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Bernoulli number identities for associated Stirling numbers and derangements

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Abstract

Denote by $b(n,k)$ the associated Stirling number of the second kind, that is, the number of partitions of $[n]$ into k blocks, where each block contains at least 2 elements. Denote by $d(n,k)$ the number of derangements of $[n]$ into k cycles. Let B_n denote the sequence of Bernoulli numbers. We establish that $\sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-2}{k-1} = -\,B_{n-1}$, for all $n \geq 3$, and $\sum_{k=1}^{n}(-1)^k d(n+k,k)/((n+k-1)(n+k-2)) = -\,B_{n-1}/(n-1)$, for all $n \geq 3$. These results are extended to the numbers $b_r(n,k)$ and $d_r(n,k)$ which, besides the defining properties of $b(n,k)$ and $d(n,k)$, satisfy also the condition that $1,2,\ldots,r$ fall in distinct blocks or cycles.

1 Introduction

Denote by $b(n,k)$ the number of partitions of $[n]$ into k blocks, where each block contains at least 2 elements, [13, A008299]. The numbers $b(n,k)$ are called the 2-associated Stirling numbers of the second kind as introduced by [5, p. 221], and are denoted variously in [12, p. 77], [5, p. 221], [8, p. 303], [4, p. 136], [6, p. 3], and [15]. The ordinary Stirling number of the second kind, $\left\{ {n \atop k} \right\}$, is defined as the number of partitions of $[n]$ into k blocks. Parallel notions for permutations are as follows. Denote by $d(n,k)$ the number of permutations of $[n]$ into k cycles such that there is no fixed point of the permutation, [13, A008306]. The number $d(n,k)$ is called the number of derangements of $[n]$ into k cycles, [5, p. 256], [4, p. 132]. The derangement numbers $d(n,k)$ are correspondingly referred to as associated Stirling numbers of the first kind, [15]. The ordinary Stirling number of the first kind, $\left[{n \atop k} \right]$, is defined as the number of permutations of $[n]$ into k cycles.

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A generating function of $b(n,k)$ is, by [5, p. 222],

$$\sum_{n,k\geq 0} b(n,k)u^k x^n/n! = \exp(u(e^x-1-x)), \tag{1.1}$$

where we define $b(0,0)=1$ by convention. The corresponding generating function for derangements is, by [5, p. 256],

$$\sum_{n,k\geq 0} d(n,k)u^k x^n/n! = e^{-ux}(1-x)^{-u}, \tag{1.2}$$

where again we define $d(0,0)=1$ by convention. One way to find (1.2) is to follow [4, p. 134]. By [5, p. 221, p. 256] we have the triangular recurrences

$$\begin{aligned} &\text{(i)}\quad b(n,k) = kb(n-1,k) + (n-1)b(n-2,k-1),\ n\geq 2, k\geq 1,\\ &\text{(ii)}\quad d(n,k) = (n-1)d(n-1,k) + (n-1)d(n-2,k-1),\ n\geq 2, k\geq 1; \end{aligned} \tag{1.3}$$

see also [4, (1),(10)] wherein the partition numbers $b(n,k)$ are denoted $e(n,k)$. Here we define $b(n,0)=0$ if $n\geq 1$, and $b(n,k)=0$ if $n<2k$ or $n<0$, and likewise for $d(n,k)$.

The Bernoulli numbers B_n, $n=0,1,2,\ldots$ may be defined simply by the generating function $\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{x^n}{n!}B_n$; [7, (6.81)], [5, pp. 48–49], [11, (15.1)]. An alternative approach is to develop power sums directly in terms of Bernoulli numbers as shown in [7, (6.78)–(6.79)]. Two versions of Bernoulli's formulae for the power sums are given by [11, (15.24)–(15.25)]; the second of these is as follows:

$$\sum_{k=0}^{N} k^p = \frac{1}{p+1}\sum_{k=0}^{p}(-1)^k\binom{p+1}{k}N^{p+1-k}B_k,\quad p\geq 1. \tag{1.4}$$

To obtain a combinatorial interpretation of the Bernoulli numbers, consider the Euler or up/down numbers U_n defined as the number of permutations $a_1a_2\ldots a_n$ of $[n]$ such that $a_1<a_2>a_3<a_4>\cdots$; [13, A000111]. Then we have

$$U_{2n-1} = (-1)^{n-1}\frac{4^n(4^n-1)}{2n}B_{2n},$$

[2, (1.3)], [5, Ex. 11, p. 258].

We investigate certain alternating sums involving the partition numbers $b(n+k,k)$, and derangement numbers $d(n+k,k)$. For example, by the recurrences (1.3), expanded for $b(n+k,k)$ in (1.7), it may be seen directly that $\sum_{k=1}^{n}(-1)^k b(n+k,k) = (-1)^n n!$ and $\sum_{k=1}^{n}(-1)^k d(n+k,k) = (-1)^n$, [5, p. 221, p. 256], [4, (7),(11)]. Bijective arguments for these identities are given by [4]. We are especially interested in proving results of this type but with Bernoulli numbers arising as the outcome of the summations. That an identity of this type exists for the partition numbers $b(n,k)$, as shown by Theorem 1.1, is perhaps not too surprising due to known formulae for the ordinary Stirling numbers of the second kind, such as the following identity of Jordan:

$$\sum_{q=0}^{m}(-1)^q\binom{m+1}{q+1}\frac{\left\{{m+q \atop q}\right\}}{\binom{m+q}{m}} = B_m, \tag{1.5}$$

where B_m is the sequence of Bernoulli numbers, [9], [11, (15.10)]. Other evidence for partition numbers is given by the formula $B_m = \sum_{k=0}^{m}(-1)^k k!\left\{{m \atop k}\right\}/(k+1)$; [5, p. 220], [11, (15.2)]. We are thus led after Theorem 1.1 to find a similar identity for a sum involving derangement numbers on one side and Bernoulli numbers on the other side of an equality. We note that the numbers B_n/n do arise in connection with the Stirling numbers of the first kind in [1].

In this paper we are motivated to prove the following results, and to generalize them to certain r-distinguished cases, as we shall explain.

Theorem 1.1 *Let $n \geq 3$. Then*

$$\sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-2}{k-1} = -B_{n-1}.$$

Theorem 1.2 *Let $n \geq 3$. Then*

$$\sum_{k=1}^{n}(-1)^k d(n+k,k)/((n+k-1)(n+k-2)) = -\frac{B_{n-1}}{n-1}.$$

We first discuss Theorem 1.1 and its extension to Theorem 1.3. The motivation for the form that Theorem 1.1 takes is its similarity to the simple identity

$$\sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-1}{k-1} = 0, \text{ for all } n \geq 2. \tag{1.6}$$

One may easily prove (1.6) by applying the triangular recurrence (1.3)(i) to write $b(n+k,k) = kb(n+k-1,k) + (n+k-1)b(n+k-2,k-1)$. Therefore, starting from the case $k = n$ and working backwards, we have

$$\begin{aligned} &b(2n,n) = nb(2n-1,n) + (2n-1)b(2n-2,n-1)\\ &b(2n-1,n-1) = (n-1)b(2n-2,n-1) + (2n-2)b(2n-3,n-2)\\ &b(2n-2,n-2) = (n-2)b(2n-3,n-2) + (2n-3)b(2n-4,n-3), \ \ldots. \end{aligned} \tag{1.7}$$

Notice that, matching terms involving $b(2n-2,n-1)$ in (1.7) for the sum (1.6), we find

$$(-1)^n\left((2n-1)/\binom{2n-1}{n-1} - (n-1)/\binom{2n-2}{n-2}\right)b(2n-2,n-1) = 0.$$

It is easy to see that the binomial coefficients in the denominators have been chosen to make the sum (1.6) telescope after the continuation of (1.7). In working backwards from $k = n$ to $k = 1$, after substituting (1.7) into the sum of (1.6), we have that the endpoint terms $(-1)^n\, n \cdot b(2n-1, n)/\binom{2n-1}{n-1} = 0$ and $(-1)^1 n \cdot b(n-1, 0)/\binom{n}{0} = 0$, as long as $n \geq 2$. So the proof of (1.6) follows. Notice that the same type of telescoping argument may be made for the identity $\sum_{k=1}^{n}(-1)^k d(n+k, k)/(n+k-1) = 0$, [5, p. 256], by applying instead (1.3)(ii).

We extend Theorem 1.1 to the case of r-distinguished associated Stirling numbers of the second kind, $b_r(n, k)$, as follows. Define $b_r(n, k)$ as the number of partitions of $[n]$ into k blocks without singleton blocks such that $1, 2, \ldots, r$ fall in distinct blocks; in particular $b_1(n, k) = b(n, k)$. The following result extends Theorem 1.1 to all $r \geq 1$.

Theorem 1.3 *Let $r \geq 1$ and $n \geq r + 2$. Then*

$$\sum_{k=r}^{n} b_r(n+k, k)(-1)^k/\binom{n+k-r-1}{k-1} = (-1)^r r! B_{n-r}.$$

We prove Theorem 1.1 in Section 2 by applying the generating function method. We apply induction to prove Theorem 1.3 in Section 2.1 by using Theorem 1.1 as the base case. The induction step relies on the extension of (1.6) to Lemma 2.3. We employ the combinatorial extension of (1.3)(i) that is given by Lemma 2.1 to obtain the proof of Lemma 2.3 by the generating function method.

We next discuss Theorem 1.2 and its extension. Since it is noted by [5, p. 256] that $\sum_{k=1}^{n}(-1)^k d(n+k, k)/(n+k-1) = 0$, in Theorem 1.2 we may drop the factor $1/(n+k-1)$ due to $1/((n+k-1)(n+k-2)) = 1/(n+k-2) - 1/(n+k-1)$. Yet it is natural to leave the factor $1/(n+k-1)$ as stated for an extension of Theorem 1.2 to the following Theorem 1.5. This extension treats r-distinguished derangement numbers $d_r(n, k)$ that are defined as the number of permutations of $[n]$ into k cycles and no fixed points with the condition that the elements $1, 2, \ldots, r$ fall in distinct cycles; here $d_1(n, k) = d(n, k)$. These r-distinguished derangement numbers are not to be confused with Comtet's r-associated Stirling numbers of the first kind, [5, p. 257]. For the statement of the extension we first define power sums $P_r(n)$ as follows.

Definition 1.4 *Let $r \geq 1$ and let $n \geq r + 2$. Define*

$$P_r(n) = (-1)^r \sum_{i=0}^{r-1} i^{n-r-1}.$$

Theorem 1.5 *Let $r \geq 1$. Then for all $n \geq r + 2$ we have*

$$\sum_{k=1}^{n}(-1)^k d_r(n+k, k)/((n+k-r)(n+k-r-1)\cdots(n+k-2r)) = P_r(n) + (-1)^r \frac{B_{n-r}}{n-r}.$$

In the statement of Theorem 1.5 we see that the product $(n+k-1)(n+k-2)$ that appears in the denominator of the alternating sum of Theorem 1.2 becomes a falling factorial power (that is a descending product), denoted $(n+k-r)^{\underline{r+1}} = \prod_{j=0}^{r}(n+k-r-j)$, [7, p. 47].

We prove Theorem 1.2 in three steps. The first step is to use a generating function argument in Section 3 to prove the following identity.

Proposition 1.6 *Let $n \geq 2$. Then we have*

$$\sum_{k=1}^{n}(-1)^k \frac{d(n+k,k)}{(n+k-1)(n+k-2)} = \sum_{k=1}^{n} \frac{(-1)^k}{n+k-2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \binom{2n-2}{n+k-2}.$$

The second step of the proof of Theorem 1.2 relies on both (1.5) and the following connection between Stirling numbers of the first and second kinds going back to Schläfli that is derived by [11, (13.32)]; see also [10, (1)].

$$\begin{bmatrix} k+m \\ k \end{bmatrix} = (-1)^m \sum_{q=0}^{m}(-1)^q \binom{k+m+q-1}{m+q}\binom{k+2m}{m-q}\begin{Bmatrix} m+q \\ q \end{Bmatrix}. \tag{1.8}$$

We establish the second step, leading to Claim 1.7, as follows. First substitute $m = n-1$ and then plug in the Schläfli formula (1.8) for $\begin{bmatrix} n+k-1 \\ k \end{bmatrix} = \begin{bmatrix} m+k \\ k \end{bmatrix}$ into the right side of Proposition 1.6, where we rewrite $\binom{2n-2}{n+k-2} = \binom{2m}{m+k-1}$ and trivially reverse the order of summation to obtain the following expression for the right side of Proposition 1.6:

$$(-1)^m \sum_{q=0}^{m}(-1)^q \sum_{k=1}^{m+1} \frac{(-1)^k}{m+k-1} \binom{k+m+q-1}{m+q}\binom{k+2m}{m-q}\binom{2m}{m+k-1}\begin{Bmatrix} m+q \\ q \end{Bmatrix}. \tag{1.9}$$

Ignoring for the moment the leading sign $(-1)^m$ in this last expression, and, in view of the formula (1.5), it now suffices for the proof of Theorem 1.2 to prove that the following holds.

Claim 1.7

$$\sum_{k=1}^{m+1} \frac{(-1)^k}{m+k-1} \binom{k+m+q-1}{m+q}\binom{k+2m}{m-q}\binom{2m}{m+k-1} = -\frac{1}{m}\binom{m+1}{q+1} \Big/ \binom{m+q}{m},$$

for all $q = 0, 1, \cdots, m$, and all $m \geq 2$.

Here we have taken the inner sum in (1.9) except for the Stirling number factor $\begin{Bmatrix} m+q \\ q \end{Bmatrix}$ and matched it with the corresponding prefactor of the same Stirling number in the Jordan's formula (1.5), with an extra factor of $-\frac{1}{n-1} = -\frac{1}{m}$ on the right side of Claim 1.7 to account for the denominator of the right side of Theorem 1.2. The reader is warned that the statement of Claim 1.7 is *not* an identity valid for all q. It is easily verified by experiment, say at $q = m+1$, that Claim 1.7 fails outside the given range $q = 0, 1, \cdots, m$. So we simply want to show that the right side of the claim interpolates the left side in the given range. Once this is done, then by Proposition 1.6, (1.9), (1.5), and Claim 1.7 we obtain the statement of the Theorem 1.2 up to the factor $(-1)^m = (-1)^{n-1}$ that we dropped from (1.9). However, since $B_{n-1} = 0$ when $n-1$ is odd and $n \geq 3$, then these results will indeed prove Theorem 1.2.

The third step of the proof of Theorem 1.2 is to verify Claim 1.7. This is done in Section 3.1 by first manipulating the forms of the left side and right sides of the claim via (3.14)–(3.15) to eliminate a common factor $\binom{2m}{m-q}$ and thus write Claim 1.7 in the equivalent form (3.16). We refer the reader to the calculations of Section 3.1 for the details. From the equivalent form (3.16) of the claim, we now treat q as a continuous variable and indeed obtain a polynomial identity in x as follows.

Proposition 1.8 *Define $P(x)$ and $\phi(x)$ by (3.17)–(3.18). Then we have the following polynomial identity in the real variable x.*

$$\begin{aligned} \phi(x) + (1+x)P(x)\sum_{k=0}^{m}(-1)^k \frac{m+1}{(m+1+k+x)(m+k)}\binom{m}{k}\binom{k+2m+1}{m+1} \\ = P(x)\Big/\binom{2m}{m+1}. \end{aligned} \tag{1.10}$$

Since $\phi(q) = 0$ for all $q = 0, 1, \ldots, m$, we have immediately from Proposition 1.8 and the fact that $P(q) \neq 0$ for all $q = 0, 1, \ldots, m$ that (3.16) holds and thus the claim. The proof of Proposition 1.8 is obtained via Lagrange interpolation and Melzak's formula [11, (7.1)]. Again see Section 3.1 for details.

The pattern of proof of the extension Theorem 1.3 for $b_r(n, k)$ is simpler than, but in broad outline parallel to, the pattern we use to prove the extension Theorem 1.5 for derangement numbers $d_r(n, k)$. First, for the case $r = 1$, generating function arguments are used to represent the sums for both Theorems 1.1 and 1.2. This representation gives a direct link to the Bernoulli numbers in the case of $b(n, k)$ but, as shown by Proposition 1.6, not for the case of $d(n, k)$. For $r \geq 2$ we mention one other key step, that is to evaluate a companion alternating sum; already we see this in the context of $r = 1$ for the associated partition numbers wherein the companion sum is given by (1.6). This companion sum is generalized for $b_r(n, k)$ by Lemma 2.3 wherein we continue to find a zero sum. The companion sum for derangement numbers $d_r(n, k)$ is denoted $T_r(n)$ in (4.3), but only in the case $r = 1$ does this companion sum evaluate to zero.

In the proof of Theorem 1.5 in Section 4 we rely on an inductive approach to find the generating function of Lemma 4.1. For this, and Lemma 4.2 as well, we require a workable recurrence for $d_r(n, k)$. Yet, despite the simplicity and natural form of the generating function (4.2), a straightforward approach to a recurrence for $d_r(n, k)$ for $r \geq 2$ yields a relation (4.1) that is too complicated to implement. Instead we prove the following useful recurrence. Its proof in Section 4 is combinatorial and somewhat surprisingly involved.

Lemma 1.9 *Let $r \geq 1$ and let $n \geq 2k \geq 2r$. Then*

$$d_r(n,k) = (n-r)d_r(n-1,k) + (n-r)d_{r-1}(n-2,k-1). \tag{1.11}$$

To prove how the integer part $P_r(n)$ of Definition 1.4 arises in Theorem 1.5, we extend the generating function argument of Proposition 1.6 by way of Lemma 4.1 to represent the companion sum $T_r(n)$ as an r-Stirling number sum; see Definition 4.3 and Lemma 4.5. We further generalize $T_r(n)$ to an r-Stirling number sum $U_r(n,N)$ defined by (4.17). We evaluate $U_r(n,N)$ in Lemma 4.6, which generalizes Corollary 3.2(ii), and thus find $T_r(n) = (-1)^r(r-1)^{n-r}$ for $n \geq r+1$. The power sum part $P_r(n)$ then falls out by working backward with the recurrence of Lemma 4.2. Since power sums are represented by Bernoulli's formula (1.4), we obtain the following corollary to Theorem 1.5.

Corollary 1.10 *Let $r \geq 1$. Then for all $n \geq r+2$ we have*

$$\sum_{k=1}^{n}(-1)^k \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r+1}}} = \frac{(-1)^r}{n-r}\sum_{k=0}^{n-r}(-1)^k \binom{n-r}{k}(r-1)^{n-r-k}B_k,$$

where in the case $r=1$ we interpret the sum on the right as $(-1)^{n-r}\ 0^0 B_{n-1} = (-1)^{n-r}B_{n-1}$.

2 Proof of Theorems 1.1 and 1.3

In Section 2.1 we prove an extension of Theorem 1.1 to the numbers $b_r(n,k)$ in Theorem 1.3. A key to that proof is the basis $r=1$ of the extension, namely Theorem 1.1 itself.

Proof of Theorem 1.1. We apply a generating function defined as follows:

$$f(u,z) = \sum_{n\geq 1, k\geq 1} \frac{b(n+k,k)}{\binom{n+k-2}{k-1}} \frac{u^k}{k!} \frac{z^{n-1}}{(n-1)!}. \tag{2.1}$$

Now reorganize the sum in (2.1) with the substitution $m = n+k$, and also substitute the simplification $k!(n-1)!\binom{n+k-2}{k-1} = k(m-2)!$. Hence we have

$$f(u,z) = \sum_{m\geq 2, k\geq 1} b(m,k)\frac{u^k}{k}\frac{z^{m-k-1}}{(m-2)!}.$$

Substitute the recurrence (1.3)(i) to find in turn that

$$f(u,z) = \sum_{m\geq 2, k\geq 1} (kb(m-1,k) + (m-1)b(m-2,k-1))\frac{(u/z)^k}{k}\frac{z^{m-1}}{(m-2)!}. \tag{2.2}$$

Write the sum involving $kb(m-1,k)$ in (2.2) via the substitution $M=m-1$ and call it I:

$$I=\sum_{M\geq 1,k\geq 1} b(M,k)(u/z)^k\frac{z^M}{(M-1)!}. \tag{2.3}$$

Next, for the other term in (2.2), write the factor $(m-1)=(m-2)+1$ and thus break up the sum involving $(m-1)b(m-2,k-1)$ into the sum of two infinite sum expressions, II and III, after the substitutions $N=m-2$, $j=k-1$ as follows:

$$II=\sum_{N\geq 0,j\geq 0} b(N,j)\frac{(u/z)^{j+1}}{j+1}\frac{z^{N+1}}{(N-1)!},\quad III=\sum_{N\geq 0,j\geq 0} b(N,j)\frac{(u/z)^{j+1}}{j+1}\frac{z^{N+1}}{N!}. \tag{2.4}$$

To calculate I, we note by (1.1) that $\sum_{M\geq 1,k\geq 1} b(M,k)v^k\,\frac{z^M}{M!}=e^{v(e^z-1-z)}-1$, so by (2.3) we have $I=z\frac{\partial}{\partial z}(e^{v(e^z-1-z)}-1)$ evaluated at $v=u/z$. Thus, with $v=u/z$, we have

$$I=zv(e^z-1)e^{v(e^z-1-z)}=(e^z-1)\cdot ue^{uB},$$

for $B=B(z)=\frac{e^z-1-z}{z}$.

Next, for the purpose of calculation write $h=h(z)=e^z-1-z$, and thus find II of (2.4) as follows:

$$\begin{aligned} II &= z^2\int_0^{u/z}\frac{\partial}{\partial z}e^{vh(z)}dv \\ &= z^2h'\int_0^{u/z} ve^{vh(z)}dv \\ &= z^2h'\left[\frac{v}{h}e^{vh}-\frac{1}{h^2}e^{vh}\right]_0^{u/z}. \end{aligned}$$

Writing $B=h/z$ as before and putting in $h'=e^z-1$, we therefore have

$$II=(e^z-1)\left(\frac{ue^{uB}}{B}-\frac{e^{uB}-1}{B^2}\right).$$

Lastly, write III of (2.4) as:

$$III=z\int_0^{u/z}e^{vh(z)}dv=z\left[\frac{e^{uh}-1}{h}\right]_0^{u/z}=\frac{e^{uB}-1}{B}.$$

Fix now $n\geq 3$. To calculate the sum of the statement of the theorem, by the definition (2.1) of f it remains to determine the left hand side of the theorem as

$$S(n)=\sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-2}{k-1}=(n-1)![z^{n-1}]\sum_{k=1}^{n}(-1)^k k![u^k]f(u,z).$$

Since $f(u,z) = I + II + III$, we expand $S(n) = (n-1)![z^{n-1}]\sum_{k=1}^{n}(-1)^k k![u^k](I + II + III)$ as follows:

$$S(n) = (n-1)![z^{n-1}]\sum_{k=1}^{n}(-1)^k k![u^k]\left((e^z-1)\left(ue^{uB}+\frac{ue^{uB}}{B}-\frac{e^{uB}-1}{B^2}\right)+\frac{e^{uB}-1}{B}\right)$$
$$= (n-1)![z^{n-1}]\left((e^z-1)\sum_{k=1}^{n}(-1)^k k!\left(\frac{B^{k-1}}{(k-1)!}+\frac{B^{k-2}}{(k-1)!}-\frac{B^{k-2}}{k!}\right)+\sum_{k=1}^{n}(-1)^k k!\frac{B^{k-1}}{k!}\right). \tag{2.5}$$

Now for the first sum on the right side of (2.5) we have

$$\sum_{k=1}^{n}\left((-1)^k kB^{k-1} - (-1)^{k-1}(k-1)B^{k-2}\right) = (-1)^n nB^{n-1}$$

as a telescoping sum. For the second sum on the right side of (2.5) we have simply a finite geometric sum $\sum_{k=1}^{n}(-1)^k B^{k-1} = \frac{-1+(-1)^n B^n}{1+B}$. Hence by these calculations following from (2.5) we have shown that

$$S(n) = (n-1)![z^{n-1}]\left((e^z-1)(-1)^n nB^{n-1} + \frac{-1+(-1)^n B^n}{1+B}\right). \tag{2.6}$$

As power series, we have $B(z) = z/2 + \cdots$, and $e^z - 1 = z + \cdots$. Therefore, since as power series in z both $(e^z-1)B^{n-1}$ and $B^n/(1+B)$ will not contribute any terms containing z^k with $k \le n-1$, by (2.6) we have

$$S(n) = (n-1)![z^{n-1}]\left(-\frac{1}{1+B}\right).$$

Finally, $-\frac{1}{1+B} = -\frac{z}{e^z-1} = -\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$, where B_n is the n-th Bernoulli number. Hence the proof is complete by $(n-1)![z^{n-1}]\left(-\sum_{n=1}^{\infty} B_n \frac{z^n}{n!}\right) = -B_{n-1}$. □

2.1 Extension to $b_r(n,k)$: Proof of Theorem 1.3

Recall the definition, just preceding the statement of Theorem 1.3, of the r-distinguished associated Stirling numbers of the second kind $b_r(n,k)$. Define $b_0(n,k)$ by $b_0(n,k) = b_1(n,k) = b(n,k)$. Note that $b_r(n,k) = 0$ for $k < r$ or $n < 2k$. We take $b_r(0,0) = 0$ for $r \ge 2$.

Lemma 2.1 *Let $r \ge 1$. Then for all $n \ge 1$ and $k \ge 1$ we have*

$$b_r(n,k) = (k-r+1)b_{r-1}(n-1,k) + (n-r)b_{r-1}(n-2,k-1).$$

Proof. The second term accounts for the number $(n-r)$ of ways of forming doubleton sets with minimal element r that we can form with one of the elements x of $[n]\setminus[r]$ to comprise the k-th block, and thus complete the required partition from the $(k-1)$

blocks of $[n] \setminus \{r, x\}$ already counted by $b_{r-1}(n-2, k-1)$. The accounting provided by the first term on the right side of the lemma arises simply by adding the minimal element r to any one of the $(k-r+1)$ blocks of $[n] \setminus \{r\}$ that are not already distinguished by having a specified minimal element i for some $i \in [r-1]$. □

Lemma 2.2 *Let $r \geq 0$. Then*

$$\sum_{n \geq 0, k \geq 0} b_r(n+r, k+r) u^k \frac{z^n}{n!} = (e^z - 1)^r e^{u(e^z - 1 - z)}. \tag{2.7}$$

Proof. We proceed by induction in r. By (1.1) the basis $r = 0$ is verified. For the induction step, assume that the statement of the lemma is true for some $r \geq 0$. Denote by $f_r(u, z)$ the left side of (2.7). Then, by Lemma 2.1 write

$$f_{r+1}(u, z) = \sum_{n \geq 0, k \geq 0} (k+1) b_r(n+r, k+r+1) u^k \frac{z^n}{n!} + \sum_{n \geq 0, k \geq 0} n b_r(n+r-1, k+r) u^k \frac{z^n}{n!}$$

In the first sum change indices by $\ell = k+1$, and in the second sum write $N = n-1$. Thus

$$f_{r+1}(u, z) = \sum_{n \geq 0, \ell \geq 1} \ell b_r(n+r, \ell+r) u^{\ell-1} \frac{z^n}{n!} + \sum_{N \geq 0, k \geq 0} (N+1) b_r(N+r, k+r) u^k \frac{z^{N+1}}{(N+1)!}$$

Thus obtain $f_{r+1}(u, z) = \frac{\partial}{\partial u} f_r(u, z) + z f_r(u, z)$. By the induction hypothesis we easily compute this last expression to find

$$f_{r+1}(u, z) = \left((e^z - 1 - z)(e^z - 1)^r + z(e^z - 1)^r\right) e^{u(e^z - 1 - z)},$$

or $f_{r+1}(u, z) = (e^z - 1)^{r+1} e^{u(e^z - 1 - z)}$, as desired. □

Lemma 2.3 *Let $r \geq 1$ and $n \geq r+1$. Then*

$$\sum_{k=1}^{n} b_r(n+k, k)(-1)^k \Big/ \binom{n+k-r}{k-1} = 0.$$

Proof. Define

$$g_r(u, z) = \sum_{n \geq 1, k \geq 1} \frac{b_r(n+k, k)}{\binom{n+k-r}{k-1}} \frac{u^k}{(k-1)!} \frac{z^n}{(n-r+1)!}.$$

Substitute $m = n+k$ and use $\binom{n+k-r}{k-1}(k-1)!(n-r+1)! = (n+k-r)! = (m-r)!$ to write

$$g_r(u, z) = \sum_{m \geq 2, k \geq 1} b_r(m, k) u^k \frac{z^{m-k}}{(m-r)!} = z^r (u/z)^r \sum_{m \geq 2, k \geq 1} b_r(m, k)(u/z)^{k-r} \frac{z^{m-r}}{(m-r)!}.$$

Thus by Lemma 2.2 we have that $g_r(u,z) = u^r(e^z-1)^r e^{(u/z)(e^z-1-z)}$. Now write $B = B(z) = \frac{e^z-1-z}{z}$. Thus we have $g_r(u,z) = u^r(e^z-1)^r e^{uB}$. Therefore, fixing $r \geq 1$ and $n \geq r+1$, we compute

$$\sum_{k=1}^{n} b_r(n+k,k)(-1)^k \Big/ \binom{n+k-r}{k-1} = (n-r+1)![z^n](e^z-1)^r \sum_{k=r}^{n} [u^{k-r}](-1)^k (k-1)! e^{uB}, \tag{2.8}$$

where in the last sum we write the summation index k starting from $k=r$ because for $r \geq 1$ we have $b_r(m,k) = 0$ unless $k \geq r$. Now expand $e^{uB} = \sum_{j\geq 0} u^j B^j/j!$ and so rewrite the right side of (2.8) as

$$(n-r+1)![z^n](e^z-1)^r \sum_{k=r}^{n} \frac{(-1)^k (k-1)! B^{k-r}}{(k-r)!}. \tag{2.9}$$

Pull out a factor $(-1)^r(r-1)!$ and thus rewrite with a binomial coefficient to obtain the sum in (2.9) as

$$(-1)^r(r-1)! \sum_{k=r}^{n} (-1)^{k-r} B^{k-r} \binom{k-1}{r-1}.$$

Finally, because as a power series $B(z) = z/2 + \cdots$, and since likewise $e^z - 1 = z + \cdots$, we may replace the finite sum by the corresponding infinite sum in this last display because the terms beyond $k=n$ in the full series do not contribute to a calculation of (2.9). Thus we have that our desired expression (2.9) is written as

$$(n-r+1)![z^n](e^z-1)^r(-1)^r(r-1)! \sum_{k=r}^{\infty} (-1)^{k-r} B^{k-r} \binom{k-1}{r-1}.$$

But the infinite series collapses as the binomial series for $(1+B)^{-r}$. Hence, because by definition $(e^z-1)^r(1+B)^{-r} = z^r$, the proof is complete by $[z^n](z^r) = 0$ for all $n \geq r+1$. $\square$

Proof of Theorem 1.3. Let $r \geq 2$. Define $S_r(n) = \sum_{k=r}^{n} (-1)^k b_r(n+k,k) \Big/ \binom{n+k-r-1}{k-1}$. By Lemma 2.1 we make the reduction $b_r(n+k,k) = (k-r+1)b_{r-1}(n+k-1,k) + (n+k-r)b_{r-1}(n+k-2,k-1)$. Hence by substituting this reduction and making a change of index $j = k+1$ in the first sum of the resulting expression for $S_r(n)$ we have that $S_r(n)$ is given by

$$\sum_{j=r+1}^{n+1} (-1)^{j-1} \frac{(j-r)b_{r-1}(n+j-2,j-1)}{\binom{n+j-r-2}{j-2}} + \sum_{k=r}^{n} (-1)^k \frac{(n+k-r)b_{r-1}(n+k-2,k-1)}{\binom{n+k-r-1}{k-1}}.$$

Then in turn, by combining the sums into one, using the fact that the terms in the first sum at both $j = n+1$ (by $b_{r-1}(2n-1,n) = 0$) and $j = r$ are zero, we rewrite $S_r(n)$ under a single summation by

$$S_r(n) = \sum_{k=r}^{n} (-1)^k b_{r-1}(n+k-2,k-1) \left(\frac{-(k-r)}{\binom{n+k-r-2}{k-2}} + \frac{n+k-r}{\binom{n+k-r-1}{k-1}} \right).$$

Now calculate

$$\frac{-(k-r)}{\binom{n+k-r-2}{k-2}}+\frac{n+k-r}{\binom{n+k-r-1}{k-1}} = (k-2)!\frac{-(k-r)(n+k-r-1)+(k-1)(n+k-r)}{(n+k-r-1)\cdots(n-r+1)}.$$

The numerator of the last fraction may be rewritten $r(n+k-r-1)-(n-r)$. Correspondingly we obtain that $S_r(n) = I + II$ where

$$I = r\sum_{k=r}^{n}(-1)^k\ \frac{b_{r-1}(n+k-2,k-1)}{\binom{n+k-r-2}{k-2}},$$

$$II = \frac{-(n-r)}{n-r+1}\sum_{k=r}^{n}(-1)^k\ \frac{b_{r-1}(n+k-2,k-1)}{\binom{n+k-r-1}{k-2}}.$$

Now change both indices $n' = n-1$ and $k' = k-1$ and also put $r' = r-1$ to obtain

$$II = \frac{n-r}{n-r+1}\sum_{k'=r'}^{n'}(-1)^{k'}\ \frac{b_{r'}(n'+k',k')}{\binom{n'+k'-r'}{k'-1}}.$$

Hence we have $II = 0$ by Lemma 2.3. Further, we have

$$I = (-r)\sum_{k'=r'}^{n'}(-1)^{k'}\ \frac{b_{r'}(n'+k',k')}{\binom{n'+k'-r'-1}{k'-1}} = (-r)S_{r'}(n').$$

Therefore we have shown $S_r(n) = (-r)S_{r'}(n')$. Hence because the case $r = 1$ in the statement of the theorem is the result of Theorem 1.1, by backward induction the proof is complete. □

3 Proof of Theorem 1.2

In this section we reduce the problem of Theorem 1.2 by a series of steps. The first step in Proposition 1.6 is to apply a generating function argument to write the sum in the statement of this theorem in terms of a sum involving ordinary Stirling numbers of the first kind. To prepare for this step we want an arithmetical identity for the ordinary Stirling numbers of the first kind as follows.

Lemma 3.1 *[3, Thm. 7] Let $m \geq 2$ and $m > k \geq 1$. Then we have*

$$\begin{bmatrix} m \\ m-k \end{bmatrix} = \sum_{1\leq i_1<i_2<\cdots<i_k<m} i_1 i_2\cdots i_k. \tag{3.1}$$

Further $\begin{bmatrix} m \\ m \end{bmatrix} = 1$, for all $m \geq 0$.

For example, $\left[{3 \atop 1}\right] = 1\cdot 2 = 2$, while $\left[{3 \atop 2}\right] = 1+2 = 3$. Likewise $\left[{4 \atop 2}\right] = 1\cdot 2+1\cdot 3+2\cdot 3 = 11$.

Proof of Proposition 1.6. Define

$$f(u,z) = \sum_{n\geq 1, k\geq 1} \frac{d(n+k,k)}{n+k-1}\, u^k \frac{z^n}{(n+k-2)!}. \tag{3.2}$$

Make the substitution $m = n+k$ and write $z^n = z^{n+k}/z^k = z^m/z^k$ to rewrite (3.2) as

$$f(u,z) = \sum_{m\geq 2, k\geq 1} d(m,k)(u/z)^k \frac{z^m}{(m-1)!} \tag{3.3}$$

Therefore, by denoting $g(u,z)$ the generating function of $d(n,k)$ defined by (1.2), we have by (3.3) that

$$f(u,z) = z\frac{\partial}{\partial z} g(v,z) \text{ evaluated at } v = u/z.$$

Here we have

$$\begin{aligned} z\frac{\partial}{\partial z} g(v,z) &= z\frac{\partial}{\partial z}\left(e^{-zv}(1-z)^{-v}\right) \\ &= zve^{-zv}\left((1-z)^{-v-1} - (1-z)^{-v}\right) \\ &= z^2 ve^{-zv}(1-z)^{-v-1}. \end{aligned}$$

Therefore, evaluation at $v = u/z$ yields

$$f(u,z) = zue^{-u}(1-z)^{-u/z-1} \tag{3.4}$$

Now expand $(1-z)^{-v-1}$ as a binomial expansion about $z = 0$ to obtain

$$(1-z)^{-v-1} = 1 + \frac{(v+1)}{1!}z + \frac{(v+1)(v+2)}{2!}z^2 + \frac{(v+1)(v+2)(v+3)}{3!}z^3 + \cdots.$$

Thus, after plugging in $v = u/z$ we find by (3.4) that

$$f(u,z) = ze^{-u}\cdot u\left(1 + \frac{(u+z)}{1!} + \frac{(u+z)(u+2z)}{2!} + \frac{(u+z)(u+2z)(u+3z)}{3!} + \cdots\right). \tag{3.5}$$

We have $(u+z)(u+2z)\cdots(u+pz) = \sum_{n=0}^{p} u^{p-n}z^n \sum_{1\leq i_1<i_2<\cdots<i_n<p+1} i_1 i_2\cdots i_n$. Therefore, after applying Lemma 3.1 to compute the inner sum, and multiplying by an additional factor of u, we have the basic representation:

$$u(u+z)(u+2z)(u+3z)\cdots(u+pz) = \sum_{n=0}^{p} \left[{p+1 \atop p+1-n}\right] z^n u^{p+1-n}. \tag{3.6}$$

Therefore by (3.4)–(3.6), putting the factor of z now under the sum we must expand $f(u,z) = e^{-u}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{n=0}^{p}\begin{bmatrix} p+1 \\ p+1-n \end{bmatrix} z^{n+1}u^{p+1-n}$. We write $e^{-u} = \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{u^{\ell}}{\ell!}$ and thus find:

$$
\begin{aligned}
f(u,z) &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\ \frac{u^{\ell}}{\ell!}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{n=0}^{p}\begin{bmatrix} p+1 \\ p+1-n \end{bmatrix} z^{n+1}u^{p+1-n} \\
&= u\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{z}{0!} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\frac{z^2}{1!} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\frac{z^3}{2!} + \cdots\right) \\
&\quad + u^2\left(\left(-\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{z}{1!0!} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\frac{z}{0!1!}\right) + \left(-\begin{bmatrix} 2 \\ 1 \end{bmatrix}\frac{z^2}{1!1!} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}\frac{z^2}{0!2!}\right) + \cdots\right) \\
&\quad + u^3\Bigg(\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{z}{2!0!} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\frac{z}{1!1!} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}\frac{z}{0!2!}\right) \\
&\qquad\quad + \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\frac{z^2}{2!1!} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}\frac{z^2}{1!2!} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}\frac{z^2}{0!3!}\right) + \cdots\Bigg) + \cdots
\end{aligned}
\tag{3.7}
$$

Here the first expanded sum for the triple sum expression is the case $p - n = 0$, $\ell = 0$, for all $n = 0, 1, 2, \cdots$; the second expanded sum is the case $p-n = 0, \ell = 1$ or $p-n = 1, \ell = 0$ for each grouped parenthetical term involving powers of z^{n+1}, where $n = 0, 1, \cdots$ for successive grouped terms; and the third expanded sum is the case $p-n = 0, \ell = 2$, or $p-n = 1, \ell = 1$ or $p-n = 2, \ell = 0$, again with $n = 0, 1, \cdots$ picking up powers of z^{n+1} in successive grouped terms. The Stirling number coefficients for the term $u^k z^{n+1}/(p!\ell!)$, where $p + \ell = n + k - 1$, have top index $n + j$ and bottom index j, for $j = k - \ell$, $0 \le \ell \le k - 1$. That is, we take $j = p + 1 - n$ for the bottom index and $n + j$ for the top index of the Stirling number coefficient. So we have $k = (p + 1 - n) + \ell = j + \ell$ for the power on u, and $n + 1$ for the power on z. Thus we convert to summation indices k, n, and j, with $k \ge 1$, $n \ge 0$ and $1 \le j \le k$. The sign of the term $u^k z^{n+1}/(p!\ell!) = u^k z^{n+1}/((n+j-1)!(k-j)!)$ is $(-1)^{\ell} = (-1)^j(-1)^k$. Therefore we have

$$
f(u,z) = \sum_{k=1}^{\infty}(-1)^k\ u^k\sum_{n=0}^{\infty} z^{n+1}\sum_{j=1}^{k}(-1)^j\begin{bmatrix} n+j \\ j \end{bmatrix}\frac{1}{(n+j-1)!(k-j)!}. \tag{3.8}
$$

We note that the $u^k z^{n+1}$ terms in the expansion of $f(u,z)$ where $n < k$ must have coefficient zero by $d(n + k, k) = 0$ in this case. For example, the $u^3 z$ term has coefficient $\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{2!0!} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\frac{1}{1!1!} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}\frac{1}{0!2!} = \frac{1}{2} - 1 + \frac{1}{2} = 0$, and the u^3z^2 term has coefficient $\begin{bmatrix} 2 \\ 1 \end{bmatrix}\frac{1}{2!1!} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}\frac{1}{1!2!} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}\frac{1}{0!3!} = \frac{1}{2} - \frac{3}{2} + 1 = 0$.

Now we rewrite $f(u,z)$ in (3.8) by replacing n by $n-1$ so as to make a power of z^n, consistent with (3.2). Correspondingly we introduce $\frac{1}{(n+j-2)!(k-j)!} = \binom{n+k-2}{n+j-2}\frac{1}{(n+k-2)!}$. We rewrite (3.8) accordingly as follows.

$$
f(u,z) = \sum_{k=1}^{\infty}(-1)^k\ u^k\sum_{n=1}^{\infty} z^{n}\sum_{j=1}^{k}(-1)^j\begin{bmatrix} n-1+j \\ j \end{bmatrix}\binom{n+k-2}{n+j-2}\frac{1}{(n+k-2)!}. \tag{3.9}
$$

We simply write, by the definition (3.2) of $f(u,z)$ and by (3.9), that for any $n \geq 1$ and $k \geq 1$,

$$(-1)^k \frac{d(n+k,k)}{(n+k-1)} = (-1)^k [u^k z^n](n+k-2)! f(u,z) = \sum_{j=1}^{k} (-1)^j \begin{bmatrix} n-1+j \\ j \end{bmatrix} \binom{n+k-2}{n+j-2}. \tag{3.10}$$

Finally, fix $n \geq 2$. Divide through (3.10) by $(n+k-2)$ and sum over k from 1 to n to find

$$\begin{aligned} \sum_{k=1}^{n} (-1)^k \frac{d(n+k,k)}{(n+k-1)(n+k-2)} &= \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j \begin{bmatrix} n-1+j \\ j \end{bmatrix} \binom{n+k-2}{n+j-2} \frac{1}{n+k-2} \\ &= \sum_{j=1}^{n} (-1)^j \begin{bmatrix} n-1+j \\ j \end{bmatrix} \sum_{k=j}^{n} \binom{n+k-2}{n+j-2} \frac{1}{n+k-2}, \end{aligned} \tag{3.11}$$

where we changed the order of summation of the double sum in k and j at the last step. However, we have the simple binomial identity $\sum_{m=a}^{b} \frac{1}{m}\binom{m}{a} = \sum_{m=a}^{b} \frac{1}{a}\binom{m-1}{a-1} = \frac{1}{a}\binom{b}{a}$, because $\binom{m-1}{a-1} = \binom{m}{a} - \binom{m-1}{a}$, so we have a telescoping sum in m. Therefore, with $m = n+k-2$, $a = n+j-2$, and $b = n+n-2 = 2n-2$, we have $\sum_{k=j}^{n} \binom{n+k-2}{n+j-2} \frac{1}{n+k-2} = \binom{2n-2}{n+j-2} \frac{1}{n+j-2}$. Thus by (3.11) we have

$$\sum_{k=1}^{n} (-1)^k \frac{d(n+k,k)}{(n+k-1)(n+k-2)} = \sum_{j=1}^{n} \frac{(-1)^j}{n+j-2} \begin{bmatrix} n-1+j \\ j \end{bmatrix} \binom{2n-2}{n+j-2}. \tag{3.12}$$

This completes the proof of the proposition. □

Corollary 3.2 *The following hold under the stated conditions.*

$$\begin{aligned} &\text{(i)} \quad \sum_{j=1}^{N-n+2} (-1)^j \begin{bmatrix} n+j-1 \\ j \end{bmatrix} \binom{N}{n+j-2} = 0, \quad \textit{for all } \ n \geq 1, \ N \geq 2n-1; \\ &\text{(ii)} \quad \sum_{j=1}^{N-n+1} (-1)^j \begin{bmatrix} n+j-1 \\ j \end{bmatrix} \binom{N}{n+j-1} = 0, \quad \textit{for all } \ n \geq 2, \ N \geq 2n-1. \end{aligned} \tag{3.13}$$

Proof. First, let $k \geq n+1$ in (3.10). Then because $d(n+k,k) = 0$, the sum on the right side of (3.10) is zero. Now put $N = n+k-2$. So the sum (3.13)(i) is zero for all $N \geq n+(n+1)-2 = 2n-1$. To prove (3.13)(ii), if $n \geq 2$ we have $\sum_{k=1}^{n} (-1)^k d(n+k,k)/(n+k-1) = 0$, [5, p. 256]. Therefore because $d(n+k,k) = 0$ for $k \geq n+1$, we have $\sum_{k=1}^{M} (-1)^k d(n+k,k)/(n+k-1) = 0$ for any $M \geq n$. Therefore by (3.10), if $M \geq n$, we have $\sum_{k=1}^{M} \sum_{j=1}^{k} (-1)^j \begin{bmatrix} n-1+j \\ j \end{bmatrix} \binom{n+k-2}{n+j-2} = \sum_{j=1}^{M} (-1)^j \begin{bmatrix} n-1+j \\ j \end{bmatrix} \sum_{k=j}^{M} \binom{n+k-2}{n+j-2} = 0$. But the inner sum of this last double sum is simply $\binom{n+M-1}{n+j-1}$. Now put $M = N-n+1$ and substitute this binomial expression in place of the inner sum of the last double sum to obtain (3.13)(ii). It only remains to check the condition on the size of N. By the definition of M in terms of N for all $N \geq 2n-1$ we have $M \geq n$. Therefore the proof is complete. $\square$

3.1 Proof of Claim 1.7 and Theorem 1.2

We first undertake to simplify the statement of Claim 1.7 by isolating the variable q. First, by simply expanding binomial coefficients and rearranging factors we have that the product of binomial factors on the left side of Claim 1.7 is as follows.

$$\binom{k+m+q-1}{m+q}\binom{k+2m}{m-q}\binom{2m}{m+k-1} = \frac{m+1}{m+k+q}\binom{2m}{m-q}\binom{m}{k-1}\binom{k+2m}{m+1}. \tag{3.14}$$

Indeed, the left side of (3.14) is

$$\begin{aligned}
&\frac{(k+m+q-1)!}{(m+q)!(k-1)!}\frac{(k+2m)!}{(k+m+q)!(m-q)!}\frac{(2m)!}{(m+k-1)!(m-k+1)!}\\
&= \frac{1}{k+m+q}\frac{(k+2m)!}{(k-1)!}\frac{(2m)!}{(m+q)!(m-q)!}\frac{m+1}{(m+k-1)!(m+1)!}\frac{m!}{(m-k+1)!}\\
&= \frac{m+1}{k+m+q}\frac{(2m)!}{(m+q)!(m-q)!}\frac{(k+2m)!}{(m+k-1)!(m+1)!}\frac{m!}{(k-1)!(m-k+1)!}
\end{aligned}$$

where we canceled $(k+m+q)!$ in numerator and denominator and multiplied and divided by $m!$ in the form $(m+1)\frac{m!}{(m+1)!}$ in the second line, and moved the factors $\frac{(k+2m)!}{(k-1)!}$ in the third line. This third line is evidently the right side of (3.14). Next we rewrite the right side of Claim 1.7 modulo the minus sign as follows:

$$\frac{1}{m}\binom{m+1}{q+1}\Big/\binom{m+q}{q} = \frac{1}{q+1}\binom{2m}{m-q}\Big/\binom{2m}{m+1}. \tag{3.15}$$

Indeed,

$$\begin{aligned}
\frac{1}{m}\frac{(m+1)!}{(q+1)!(m-q)!}\frac{q!m!}{(m+q)!} &= \frac{1}{q+1}\frac{m+1}{m}\frac{m!m!}{(2m)!}\frac{(2m)!}{(m-q)!(m+q)!}\\
&= \frac{1}{q+1}\frac{(m-1)!(m+1)!}{(2m)!}\frac{(2m)!}{(m-q)!(m+q)!},
\end{aligned}$$

where we multiplied and divided by $(2m)!$. So (3.15) is verified. Finally, by plugging in (3.14) and (3.15) respectively into the left and right sides of Claim 1.7, and canceling the common factor $\binom{2m}{m-q}$ on the two sides, and finally changing the index of summation with k in place of $k-1$ running from 0 to m, and thus introducing a minus sign on the left side, we see that Claim 1.7 is equivalent to the following statement:

$$\sum_{k=0}^{m}(-1)^k \frac{m+1}{(m+1+k+q)(m+k)}\binom{m}{k}\binom{k+2m+1}{m+1} = \frac{1}{q+1}\frac{1}{\binom{2m}{m+1}}, \tag{3.16}$$

for all $q = 0, 1, \cdots, m,$ and all $m \geq 2$.

We now treat q in (3.16) as a real variable. Define a polynomial of degree $m+1$ by

$$P(x) = \prod_{k=0}^{m}(m+1+k+x), \quad \text{for all real } x. \tag{3.17}$$

Since we want to prove that the two sides of (3.16) indeed interpolate each other at the points $q = 0, 1, \cdots, m$ and since multiplication of both sides by $(1+q)P(q)$ leaves a polynomial of degree $(m+1)$ in q on the left and a constant multiple of $P(q)$ on the right, there would be a polynomial of degree $(m+1)$ that makes up the difference. Define the polynomial

$$\phi(x) = (-1)^{m+1}x(x-1)(x-2)\cdots(x-m) = \prod_{j=0}^{m}(j-x). \tag{3.18}$$

Then we see that the problem of the verification of (3.16) will be solved by Proposition 1.8. Before proving this proposition we state some useful formulae for polynomials from [11, Chp. 7].

Theorem 3.3 *Lagrange Interpolation Formula [11, Thm. 7.1] Let* $\phi(x) = \sum_{i=0}^{n} a_i x^i$ *with* $n \geq 1$. *Then*

$$\phi(x) = \sum_{k=0}^{n} \phi(x_k) \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i},$$

whenever $\{x_i\}_{i=0}^{n}$ *is a set of cardinallity* $n+1$.

Theorem 3.4 *Melzak's Formula [11, (7.1)] Let* $f(x) = \sum_{i=0}^{n} a_i x^i$ *with* $n \geq 0$. *Let* y *be an arbitrary complex number with* $y \neq 0, -1, -2, \ldots, -n$. *Then*

$$f(x+y) = y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}.$$

Proof of Proposition 1.8. The idea of the proof is to apply the Lagrange interpolation formula to $\phi(x)$ with $m+1$ in place of n and with the interpolation points $\{x_i\}_{i=0}^{m+1}$ chosen in such a way that the polynomial term $(1+x)P(x)/(m+1+k+x) = (1+x)\prod_{i=0,\ i\neq k}^{m}(m+1+i+x)$ is matched in the formula for ϕ instead by $\prod_{i=0,\ i\neq k}^{m+1}(m+1+i+x)$, where the upper index $m+1$ in this last product agrees with the Lagrange formula.

By Theorem 3.3 and the definition (3.18) of $\phi(x)$ this means that we simply take as interpolation points $x_i = -(m+1+i)$, for all $i = 0, 1, \ldots, , m+1$. Now we develop a combinatorial formula for $\phi(x)$ based on Lagrange's formula. Define

$$Q(x) = \prod_{i=0}^{m+1} (m+1+i+x), \tag{3.19}$$

so $Q(x)$ has degree $n+1 = m+2$. Then by (3.18) and our choice of the interpolation points x_i in the Lagrange formula, the k-th summand of the interpolation formula for $\phi(x)$ may be written as

$$\phi(x_k) \prod_{i=0,\ i\neq k}^{m+1} \frac{x-x_i}{x_k-x_i} = \frac{\prod_{j=0}^{m}(j-x_k)}{\prod_{i=0,\ i\neq k}^{m+1}(x_k-x_i)} \frac{Q(x)}{m+1+k+x}, \quad k = 0, 1, 2, \ldots, m+1, \tag{3.20}$$

since by (3.19) we have $\prod_{i=0,\ i\neq k}^{m+1}(x-x_i) = Q(x)/(m+1+k+x)$. Since $(j-x_k) = (m+1+k+j)$, and $(x_k - x_i) = (i-k)$, for our representation of $\phi(x)$ it remains to evaluate the coefficient

$$\frac{\prod_{j=0}^{m}(j-x_k)}{\prod_{i=0,\ i\neq k}^{m+1}(x_k-x_i)} = \frac{\prod_{j=0}^{m}(m+1+k+j)}{\prod_{i=0,\ i\neq k}^{m+1}(i-k)}. \tag{3.21}$$

It is easy to see that the denominator of (3.21) is $(-1)^k k!(m+1-k)!$. Moreover the numerator of this fraction is the falling factorial $(k+2m+1)^{\underline{m+1}}$ introduced after Theorem 1.5. Hence (3.21) becomes, after multiplying and dividing by $(m+1)!$,

$$\frac{(m+1)!}{(-1)^k k!(m+1-k)!} \frac{(k+2m+1)^{\underline{m+1}}}{(m+1)!} = (-1)^k \binom{m+1}{k} \binom{k+2m+1}{m+1}. \tag{3.22}$$

Therefore by Theorem 3.3 and (3.20)–(3.22) the assertion (1.10) of the proposition may be rewritten as follows.

$$\begin{aligned} &\sum_{k=0}^{m+1} (-1)^k \frac{Q(x)}{m+1+k+x} \binom{m+1}{k} \binom{k+2m+1}{m+1} \\ &+ \sum_{k=0}^{m} (-1)^k \frac{(m+1)(1+x)P(x)}{(m+1+k+x)(m+k)} \binom{m}{k} \binom{k+2m+1}{m+1} = P(x) / \binom{2m}{m+1}. \end{aligned} \tag{3.23}$$

Notice that in the second sum of (3.23) the binomial coefficient $\binom{m}{k}$ is automatically zero when $k = m+1$, so we may regard the sum of the two sums on the left side as a single sum as k runs from 0 to $m+1$. We claim that we can reduce the sum of the two corresponding k-th summands modulo the common factor $(-1)^k\binom{k+2m+1}{m+1}$ as follows.

$$\frac{Q(x)}{m+1+k+x}\binom{m+1}{k}+\frac{(m+1)(1+x)P(x)}{(m+1+k+x)(m+k)}\binom{m}{k} \\ = P(x)\left(\frac{m+1}{m+k}\binom{m}{k}+\binom{m+1}{k}\right), \tag{3.24}$$

so that the two summands add simply to a combinatorial multiple of $P(x)$. Now by the definitions (3.17) and (3.19) of $P(x)$ and $Q(x)$, we have $Q(x) = (2m+2+x)P(x)$. Thus to verify (3.24) we must simply check, after transposing the binomial terms on the right side of (3.24) to their matching terms on the left, that

$$\left(\frac{2m+2+x}{m+1+k+x}-1\right)\binom{m+1}{k}+\left(\frac{(m+1)(1+x)}{(m+1+k+x)(m+k)}-\frac{m+1}{m+k}\right)\binom{m}{k}=0. \tag{3.25}$$

After obtaining common denominators in the two differences in (3.25) we must simply verify that $\frac{m+1-k}{m+1+k+x}\binom{m+1}{k}-\frac{m+1}{(m+1+k+x)}\binom{m}{k}=0$. Obviously we can dispense with the denominators in this last difference so the verification boils down to $(m+1)\binom{m+1}{k}-(m+1)\binom{m}{k}-k\binom{m+1}{k}=(m+1)\binom{m+1}{k-1}-k\binom{m+1}{k}=0$. Thus (3.24) has been verified. Hence, by (3.24) the assertion (3.23), which we recall is equivalent to the statement of the proposition, takes the form

$$P(x)\sum_{k=0}^{m+1}(-1)^k\left(\frac{m+1}{m+k}\binom{m}{k}+\binom{m+1}{k}\right)\binom{k+2m+1}{m+1}=P(x)/\binom{2m}{m+1}. \tag{3.26}$$

We now dispense with the common factor $P(x)$, and so to complete the proof of the proposition we are left in (3.26) with a final combinatorial identity to prove. To handle the proposed combinatorial identity (3.26) we will first rewrite $\frac{m+1}{m+k}\binom{m}{k}=\frac{k+1}{m+k}\binom{m+1}{k+1}$, and then change index by $j=k+1$. Thus (3.26) becomes

$$-\sum_{j=0}^{m+1}(-1)^j\frac{j}{m-1+j}\binom{m+1}{j}\binom{j+2m}{m+1}+\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\binom{k+2m+1}{m+1}=1/\binom{2m}{m+1}. \tag{3.27}$$

Here, for the first sum on the left, the term $j=0$ has contribution zero and the term $j=m+2$ has contribution zero, so we've included the case $j=0$ and excluded the case $j=m+2$ to match the same initial and final indices of the sum over k. Thus convert both sums to sums over k and rewrite $\frac{k}{m-1+k}=1-\frac{m-1}{m-1+k}$ in the first sum and thus obtain that the left side of (3.27) is written

$$(m-1)\sum_{k=0}^{m+1}(-1)^k\frac{1}{m-1+k}\binom{m+1}{k}\binom{k+2m}{m+1} \\ +\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\left(\binom{k+2m+1}{m+1}-\binom{k+2m}{m+1}\right)=I+II. \tag{3.28}$$

Now by the binomial recurrence we have $II = \sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\binom{k+2m}{m}$. But by [11, (6.35)] we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+x}{m} = (-1)^n\binom{x}{m-n}$. Thus with $n = m+1$ and $x = 2m$ we find that $II = 0$.

Finally to handle I of (3.28) we apply Melzak's formula. The procedure is similar to that shown by [11, (7.29)–(7.30)]. To match the form of Melzak's formula in Theorem 3.4, put back $n = m+1$ and define $y = m-1 = n-2$. Also take the polynomial function of degree n as $f(x) = \binom{x+2n-2}{n}$. For our application of Theorem 3.4 we choose the real variable x by $-x = 3n-3$. To motivate this choice of x, notice that to apply Melzak's formula as stated we must apply the -1 transformation $\binom{x}{n} = (-1)^n\binom{-x+n-1}{n}$ to rewrite $f(x-k)$ as follows. We have $f(x-k) = (-1)^n\binom{-(x-k+2n-2)+n-1}{n} = (-1)^n\binom{k+2n-2}{n}$, where we applied our choice of x. On the other hand, using the original formula for $f(x)$, we have $f(x+y) = \binom{-(3n-3)+2n-2+n-2}{n} = \binom{-1}{n} = (-1)^n$. By our choices of n, x, and y the term I from (3.28) is indeed written in the form

$$I = y(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x-k)/(y+k).$$

Therefore we have by Melzak's formula that

$$I = (-1)^n f(x+y)/\binom{y+n}{n} = 1/\binom{2n-2}{n}, \tag{3.29}$$

since $f(x+y) = (-1)^n$ and $y+n = 2n-2$. But $2n-2 = 2m$ and $n = m+1$, so $I = 1/\binom{2m}{m+1}$, as desired. That is, by (3.27)–(3.29), and using $II = 0$, we have verified (3.26). Thus the proposition is proved. □

Proof of Claim 1.7. Let $q \in \{0, 1, 2, \ldots, m\}$. Plug in $x = q$ in the polynomial identity of Proposition 1.8. By definition (3.18) we have $\phi(q) = \prod_{j=0}^{m}(j-q) = 0$. Since the term $\phi(q)$ vanishes, and since $P(q) \neq 0$ for $P(x)$ defined by (3.17), and since $P(q)$ is thus a common nonzero factor of the remaining equality of (1.10), after canceling this factor and dividing both sides by $1+q$ we obtain that the statement (3.16) holds for the given q. Therefore since it was shown by (3.14)–(3.15) that statement (3.16) is equivalent to the statement of the claim, the proof is complete. □

Proof of Theorem 1.2. We already argued in the Introduction that, via Proposition 1.6, (1.5), and (1.8)–(1.9), the theorem follows from Claim 1.7. □

4 Extension to $d_r(n, k)$: Proofs of Theorem 1.5 and Corollary 1.10

Let $d_r(n, k)$ be the number of permutations of $[n]$ into k cycles with no fixed points such that $1, 2, \ldots, r$ fall in distinct cycles; this is the r-distinguished case for derangements. Define $d_0(n, k) = d_1(n, k) = d(n, k)$. In this section we obtain an extension of

Theorem 1.2 to $d_r(n,k)$ for all $r \geq 1$. The pattern of proof follows roughly the proof of Theorem 1.3 for the case of the partition numbers $b_r(n,k)$. For the derangement numbers we work backwards to the $r = 1$ case proved in Theorem 1.2 by applying Lemma 4.2. However, in the case of $d_r(n,k)$, besides the appearance of the Bernoulli numbers in the form $B_{n-r}/(n-r)$, we get a nonzero integer part $P_r(n)$ for the evaluation of the appropriately extended alternating sum. It is not hard to find this integer part experimentally, but in contrast to the case of partitions, the proof of its form takes some work that does not appear to be readily available in the literature.

The first step in the pattern of Section 2.1 is to find a suitable recurrence for $d_r(n,k)$. A recurrence obtained by emulating a proof of the triangular recurrence (1.3)(ii) is as follows:

$$d_r(n,k) = rd_{r-1}(n-2,k-1) + (n-r-1)d_r(n-2,k-1) + (n-1)d_r(n-1,k) \quad (4.1)$$

Indeed we have the following three disjoint and exhaustive possibilities.

1. Element n makes a 2-cycle with one of the elements of $x \in [r]$, where the other $(r-1)$ elements $[r]\setminus\{x\}$ are already distinguished for $(k-1)$-cycle derangements of $[n-1] \setminus \{x\}$.
2. Element n makes a 2-cycle with one of the elements $y \in [n-1] \setminus [r]$, where there are already r distinguished cycles from $(k-1)$-cycle derangements of $[n-1] \setminus \{y\}$.
3. We already have derangements of $[n-1]$ into k cycles with r distinguished cycles and we place element n in any one of the $(n-1)$ places available, one place after each element of $[n-1]$, in any such derangement.

The value of (4.1) lies in the simplicity of its proof. Since $d_0(n,k)$ is the same as $d_1(n,k) = d(n,k)$ we recover (1.3)(ii) by the case $r = 1$. Yet, despite its straightforward derivation, the recurrence (4.1) does not lend itself easily to developing a generating function for $d_r(n,k)$. Moreover, besides wanting a nicer recurrence that we can use to establish a generating function, we want a nicer recurrence for our proofs to follow. Surprisingly, even though Lemma 1.9 does yield such a recurrence, our proof of it requires some construction.

Proof of Lemma 1.9. We must prove that for all $r \geq 1$ and all $n \geq 2k \geq 2r$ we have

$$d_r(n,k) = (n-r)d_r(n-1,k) + (n-r)d_{r-1}(n-2,k-1).$$

Denote $\Pi(r,n,k)$ as the derangements of $[n]$ into k cycles such that the elements of $[r]$ fall in disjoint cycles. We break up $\Pi(r,n,k)$ into two disjoint sets A and B as follows. Extend each derangement $\alpha \in \Pi(r,n-1,k)$ for each $y \in [r,n-1]$ to obtain a derangement $\alpha^+ \in \Pi(r,n,k)$ by inserting the element n after the element y in α. The set A consists of all these extended derangements α^+ as y ranges between r and $n-1$. In particular we never insert the element n immediately after one of the elements i with $1 \le i \le r-1$. We get accordingly the cardinality of the subset A as $|A| = (n-r)|\Pi(r, n-1, k)| = (n-r)d_r(n-1, k)$.

The definition of the set B proceeds in several cases.

Case *b.1*: First consider $y = n$. Consider the derangements of $[n] \setminus \{r, y\} = [n] \setminus \{r, n\}$ into $(k-1)$ cycles such that the elements of $[r-1]$ fall in disjoint cycles. For each such derangement β simply extend it to a derangement $\beta^+ \in \Pi(r, n, k)$ by adding the 2-cycle (r, n) as a k-th cycle. We take β^+ to belong to B.

Case *b.2*: Consider next $r+1 \le y \le n-1$. Let β be a derangement of $[n] \setminus \{y, n\}$ into $(k-1)$ cycles such that the elements of $[r-1]$ fall in disjoint cycles. Form the doubleton cycle (y, n) as a k-th cycle that we add to β to obtain a derangement β^* that is a k-cycle decomposition of $[n]$ such that the elements of $[r-1]$ fall in disjoint cycles: $\beta^* \in \Pi(r-1, n, k)$.

Now consider the main subcases *b.2.I* and *b.2.II* under case *b.2*.

Subcase *b.2.I*: In this subcase the element r (where of course $r \le k \le n/2 < n$) does not belong to any of the first $(r-1)$ distinguished cycles of β^*. In this case we define $\beta^+ = \beta^* \in \Pi(r, n, k)$ and take this derangement β^+ to belong to B. There are no subsubcases under *b.2.I*.

Subcase *b.2.II*: In this subcase the element r does belong to one of the $(r-1)$ distinguished cycles of β^*. Say r belongs to the cycle with leading entry i where $1 \le i \le r-1$. There are two further subsubcases under subcase *b.2.II*.

Subsubcase *b.2.II.i*: If the cycle containing both i and r has at least 3 elements, then starting from β^*, we move element n to just follow element i and simultaneously move element r to take the former place of element n, so that we end with the doubleton (r, y) in place of (y, n). The cycle containing element i as a minimal element that remains after the swap now has the element i directly preceding element n and this revised cycle still has at least three elements. This revision of β^* is thus a derangement $\beta^+ \in \Pi(r, n, k)$ that we define to be an element of B.

Subsubcase *b.2.II.ii*: If the cycle containing both i and r in β^* is a doubleton cycle, then switch the element n of the doubleton (y, n) with the element i so as to form two doubletons (r, y) and (i, n) in the final revised derangement $\beta^+ \in \Pi(r, n, k)$, which we take to belong to B.

We now argue that A and B are disjoint and that $A \cup B = \Pi(r, n, k)$, and that there is no overlap in the various cases for the definition of the set B.

To prove A and B are disjoint, consider first by definition that every derangement in A has element n in a cycle of length at least 3. Therefore there is no overlap with A and the derangements from case *b.1*, subcase *b.2.I*, or subsubcase *b.2.II.ii* because in all these cases n belongs to a doubleton cycle. In subsubcase *b.2.II.i* we have that element i with $1 \le i \le r-1$ precedes element n in a cycle c of at least 3 elements, and this cannot occur for any derangement in A, again by definition of A, since a cycle of length at least 3 containing both i and n where n does not immediately succeed i is not equivalent to c. Thus A and B are disjoint.

We next prove that $A \cup B$ exhausts all derangements in $\Pi(r, n, k)$. Let $\pi \in$

$\Pi(r, n, k)$. Then element n belongs to one of the k cycles of π. If element n belongs to a cycle of at least 3 elements, then this is handled by set A except for the caveat that n must not directly succeed an element i with $1 \le i \le r-1$ in such a cycle; it does handle cycles of length at least 3 for minimal element r that also contain element n whether n follows r or not. If element n belongs to a 2-cycle then this is handled for all cases by the union of cases *b.1*, *b.2.I*, and *b.2.II.ii*. If finally n belongs to a cycle of length at least 3 and directly succeeds an element i with $1 \le i \le r-1$, then this is covered by *b.2.II.i*.

Finally we prove that there is no overlap among the cases for the construction of the set B. Subsubcases *b.2.II.i* and *b.2.II.ii* are disjoint by construction; n is in a doubleton cycle under *b.2.II.ii* but not under *b.2.II.i*. The cases *b.2.II.ii*, *b.2.I*, and *b.1* are mutually disjoint because we have a doubleton cycle (i, n) with $1 \le i \le r-1$ in case *b.2.II.ii*, while we have a doubleton cycle (y, n) with $y \in [r+1, n-1]$ in case *b.2.I*, and we have a doubleton cycle (r, n) in case *b.1*. The case *b.2.II.i* is obviously disjoint from both *b.1* and *b.2.I* again because n is not in a doubleton cycle under *b.2.II.i*. This completes the discussion showing no overlap in the various cases for the set B. Since there is only one y-value for case *b.1* while there are $(n-r-1)$ such y-values for case *b.2*, we obtain that the cardinality of B is $(n-r)|\Pi(r-1, n-2, k-1)| = (n-r)d_{r-1}(n-2, k-1)$. Thus the proof of the lemma is complete. $\square$

With Lemma 1.9 in hand we obtain a generating function formula for the r-distinguished derangement numbers in parallel to the corresponding case for partitions of Lemma 2.2.

Lemma 4.1 *Let $r \ge 0$. Then for all $r \ge 0$ we have*

$$\sum_{n \ge 0, k \ge 0} d_r(n+r, k+r) u^k \, \frac{z^n}{n!} = \left(\frac{z}{1-z} \right)^r e^{-uz} (1-z)^{-u} \tag{4.2}$$

Proof. We proceed by induction. By (1.2) the basis $r = 0$ is verified. For the induction step, assume that the statement of the lemma is true for some $r \ge 0$. Denote by $f_r(u, z)$ the left side of (4.2). Then, since $f_{r+1}(u, z) = \sum_{n \ge 0, k \ge 0} d_{r+1}(n + r + 1, k + r + 1) u^k \, \frac{z^n}{n!}$, by Lemma 1.9 we obtain

$$f_{r+1}(u, z) = \sum_{n \ge 0, k \ge 0} n d_{r+1}(n+r, k+r+1) u^k \, \frac{z^n}{n!} + \sum_{n \ge 0, k \ge 0} n d_r(n+r-1, k+r) u^k \, \frac{z^n}{n!}$$

We change indices by $N = n-1$ in both sums. Thus

$$f_{r+1}(u, z) = \sum_{N \ge 0, k \ge 0} d_{r+1}(N+r+1, k+r+1) u^k \, \frac{z^{N+1}}{N!} + \sum_{N \ge 0, k \ge 0} d_r(N+r, k+r) u^k \, \frac{z^{N+1}}{N!}$$

Thus obtain $f_{r+1}(u, z) = z f_{r+1}(u, z) + z f_r(u, z)$. Hence $f_{r+1}(u, z) = \frac{z}{1-z} f_r(u, z)$. By the induction hypothesis we therefore have $f_{r+1}(u, z) = \left(\frac{z}{1-z}\right)^{r+1} e^{-uz}(1-z)^{-u}$, as desired. $\square$

To motivate the remaining steps we take to prove Theorem 1.5, we first show the backwards recurrence step we will apply. To help make this step clear we introduce two notations as follows. Denote

$$\begin{aligned} S_r(n) &= \sum_{k=1}^{n} (-1)^k \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r+1}}}, \\ T_r(n) &= \sum_{k=1}^{n} (-1)^k \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r}}}. \end{aligned} \tag{4.3}$$

Notice that the sum $S_r(n)$ is the sum appearing in Theorem 1.5.

Lemma 4.2 *Let $r \geq 2$ and $n \geq r+2$. Then we have*

$$S_r(n) = T_r(n-1) - S_{r-1}(n-1). \tag{4.4}$$

Proof. Apply Lemma 1.9 to write $S_r(n)$ as

$$\sum_{k=1}^{n-1} (-1)^k \frac{n+k-r}{(n+k-r)^{\underline{r+1}}} d_r(n+k-1,k) + \sum_{k=2}^{n-1} (-1)^k \frac{n+k-r}{(n+k-r)^{\underline{r+1}}} d_{r-1}(n+k-2,k-1). \tag{4.5}$$

Call the first sum in (4.5) as I and the second sum as II. In I, by canceling the first factor $(n+k-r)$ of the denominator, and then by putting $m = n-1$, we have

$$I = \sum_{k=1}^{n-1} (-1)^k \frac{d_r(n+k-1,k)}{(n+k-r-1)^{\underline{r}}} = \sum_{k=1}^{m} (-1)^k \frac{d_r(m+k,k)}{(m+k-r)^{\underline{r}}}.$$

Also put $m = n-1$ in II and in addition change the index of summation by $j = k-1$ and denote $s = r-1$, so that $(n+k-r-1) = (m+j-s)$, $r = s+1$, and $(-1)^k = (-1)(-1)^j$. Thus

$$II = \sum_{k=2}^{n-1} (-1)^k \frac{d_{r-1}(n+k-2,k-1)}{(n+k-r-1)^{\underline{r}}} = (-1) \sum_{j=1}^{m} (-1)^j \frac{d_s(m+j,j)}{(m+j-s)^{\underline{s+1}}}.$$

Clearly by our notation (4.3) we have $I = T_r(m) = T_r(n-1)$ and $II = -S_s(m) = -S_{r-1}(n-1)$. Thus by $S_r(n) = I + II$ the proof is complete. $\square$

We want to prove an evaluation of $T_r(n)$ defined by (4.3). To do this we first find a certain Stirling number representation of $T_r(n)$ by an extension of the generating function argument of Proposition 1.6. The Stirling numbers that arise naturally are the r-Stirling numbers of the first kind introduced by [3].

Definition 4.3 *[3, p. 241].* Let r, k, and $n \geq 0$. Define the r-Stirling number of the first kind, $\left[{n \atop k}\right]_r$, as the number of permutations of $[n]$ into k cycles such that each of $1, 2, \ldots, r$ appears in a different cycle. The ordinary Stirling number of the second kind, namely the case $r = 1$, is denoted without subscript. By convention, $\left[{n \atop k}\right]_0 = \left[{n \atop k}\right]$. Further $\left[{n \atop k}\right]_r = 0$, if $n < r$ or $k < r$, while $\left[{r \atop k}\right]_r = \delta_{k,r}$, for $r \geq 0$, and $\left[{n \atop 0}\right]_r = 0$, for $n > r$.

We have the following recurrences.

Lemma 4.4 *[3, Theorems 1 and 3]*

$$\begin{aligned} &\textit{(triangular recurrence)} \quad \begin{bmatrix} n \\ m \end{bmatrix}_r = (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}_r + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_r, \quad n > r. \\ &\textit{(cross recurrence)} \quad (r-1)\begin{bmatrix} n \\ m \end{bmatrix}_r = \begin{bmatrix} n \\ m-1 \end{bmatrix}_{r-1} - \begin{bmatrix} n \\ m-1 \end{bmatrix}_r, \quad n \geq r > 1. \end{aligned} \tag{4.6}$$

We also have the following r-Stirling extension of Lemma 3.1 given by [3, Thm. 7]:

$$\begin{bmatrix} m \\ m-k \end{bmatrix}_r = \sum_{r \leq i_1 < i_2 < \cdots < i_k < m} i_1 i_2 \cdots i_k.$$

Therefore, in the same manner that we obtain (3.6) we find:

$$u^r(u+rz)(u+(r+1)z)(u+(r+2)z)\cdots(u+(r+p-1)z) = \sum_{n=0}^{p} \begin{bmatrix} p+r \\ p+r-n \end{bmatrix}_r z^n u^{p+r-n}. \tag{4.7}$$

Lemma 4.5 *Define $T_r(n)$ by (4.3). Let $n \geq r \geq 1$. Then we have*

$$T_r(n) = \sum_{j=r}^{n} (-1)^j \begin{bmatrix} n+j-r \\ j \end{bmatrix}_r \binom{2n-2r+1}{n+j-2r+1}. \tag{4.8}$$

Proof. Define

$$f_r(u,z) = \sum_{n,k \geq r} \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r}}}\, u^k \frac{z^n}{(n+k-2r)!}. \tag{4.9}$$

Make the substitution $m = n+k$, and write $u^k z^n = (u/z)^k z^m = u^r (u/z)^{k-r} z^{m-r}$, to rewrite

$$f_r(u,z) = \sum_{m \geq k \geq r} d_r(m,k)(u/z)^k \frac{z^m}{(m-r)!} = u^r \sum_{m \geq k \geq r} d_r(m,k)(u/z)^{k-r} \frac{z^{m-r}}{(m-r)!} \tag{4.10}$$

where we have incorporated $(m-r)^{\underline{r}} \cdot (m-2r)! = (m-r)!$. Therefore by (4.10) and Lemma 4.1 we have (compare the case $r = 1$ in (3.4)):

$$f_r(u,z) = u^r z^r e^{-u}(1-z)^{-u/z-r}. \tag{4.11}$$

Develop $(1-z)^{-v-r}$ as a binomial expansion about $z = 0$ to obtain

$$(1-z)^{-v-r} = 1 + \frac{(v+r)}{1!}z + \frac{(v+r)(v+r+1)}{2!}z^2 + \frac{(v+r)(v+r+1)(v+r+2)}{3!}z^3 + \cdots.$$

Thus, after plugging in $v = u/z$ we find by (4.11) that

$$f_r(u,z) = z^r e^{-u} \cdot u^r \left(1 + \frac{(u+rz)}{1!} + \frac{(u+rz)(u+(r+1)z)}{2!} + \cdots\right). \tag{4.12}$$

Hence by (4.7) applied to (4.12), incorporating the factor z^r under the sum, we have

$$f_r(u,z) = \sum_{\ell=0}^{\infty} (-1)^\ell \, \frac{u^\ell}{\ell!} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{n=0}^{p} \begin{bmatrix} p+r \\ p+r-n \end{bmatrix}_r z^{n+r} u^{p+r-n}. \tag{4.13}$$

Following the development after (3.7) but now applied to (4.13) with here $k = \ell + p + r - n$ as the power of u and $j = k - \ell$ as the lower index of the r-Stirling coefficient, we have the formula

$$f_r(u,z) = \sum_{k=r}^{\infty} (-1)^k \, u^k \sum_{n=0}^{\infty} z^{n+r} \sum_{j=r}^{k} (-1)^j \begin{bmatrix} n+j \\ j \end{bmatrix}_r \frac{1}{(n+j-r)!(k-j)!}; \tag{4.14}$$

compare the case $r = 1$ of (4.14) in (3.8). Now we rewrite $f_r(u,z)$ in (4.14) by replacing n by $n - r$ so as to make a power of z^n, consistent with (4.9). Correspondingly we introduce $\frac{1}{(n+j-2r)!(k-j)!} = \binom{n+k-2r}{n+j-2r} \frac{1}{(n+k-2r)!}$. We rewrite (4.14) accordingly as follows.

$$f_r(u,z) = \sum_{k=r}^{\infty} (-1)^k \, u^k \sum_{n=r}^{\infty} z^n \sum_{j=r}^{k} (-1)^j \begin{bmatrix} n-r+j \\ j \end{bmatrix}_r \binom{n+k-2r}{n+j-2r} \frac{1}{(n+k-2r)!}. \tag{4.15}$$

Fix $n \geq r+1$. We simply write, by the definition (4.9) of $f_r(u,z)$ and by (4.15), that for any $k \geq 1$,

$$\begin{aligned} (-1)^k \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r}}} &= (-1)^k [u^k z^n](n+k-2r)! f_r(u,z) \\ &= \sum_{j=r}^{k} (-1)^j \begin{bmatrix} n+j-r \\ j \end{bmatrix}_r \binom{n+k-2r}{n+j-2r}. \end{aligned} \tag{4.16}$$

Finally by constructing $T_r(n)$ by its definition (4.3) as a sum over k of the left side of (4.16), where we may take k running from r to n, and interchanging the order of summation in the resulting double sum following from the right side of (4.16), the proof is complete by noting the binomial identity $\sum_{k=j}^{n} \binom{n+k-2r}{n+j-2r} = \binom{2n-2r+1}{n+j-2r+1}$. $\square$

We will compute $T_r(n)$ by using the Stirling number representation of Lemma 4.5. We generalize slightly this representation by introducing $U_r(n,N)$ with $T_r(n) = U_r(n, 2n-(2r-1))$ by introducing a parameter N in the upper index of the binomial coefficient. We also define a companion sum $V_r(n,N)$ as follows. For all parameters

r, n, and N as shown, we define

$$\begin{aligned} U_r(n,N) &= \sum_j (-1)^j \begin{bmatrix} n+j-r \\ j \end{bmatrix}_r \binom{N}{n+j-2r+1}; \\ &\qquad r \ge 1, n \ge r+1, N \ge 2n-2r+1 \\ V_r(n,N) &= \sum_j (-1)^j \begin{bmatrix} n+j-r \\ j \end{bmatrix}_r \binom{N}{n+j-2r}; \\ &\qquad r \ge 1, n \ge r, N \ge 2n-2r+1. \end{aligned} \tag{4.17}$$

Here and in the sequel, the sums defining $U_r(n,N)$ and $V_r(n,N)$ are finite and extend to all nonzero values of the summand. By (3.13)(i) we have evaluated $V_1(n,N) = 0$ for all $n \ge 1$ and $N \ge 2n-1$. Also, by (3.13)(ii) we have evaluated $U_1(n,N) = 0$ for all $n \ge 2$ and $N \ge 2n-1$.

Lemma 4.6 *Define $U_r(n,N)$ and $V_r(n,N)$ by (4.17). Then we have*

$$\begin{aligned} &(i) \quad V_r(n,N) = 0, \quad \text{for all } \ r \ge 1,\ n \ge r,\ N \ge 2n-2r+1; \\ &(ii) \quad U_r(n,N) = (-1)^r (r-1)^{n-r}, \quad \text{for all } \ r \ge 1,\ n \ge r+1,\ N \ge 2n-2r+1. \end{aligned} \tag{4.18}$$

Additionally, for $r \ge 2$ we have $U_r(r,N) = (-1)^r$ when $N \ge 1$.

Proof. We prove statement (i) of the lemma by applying the proof of (3.13)(i) to (4.16). Indeed let $r \ge 1$ and $k \ge n+1$ in (4.16). Then $d_r(n+k,k) = 0$. Hence the sum on the right side of (4.16) is zero. Now put N in place of $n+k-2r$ in the upper index of the binomial coefficient for the sum on the right side of (4.16). Then we have that this sum is zero for all $N \ge n+(n+1)-2r$. Hence by the definition of $V_r(n,N)$ in (4.17) we have verified that $V_r(n,N) = 0$ under the constraints shown in (4.18)(i).

To prove statement (ii) of the lemma, we first apply the cross recurrence of (4.6) to $U_r(n,N)$ defined by (4.17). For the application of the recurrence we transpose the terms to write $\begin{bmatrix} n \\ m-1 \end{bmatrix}_r = -(r-1)\begin{bmatrix} n \\ m \end{bmatrix}_r + \begin{bmatrix} n \\ m-1 \end{bmatrix}_{r-1}$. Hence by (4.17) and this last relation we have

$$\begin{aligned} U_r(n,N) &= -(r-1)\sum_j (-1)^j \begin{bmatrix} n+j-r \\ j+1 \end{bmatrix}_r \binom{N}{n+j-2r+1} \\ &+ \sum_j (-1)^j \begin{bmatrix} n+j-r \\ j \end{bmatrix}_{r-1} \binom{N}{n+j-2r+1}. \end{aligned} \tag{4.19}$$

Now put $k = j+1$ and $m = n-1$ in the first sum on the right side of (4.19) and put $m = n - 1$ in the second sum. Thus

$$U_r(n,N) = +(r-1)\sum_k (-1)^k \begin{bmatrix} m+k-r \\ k \end{bmatrix}_r \binom{N}{m+k-2r+1}$$
$$+\sum_j (-1)^j \begin{bmatrix} m+j-(r-1) \\ j \end{bmatrix}_{r-1} \binom{N}{m+j-2r+2} = (r-1)U_r(m,N) + V_{r-1}(m,N). \tag{4.20}$$

By $m = n - 1$ in (4.20) we have thus shown

$$U_r(n,N) = (r-1)U_r(n-1,N) + V_{r-1}(n-1,N), \quad \text{for any} \quad N. \tag{4.21}$$

To complete the proof of (ii), we first observe that the case $r = 1$ for $U_r(n,N)$ is verified by (3.13)(ii). Thus let $r \geq 2$. Assume that $n \geq r+1$ and $N \geq 2n - 2r + 1$. By (4.21) we have

$$U_r(n,N) = (r-1)U_r(n-1,N) + V_{r-1}(n-1,N) = (r-1)U_r(n-1,N), \tag{4.22}$$

where we have applied $V_{r-1}(n-1,N) = 0$. We can make this application by (4.18)(i) because $n-1 \geq r \geq r-1$ and $N \geq 2n-2r+1 = 2(n-1)-2(r-1)+1$, as required. Now iterate (4.22), which is possible for the evaluation of $V_{r-1}(n',N) = 0$ with $n' = n-k$ because the inequality constraint $N \geq 2(n'-1)-2(r-1)+1$ is satisfied for $k \geq 1$ and N fixed. Therefore, by iterating k times with $k = n - r$ we obtain

$$U_r(n,N) = (r-1)^{n-r} U_r(r,N). \tag{4.23}$$

Finally we compute $U_r(r,N)$ by using the definition (4.17) with $r+j-r = j$ in the top index of the r-Stirling number and $r+j-2r+1 = j-r+1$ in the bottom index of the binomial coefficient. Thus we have

$$\begin{aligned} U_r(r,N) &= \sum_{j\geq r} (-1)^j \begin{bmatrix} j \\ j \end{bmatrix}_r \binom{N}{j-r+1} \\ &= (-1)^r \sum_{k=0}^{N-1} (-1)^k \binom{N}{k+1} \\ &= (-1)^r (1-(1-1)^N) = (-1)^r, \end{aligned}$$

where we made the change of index $k = j - r$. Hence, $U_r(n,N) = (-1)^r (r-1)^{n-r}$. Thus we have verified (4.18). The additional statement of the lemma has been proved by our evaluation of $U_r(r,N) = (-1)^r$ for all $r \geq 2$. Hence the lemma is proved. □

Proof of Theorem 1.5. We compute $S_r(n)$, which is defined by (4.3), and which stands as the left side of the statement of the theorem, as follows. Let $r \geq 2$ and let $n \geq r+2$. By Lemma 4.2 we have that the backward recurrence (4.4) holds, which we restate here:

$$S_r(n) = T_r(n-1) - S_{r-1}(n-1),$$

for $T_r(n)$ also defined by (4.3). By Lemma 4.5 and the definition (4.17) of $U_r(n,N)$ we have $T_r(n) = U_r(n, 2n-2r+1)$. So by Lemma 4.6, putting $n-1$ in place of n in this last relation, we have $T_r(n-1) = U_r(n-1, 2(n-1)-2r+1) = (-1)^r(r-1)^{n-1-r}$. Therefore in turn, recalling by Definition 1.4 that $P_r(n) = (-1)^r \sum_{i=0}^{r-1} i^{n-r-1}$, we have $T_r(n-1) = P_r(n) - (-1)P_{r-1}(n-1) = P_r(n) + P_{r-1}(n-1)$. Hence, by the backward recurrence (4.4) for $S_r(n)$, we have

$$S_r(n) = P_r(n) + P_{r-1}(n-1) - S_{r-1}(n-1), \text{ for all } \ r \geq 2, \ n \geq r+2. \tag{4.24}$$

Put $n' = n-1$ and $r' = r-1$. Since $n' = n-1 \geq r+1 = r'+2$, we may iterate (4.24) and thus obtain

$$\begin{aligned} S_r(n) &= P_r(n) + P_{r-1}(n-1) - S_{r-1}(n-1) \\ &= P_r(n) + P_{r-1}(n-1) - (P_{r-1}(n-1) + P_{r-2}(n-2)) + S_{r-2}(n-2) \\ &= (-1)^{r-1}S_1(n-r+1) + \sum_{k=0}^{r-2}(-1)^k(P_{r-k}(n-k) + P_{r-k-1}(n-k-1)) \\ &= (-1)^{r-1}S_1(n-r+1) + P_r(n) + (-1)^{r-2}P_1(n-r+1), \end{aligned} \tag{4.25}$$

where we used that the sum in the second to last line is telescoping. Finally, $P_1(n-r+1) = (-1)^1 0^{n-r+1} = 0$, and by the definition (4.3) of $S_1(n-(r-1))$ and Theorem 1.2 we have $(-1)^{r-1}S_1(n-r+1) = (-1)^{r-1}(-B_{n-r}/(n-r))$. Therefore by (4.25) we have $S_r(n) = P_r(n) + (-1)^r B_{n-r}/(n-r)$. By the definition of $S_r(n)$, this is what we wanted to prove. □

Proof of Corollary 1.10. Recall the Bernoulli's formula (1.4) which we rewrite here:

$$\sum_{k=0}^{N} k^p = \frac{1}{p+1}\sum_{k=0}^{p}(-1)^k \binom{p+1}{k} N^{p+1-k}B_k, \quad p \geq 1.$$

By the statement of Theorem 1.5, and by taking $p = n-r-1$ and $N = r-1$ in the power sum to represent $P_r(n)$ of Definition 1.4, and because only when n and r have the same parity with $n-r \geq 2$ do we have $B_{n-r} \neq 0$, we obtain the corollary by adding a last term $k = p+1 = n-r$ to the Bernoulli number representation of the power sum. □

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