

5 Examples

Here we consider some special cases as applications of our results. First we consider the case of an asymmetric quiver Q with no undirected closed walk. Namely the underlying graph of Q is a forest. In this case, we have

$$\begin{aligned}
|\mathrm{vert}(\mathrm{DE}(Q))| &= |Q_1| \\
&= |Q_0| - |\pi_0(Q)| \\
&= c(Q).
\end{aligned}$$

On the other hand, Q has a rank function and we have $\dim(\mathrm{DE}(Q)) = c(Q) - 1$ by Theorem 1.1. Hence we have the following:

Corollary 5.1. *Let Q be an asymmetric quiver with $Q_1 \neq \emptyset$. If the underlying graph of Q is acyclic, then $\mathrm{DE}(Q)$ is a simplex of dimension $|Q_1| - 1 = c(Q) - 1$.*

Example 5.2. Let Q be an asymmetric quiver whose underlying graph is the Dynkin graph A_{n+1}. Denote $Q_0 = \{ 0, 1, \ldots, n \}$ and $Q_1 = \{ e_1, \ldots, e_n \}$ so that $e_i = (i-1, i)$ or $(i, i-1)$ for $i = 1, \ldots, n$. Then the directed edge polytope $\mathrm{DE}(Q)$ is an $(n-1)$-simplex. Hence $\mathrm{DE}(R)$ is a face for any luff proper subquiver R of Q.

One of the simplest cases in which the underlying graph is not a tree is the following.

Example 5.3. Let Q be an asymmetric quiver whose underlying graph is the boundary of a $2n$-gon. Denote $Q_0 = \mathbb{Z}/2n\mathbb{Z} = \{ \overline{1}, \ldots, \overline{2n} = \overline{0} \}$ and $Q_1 = \{ e_1, \ldots, e_{2n} \}$ so that $e_i = (\overline{i-1}, \overline{i})$ or $(\overline{i}, \overline{i-1})$ for each $i \in \{ 1, \ldots, 2n \}$.

Define Q_1^+ and Q_1^- by

$$Q_1^+ = Q_1 \cap \left\{ (\overline{i-1}, \overline{i}) \;\middle|\; \overline{i} \in Q_0 \right\},$$
$$Q_1^- = Q_1 \cap \left\{ (\overline{i}, \overline{i-1}) \;\middle|\; \overline{i} \in Q_0 \right\}.$$

If $\left|Q_1^+\right| = \left|Q_1^-\right| = n$, then Q has a rank function. By Theorem 1.1,

$$\dim(\mathrm{DE}(Q)) = c(Q) - 1 = |Q_0| - |\pi_0(Q)| - 1 = 2n - 2.$$

Since $|\mathrm{vert}(\mathrm{DE}(Q))| = |Q_1| = 2n$, $\mathrm{DE}(Q)$ is not a simplex.

Let R be a lluf subquiver of Q whose directed edge polytope $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$. Since R also has a rank function,

$$2n - 3 = \dim(\mathrm{DE}(R)) = |Q_0| - |\pi_0(R)| - 1 = 2n - 1 - |\pi_0(R)|,$$

which implies that $Q_1 \setminus R_1$ consists of two disjoint edges. Let $Q_1 \setminus R_1 = \{ e', e'' \}$. The acyclicity of Q/R following from Theorem 1.4 allows us to assume that $e' \in Q_1^+$ and $e'' \in Q_1^-$. This is a characterization of facets of $\mathrm{DE}(Q)$.

For such a subquiver R, $\mathrm{DE}(R)$ is a simplex of dimension $2n-3$ by Corollary 5.1. Since faces of a simplex are in one-to-one correspondence with subsets of the vertex set, for a lluf subquiver S of Q, $\mathrm{DE}(S)$ is a proper face of $\mathrm{DE}(Q)$ of dimension d if and only if $\left|S_1 \cap Q_1^+\right| < n$, $\left|S_1 \cap Q_1^-\right| < n$, and $|S_1| = d+1$. Hence the number f_d of faces of $\mathrm{DE}(Q)$ of dimension d is given by

$$f_d = \binom{2n}{d+1} - 2\binom{n}{d+1-n},$$

where the binomial coefficient $\binom{m}{k}$ equals 0 if $m < k$ or $k < 0$.

Finally we consider the case of symmetric edge polytopes. For a finite graph G, the symmetric edge polytope $\mathrm{SE}(G)$ of G introduced by Matsui et al. [10] is, by definition, the directed edge polytope $\mathrm{DE}(D(G))$ of the double $D(G)$ of G. Note that any symmetric quiver in our sense is of the form $D(G)$ for a finite graph G.

Since $D(G)$ is a symmetric quiver, $D(G)/R$ is not acyclic for any proper subquiver R of $D(G)$. Hence we have the following:

Corollary 5.4. *Let G be a finite simple graph whose vertex set is denoted by G_0. For a lluf subquiver R of $D(G)$ with* $\dim(\mathrm{DE}(R)) = \dim(\mathrm{SE}(G)) - 1$*, the following are equivalent:*

1. $\mathrm{DE}(R)$ *is a facet of* $\mathrm{SE}(G)$.
2. $c(R) = c(D(G))$ *and there exists a rank function ρ of R such that*
$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0$$
for any $(v, w), (v', w') \in D(G)_1 \setminus R_1$.
3. $c(R) = c(D(G))$ *and there exists a function $\rho \in \mathbb{R}^{G_0}$ such that*
$$\rho(v) - \rho(w) = \begin{cases} 1 & ((v, w) \in R_1) \\ -1 & ((w, v) \in R_1) \\ 0 & (\textit{otherwise}) \end{cases}$$
for $(v, w) \in D(G)_1$.

Proof. The equivalence of the first two conditions follows from Theorem 1.3. The third condition is easily seen to be equivalent to the second condition. ☐

Remark 5.5. In [7], Higashitani, Jochemko and Mateusz obtained a characterization of facets of symmetric edge polytopes for a connected simple graph G as the existence of a function $\rho : G_0 \to \mathbb{Z}$ satisfying the following two conditions:

1. $\rho(v) - \rho(w) \in \{ -1, 0, 1 \}$ for any edge $(v, w) \in D(G)$, and
2. the underlying graph of the quiver E^ρ defined by
$$E^\rho_1 = \{ (v, w) \in \mathrm{SE}(G)_1 \mid \rho(v) = \rho(w) + 1 \}$$
is a spanning subgraph of G.

Their characterization can be obtained from Corollary 5.4 as follows. Let $\rho \in \mathbb{R}^{G_0}$ be a function satisfying the third condition of Corollary 5.4. Then $\rho(v) - \rho(w) \in \{ -1, 0, 1 \}$ for all $(v, w) \in Q_1$ and we may assume that ρ takes values in $\mathbb{Z}$. For such a function ρ, the quiver E^ρ can be easily seen to coincide with our subquiver R in Corollary 5.4. Since G is connected, so is $\mathrm{SE}(G)$. The condition $c(R) = c(D(G))$ implies that R is also connected. Since R is a lluf subquiver, it implies that the underlying graph of $E^\rho = R$ is a spanning subgraph of G.

Conversely, let $\rho : G_0 \to \mathbb{Z}$ be a function satisfying $\rho(v) - \rho(w) \in \{ -1, 0, 1 \}$ for all $(v, w) \in Q_1$ and suppose that the underlying graph of E^ρ is a spanning subgraph of G. In particular, E^ρ is a lluf subquiver of $D(G)$. By the connectivity of E^ρ, we see $c(E^\rho) = c(D(G))$. Since ρ is a rank function on E^ρ, it satisfies the condition of the first case of the third condition of Corollary 5.4. The condition that $\rho(v) - \rho(w) \in \{ -1, 0, 1 \}$ for $(v, w) \in D(G)$, then, implies the condition of the other cases.

Remark 5.6. The Kantorovich–Rubinstein polytope $\mathrm{KR}([n], d)$ is defined as the convex hull of the set E of vectors $\epsilon^d_{(i,j)} = \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{d(i,j)}$ indexed by $i \neq j \in [n]$. Note that the set of vertices of $\mathrm{KR}([n], d)$ might be a proper subset of E. For a face F of $\mathrm{KR}([n], d)$, the quiver with the edge set consisting of (i, j) satisfying $\epsilon^d_{(i,j)} \in F$ is called the face digraph. Note that the set of vertices of a facet of $\mathrm{KR}([n], d)$ is a subset of the set of vectors $\epsilon^d_{(i,j)}$ corresponding to edges (i, j) in the facet digraph. The face graph is the undirected graph obtained from the face digraph by forgetting the orientation of edges. In Gordon–Petrov [5, Theorem 3], subgraphs contained by a facet graph of $\mathrm{KR}([n], d)$ are characterized by using existence of 1-Lipschitz functions. In the case of the symmetric edge polytope $\mathrm{SE}(G)$, we define face and facet graphs as subgraphs of G. We call them face and facet subgraphs of G. In Chen–Davis–Korchevskaia [1], face and facet subgraphs are characterized in terms of bipartite graphs. These characterization are essentially the same as the one in this article.

The following two examples were first studied by the second author in an elementary method analogous to that of Cho's [2], whose analysis led to the current work.

Example 5.7. Let C_{2n} be the boundary of a $2n$-gon regarded as a graph of $2n$ vertices and edges. As in the case of Example 5.3, the vertex set is identified with $\mathbb{Z}/2n\mathbb{Z} = \left\{ \overline{1}, \ldots, \overline{2n} = \overline{0} \right\}$. The symmetric edge polytope $\mathrm{SE}(C_{2n}) = \mathrm{DE}(D(C_{2n}))$ is a $(2n-1)$-dimensional polytope by Theorem 1.1. The faces of $\mathrm{SE}(C_{2n})$ can be determined as follows.

Denote $Q = D(C_{2n})$ for simplicity and define

$$\begin{aligned} Q_1^+ &= \left\{ (\overline{i-1}, \overline{i}) \;\middle|\; i = 1, \ldots, 2n \right\}, \\ Q_1^- &= \left\{ (\overline{i}, \overline{i-1}) \;\middle|\; i = 1, \ldots, 2n \right\} \end{aligned}$$

so that $Q_1 = Q_1^+ \cup Q_1^-$.

By (3) of Corollary 5.4, for a lluf subquiver R of Q with

$$\dim(\mathrm{DE}(R)) = \dim(\mathrm{SE}(C_{2n})) - 1 = 2n - 2,$$

$\mathrm{DE}(R)$ is a facet of $\mathrm{SE}(C_{2n})$ if and only if $c(R) = c(Q) = 2n - 1$ and there exists a function $\rho : \mathbb{Z}/2n\mathbb{Z} \to \mathbb{R}$ such that

$$\rho(\overline{i-1}) - \rho(\overline{i}) = \begin{cases} 1 & ((\overline{i-1}, \overline{i}) \in R_1) \\ -1 & ((\overline{i}, \overline{i-1}) \in R_1) \\ 0 & (\text{otherwise}), \end{cases}$$

which implies that only one of $(\overline{i-1}, \overline{i})$ or $(\overline{i}, \overline{i-1})$ belongs to R_1 for each i. Denote

$$\begin{aligned} R_1^+ &= R_1 \cap Q_1^+ = \left\{ (\overline{i-1}, \overline{i}) \;\middle|\; i \in I_+ \right\} \\ R_1^- &= R_1 \cap Q_1^- = \left\{ (\overline{i}, \overline{i-1}) \;\middle|\; i \in I_- \right\}. \end{aligned}$$

Then

$$\begin{aligned}0 &= \sum_{i=1}^{2n}(\rho(\overline{i-1}) - \rho(\bar{i})) \\ &= \sum_{i\in I_+} 1 + \sum_{i\in I_-}(-1) + \sum_{i\notin I_+\cup I_-} 0 \\ &= |I_+| - |I_-|\end{aligned}$$

and we have $|I_+| = |I_-|$.

Since $\mathrm{DE}(R)$ is of dimension $2n-2$,

$$|R_1| = |\mathrm{vert}(\mathrm{DE}(R))| \geq 2n-1.$$

By the condition on ρ, we see that the underlying graph of R must be the whole C_{2n}. Thus we have $\left|R_1^+\right| = \left|R_1^-\right| = n$ and $R_1^+ \cap (-R_1^-) = \emptyset$, where $-R_1^- = \left\{\, (\overline{i-1}, \bar{i}) \;\middle|\; (\bar{i}, \overline{i-1}) \in R_1^- \,\right\}$. In other words, facets of $\mathrm{SE}(C_{2n})$ are in bijective correspondence with subsets of cardinality n in $Q_1 = \left\{ (\overline{i-1}, \bar{i}), (\bar{i}, \overline{i-1}) \;\middle|\; i = 1, \dots, 2n \right\}$. Hence the number f_{2n-2} of facets of $\mathrm{SE}(C_{2n})$ is given by

$$f_{2n-2} = \binom{2n}{n}.$$

Note that facets of $\mathrm{SE}(C_{2n})$ are polytopes in Example 5.3. In particular, faces of codimension 2 in $\mathrm{SE}(C_{2n})$ are simplices of dimension $(2n-3)$, which means that all faces of $\mathrm{SE}(C_{2n})$ except for facets are simplices. In other words, for $d < 2n-2$ and a lluf subquiver R of Q, $\mathrm{DE}(R)$ is a face of dimension d in $\mathrm{SE}(C_{2n})$ if and only if $|R_1| = d+1$, $\left|R_1 \cap Q_1^+\right| < n$, $\left|R_1 \cap Q_1^-\right| < n$, and $R_1^+ \cap (-R_1^-) = \emptyset$. Hence the number f_d of faces of $\mathrm{DE}(Q)$ of dimension d is given by

$$\begin{aligned}f_d &= \sum_{i\in I}\binom{2n}{i}\binom{2n-i}{d+1-i} \\ &= \binom{2n}{d+1}\sum_{i\in I}\binom{d+1}{i},\end{aligned}$$

where $I = \{\, i \in \mathbb{Z} \mid i < n, d+1-i < n \,\}$. If $d+1 < n$, then we have $\sum_{i\in I}\binom{d+1}{i} = 2^{d+1}$, which implies

$$f_d = \binom{2n}{d+1}2^{d+1}.$$

We remark that D'Ali, Delucchi, and Michałek [3] also performed the same computation based on the characterization of facets by Higashitani et al. [7].

Example 5.8. Consider the case of an odd cycle C_{2n+1}. As is the case of Example 5.7, we identify the vertex set with $\mathbb{Z}/(2n+1)\mathbb{Z} = \left\{\, \bar{1}, \dots, \overline{2n}, \overline{2n+1} = 0 \,\right\}$. For simplicity, we denote $Q = D(C_{2n+1})$ and

$$\begin{aligned}Q_1^+ &= \left\{\, (\overline{i-1}, \bar{i}) \;\middle|\; i = 1, \dots, 2n+1 \,\right\}, \\ Q_1^- &= \left\{\, (\bar{i}, \overline{i-1}) \;\middle|\; i = 1, \dots, 2n+1 \,\right\}.\end{aligned}$$

The symmetric edge polytope $\mathrm{SE}(C_{2n+1}) = \mathrm{DE}(Q)$ is a $2n$-dimensional polytope by Theorem 1.1.

For a lluf subquiver R of Q, suppose that $\dim(\mathrm{DE}(R)) = 2n-1$. By the same argument as in Example 5.7, $\mathrm{DE}(R)$ is a facet of $\mathrm{SE}(C_{2n+1})$ if and only if $c(R) = c(Q)$, $\left|R_1^+\right| = \left|R_1^-\right| = n$, and $R_1^+ \cap (-R_1^-) = \emptyset$. Since $|\mathrm{vert}(\mathrm{DE}(R))| = 2n$, $\mathrm{DE}(R)$ is a simplex of dimension $2n-1$ and all faces of $\mathrm{SE}(C_{2n+1})$ are simplices. We also see that facets are in one-to-one correspondence with a pair (E, e) of a subset E of Q_1^+ of cardinality n and an element $e \in Q_1^- \setminus (-E)$ and the number f_{2n-1} of facets of $\mathrm{DE}(Q)$ is given by

$$f_{2n-1} = (n+1)\binom{2n+1}{n} = \frac{(2n+1)!}{n!n!} = (2n+1)\binom{2n}{n}.$$

Thus, for $d < 2n-2$ and a lluf subquiver R of Q, $\mathrm{DE}(R)$ is a face of $\mathrm{SE}(C_{2n+1})$ of dimension d if and only if $|R_1| = d+1$, $\left|R_1^+\right| < n$, $\left|R_1^-\right| < n$, and $R_1^+ \cap (-R_1^-) = \emptyset$. Hence the number f_d of faces of dimension d in $\mathrm{SE}(C_{2n+1})$ is given by

$$\begin{aligned} f_d &= \sum_{i \in I} \binom{2n+1}{i}\binom{2n+1-i}{d+1-i} \\ &= \binom{2n+1}{d+1} \sum_{i \in I} \binom{d+1}{i}, \end{aligned}$$

where $I = \{\, i \in \mathbb{Z} \mid i < n, d+1-i < n \,\}$. If $d+1 < n$, then we have $\sum_{i \in I}\binom{d+1}{i} = 2^{d+1}$, which implies

$$f_d = \binom{2n+1}{d+1} 2^{d+1}.$$

Remark 5.9. In [13], Ohsugi and Shibata consider the centrally symmetric configurations and the convex hull of column vectors of them. The polytopes are the symmetric edge polytopes of cycles. They calculate the Ehrhart polynomails and h-vectors for them. The formulas for f-vectors in Examples 5.7 and 5.8 imply the same h-vetors.

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