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Packings and coverings of lambda-fold line graphs of the complete graph with k-cycles, for $k = 4, 6$

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Abstract

Let $L(K_n)(\lambda)$ denote the λ-fold line graph of the complete graph K_n. In this paper, we obtain a maximum packing of $L(K_n)(\lambda)$ with k-cycles, $k \in \{4, 6\}$, with every possible leave, and also obtain a minimum covering of $L(K_n)(\lambda)$ with k-cycles, $k \in \{4, 6\}$, with every possible padding.

1 Introduction

For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G. A *k-cycle* is the cycle on k vertices; we denote it by C_k. The complete graph on n vertices is denoted by K_n and the complete bipartite graph with bipartition (X, Y), where $|X| = m$ and $|Y| = n$, is denoted by $K_{m,n}$. The complete m-partite graph in which each of its partite sets has n vertices is denoted by $K_m \circ \overline{K}_n$. For a positive integer k, let kG denote k pairwise vertex-disjoint copies of G. For a graph G, the graph $G(\lambda)$ is obtained by replacing each edge of G by λ parallel edges. The graph $G(\lambda)$ is called the *λ-fold* copy of the graph G. For disjoint subsets A and B of the vertex set $V(G)$ of G, let $E(A, B)$ denote the set of all edges of G each having one end in A and the other end in B. For $S \subseteq V(G)$ and $E' \subseteq E(G)$, let $\langle S \rangle$ and $\langle E' \rangle$ denote the subgraphs induced by S and E' respectively. A graph G is said to be *H-decomposable* or $H|G$ if the edge set of G can be partitioned into $E_1, E_2, \ldots, E_k$ such that for each $1 \leq i \leq k$, $\langle E_i \rangle \simeq H$; if each $\langle E_i \rangle \simeq C_r$, then we say that G has a *C_r-decomposition* or an *r-cycle decomposition* and in this case we write $C_r|G$.

The *line graph* of a graph G, denoted by $L(G)$, is the graph with vertex set $V(L(G)) = E(G)$ and $e_i e_j \in E(L(G))$ if and only if the edges e_i and e_j in G are incident at a vertex of G. For a non-empty set S, let $\mathcal{P}_2(S)$ denote the set of all two-element subsets of S. The *bowtie* is a graph with five vertices, six edges, and having two edge-disjoint 3-cycles with exactly one common vertex, and it is denoted by B. A *kite* is the simple graph on four vertices, four edges, and having a triangle and an edge incident with the triangle, and it is denoted by K. A graph with vertices

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a, b, c, d and edges ab, bc, ca, cd, cd is denoted by F_1; that is, the graph consisting of a triangle with a double edge attached, on 4 vertices and 5 edges. A graph with vertices a, b, c, d, e and edges ab, bc, ca, de, de is denoted by F_2; that is, the graph having a triangle with a disjoint double edge, on 5 vertices and 5 edges. A graph with vertices a, b, c and edges ab, bc, ca, ca, ca is denoted by F_3.

For graphs G and H, the *Cartesian product* of G and H, denoted by $G\Box H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is adjacent to (g_2, h_2) if and only if $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$.

A *k-cycle packing* of the graph G is a triple (V, E, L), where V is the vertex set of G, E is a set of edge-disjoint k-cycles of G, and L is the set of edges of G not belonging to any of the k-cycles of E. The collection of edges L is the *leave*. If $|E|$ is as large as possible, or equivalently if $|L|$ is as small as possible, then (V, E, L) is called a *maximum packing* of G with k-cycles; see Chapter 4 of [24]. A *k-cycle covering* of the graph G is a triple (V, E, P), where V is the vertex set of G, P is a subset of the edge set of $G(\lambda)$, and E is a set of edge-disjoint k-cycles which partitions the union of P and the edge set of G. The collection of edges P is called the *padding*. If $|P|$ is as small as possible, then (V, E, P) is called a *minimum covering* of G with k-cycles; see Chapter 4 of [24]. Definitions which are not given here can be found in [24, 31].

Maximum packings of K_n with graphs K_4 and certain graphs on five vertices are studied in [4, 33]. Maximum packings and minimum coverings of K_n with 4-cycles, 5-cycles, 6-cycles, cubes and the graphs having four or fewer vertices are studied in [1, 18, 19, 20, 26, 27, 28]. Maximum packings and minimum coverings of $K_{n,n}(\lambda)$ with 4-cycles and $K_{1,4}$ are studied in [21]. In [22, 23], the existence of maximum packings and minimum coverings of K_{2n+1} and $K_{m,n}$ with 8-cycles are established. Maximum packings of the λ-fold complete multipartite graph $(K_{a_1,a_2,\ldots,a_n})(\lambda)$ with 4-cycles have been studied in [2, 3]. Also, maximum packings and minimum coverings of λ-fold complete equipartite graphs with triangles or kites are obtained in [16, 32]. Maximum packings and minimum coverings of the complete equipartite graph with $K_4 - e$ are studied in [11, 12]. In [17], the existence of a maximum packing of $K_m \circ \overline{K}_n$ with 5-cycles for an odd integer m is established. For $k \in \{6, 2^l, \binom{n}{2}\}$, existence of a k-cycle decomposition of the graph $L(K_n)$ has been studied in [6, 7, 14, 30]. In fact, in [5, 9, 13], the existence of a k-cycle decomposition of $L(K_n)(\lambda), k \in \{4, 5\}$ has been obtained. Maximum packings of the graph $L(K_n)$ with bowties has been completely settled in [10]. Also, maximum packings and minimum coverings of $L(K_n)(\lambda)$ with kites have been considered in [25]. In this paper, existence of a maximum k-cycle packing and a minimum k-cycle covering of $L(K_n)(\lambda)$, $k \in \{4, 6\}$, with every possible leave and padding, is established.

If $n \geq 4$ and $4|E(L(K_n)(\lambda))$, then $L(K_n)(\lambda)$ has a 4-cycle decomposition. If $4 \nmid E(L(K_n)(\lambda))$, then we look into a 4-cycle decomposition of $L(K_n)(\lambda) - E(L)$ and $L(K_n)(\lambda) \cup E(P)$, that is, the minimum number edges whose removal from $L(K_n)(\lambda)$ gives a 4-cycle decomposition, and the minimum number of edges whose addition to $L(K_n)(\lambda)$ gives a 4-cycle decomposition, where L is a leave and P is a padding. Note that L and P are even graphs as the graph $L(K_n)(\lambda)$ has regularity $2\lambda(n-2)$. In

Table 1, for $\lambda = 1$ and $n \equiv 5 \pmod 8$, $|E(L(K_n)(\lambda))| \equiv 6 \pmod 8$. Since $L(K_n)$ is a simple graph, $|E(L)| = 6$. The possible leaves are a 6-cycle or B or $2C_3$, and $|E(P)| = 2$ with possible padding $K_2(2)$. For $\lambda \equiv 1 \pmod 5 > 1$ and $n \equiv 5 \pmod 8$, $|E(L(K_n)(\lambda))| \equiv 6 \pmod 8$. Since $L(K_n)(\lambda)$ is a multigraph, $|E(L)| = 2$. The only possible leave is $K_2(2)$, and $|E(P)| = 2$ with possible padding $K_2(2)$. It is easy to observe that the possible leaves and paddings of the remaining n and λ are listed in Table 1.

We prove the following main results.

Theorem 1.1. *The graph $L(K_n)(\lambda)$ admits a maximum 4-cycle packing and a minimum 4-cycle covering with every possible leave and padding. The possible leaves and paddings are shown in Table 1.*

$\lambda \equiv$	$n \geq 4$ and $n \equiv$	Leave	Padding
0 (mod 4)	all n	$\emptyset$	$\emptyset$
all λ	0 (mod 2) or 1 (mod 8)	$\emptyset$	$\emptyset$
0 (mod 2)	5 (mod 8)	$\emptyset$	$\emptyset$
1 (mod 4)	3 (mod 8)	C_3	C_5, F_1, F_2, F_3
	5 (mod 8)	$\{C_6, B, 2C_3$ if $\lambda = 1\}$, $\{K_2(2)$ if $\lambda \geq 5\}$	$K_2(2)$
	7 (mod 8)	$\{C_5$ if $\lambda = 1\}$, $\{C_5, F_1, F_2, F_3$ if $\lambda \geq 5\}$	C_3
2 (mod 4)	3 (mod 8)	$K_2(2)$	$K_2(2)$
	5 (mod 8)	$\emptyset$	$\emptyset$
	7 (mod 8)	$K_2(2)$	$K_2(2)$
3 (mod 4)	3 (mod 8)	C_5, F_1, F_2, F_3	C_3
	5 (mod 8)	$K_2(2)$	$K_2(2)$
	7 (mod 8)	C_3	C_5, F_1, F_2, F_3

Table 1: Leaves and paddings of $L(K_n)(\lambda)$ with 4-cycle packings and 4-cycle coverings

Theorem 1.2. *The graph $L(K_n)(\lambda)$ admits a maximum 6-cycle packing and a minimum 6-cycle covering with every possible leave and padding. The possible leaves and paddings are shown in Table 2.*

$\lambda \equiv$	$n \geq 4$	Leave	Padding
1 (mod 2)	$n \not\equiv 3 \pmod 4$	$\emptyset$	$\emptyset$
	$n \equiv 3 \pmod 4$	C_3	C_3
0 (mod 2)	all n	$\emptyset$	$\emptyset$

Table 2: Leaves and paddings of $L(K_n)(\lambda)$ with 6-cycle packings and 6-cycle coverings

We state the following known results for our future reference.

Theorem 1.3. *[5] The graph $L(K_n)(\lambda)$ has a 4-cycle decomposition if and only if n and λ satisfy the following conditions:*

(i) *n even, or*

(ii) $n \equiv 1 \pmod 4$ *and* $\lambda \equiv 0 \pmod 2$, *or*

(iii) $n \equiv 3 \pmod 4$ *and* $\lambda \equiv 0 \pmod 4$, *or*
(iv) $n \equiv 1 \pmod 8$ *and* λ *is odd.*

Theorem 1.4. *[6] The graph* $L(K_n)$ *has a* 6*-cycle decomposition if and only if* $n \not\equiv 3 \pmod 4$.

The following lemma is an easy observation.

Lemma 1.5. *If* $H|G$, *then* $H|G(\lambda)$ *for any* $\lambda \geq 2$.

The following corollary is a consequence of Lemma 1.5 and Theorem 1.4.

Corollary 1.6. *If* $n \not\equiv 3 \pmod 4$, $n \geq 4$ *and* $\lambda \geq 1$, *then the graph* $L(K_n)(\lambda)$ *has a* 6*-cycle decomposition.*

Theorem 1.7. *[29] The complete bipartite graph* $K_{m,n}$ *has a* $2k$*-cycle decomposition if and only if* m *and* n *are even,* $m \geq k, n \geq k$, *and* $2k$ *divides* mn.

Theorem 1.8. *[15] The graph* $K_m \Box K_n$ *has a* 4*-cycle decomposition if and only if one of the following holds.*

(i) $m, n \equiv 0 \pmod 2$;
(ii) $m, n \equiv 1 \pmod 8$;
(iii) $m, n \equiv 5 \pmod 8$.

Theorem 1.9. *[8] The graph* $K_m \Box K_n$ *has a* 6*-cycle decomposition if and only if*

1. m, n *are even, and*
   (*a*) $6|m$ *or* $6|n$, *or*
   (*b*) $m + n \equiv 2 \pmod 3$; *or*
2. m, n *are odd, and*
   (*a*) *if* $m, n \not\equiv 0 \pmod 3$, *then* $(m + n) \equiv 2 \pmod{12}$, *or*
   (*b*) *if* $m \equiv 0 \pmod 3$ *or* $n \equiv 0 \pmod 3$, *then* $m + n \equiv 2 \pmod 4$.

2 Existence of a maximum packing and a minimum covering of $L(K_n)(\lambda)$ with 4-cycles

In this section, we prove the existence of a 4-cycle packing and a 4-cycle covering of $L(K_n)(\lambda)$ with every possible leave and padding.

Observation 2.1. Consider $k \geq 2$ and $n \geq 5$. Let $V(K_n) = \{1, 2, \ldots, n\}$. Then the vertex set of $L(K_n)$ can be given as $V(L(K_n)) = \mathcal{P}_2(\{1, 2, \ldots, n-1, n\})$, that is, the set of all two-element subsets of $\{1, 2, \ldots, n-1, n\}$. We partition the vertex set of $L(K_n)$ into three sets A_1, A_2 and A_3, where $n > k+1$, $A_1 = \mathcal{P}_2(\{1, 2, \ldots, k, n\})$, $A_2 = \mathcal{P}_2(\{k+1, k+2, \ldots, n-1, n\})$ and $A_3 = \{\{i, j\} | 1 \leq i \leq k,\ k+1 \leq j \leq n-1\}$. The subgraphs of $L(K_n)$ induced by A_1, A_2 and A_3 are isomorphic to $L(K_{k+1})$,

$L(K_{n-k})$ and $K_k \Box K_{n-k-1}$, respectively, where $\Box$ denotes the cartesian product of graphs. Clearly, $\langle E(A_1, A_2)\rangle = \langle\{\{i,n\}\{j,n\}\,;\, 1 \le i \le k, k+1 \le j \le n-1\}\rangle = K_{k,n-k-1}$; we denote the graph $\langle E(A_1, A_2)\rangle$ by A'. For $1 \le i \le k, k+1 \le j \le n-1$, let $R_i = \{\{i, k+1\}, \{i, k+2\}, \dots, \{i, n-1\}\}$ and let $Q_j = \{\{1, j\}, \{2, j\}, \dots, \{k, j\}\}$. Clearly, $\langle E(R_i, A_1)\rangle \cong K_{n-k-1,k}$ and $\langle E(Q_j, A_2)\rangle \cong K_{k,n-k-1}$. The induced subgraph $H = \langle \cup_{i=1}^{k}\{E(R_i, A_1)\} \cup_{j=k+1}^{n-1} \{E(Q_j, A_2)\}\rangle = \underbrace{K_{k,n-k-1} \oplus \cdots \oplus K_{k,n-k-1}}_{(n-1)\text{ copies}}$. Thus

$$
\begin{aligned}
L(K_n) &= \langle A_1\rangle \oplus \langle A_2\rangle \oplus \langle A_3\rangle \oplus \langle E(A_1, A_2)\rangle \oplus \langle E(A_3, A_1)\rangle \oplus \langle E(A_3, A_2)\rangle \\
&= L(K_{k+1}) \oplus L(K_{n-k}) \oplus (K_k \Box K_{n-k-1}) \oplus A' \oplus \langle \cup_{i=1}^{k} E(R_i, A_1)\rangle \oplus \\
&\qquad\qquad \langle \cup_{j=k+1}^{n-1} E(Q_j, A_2)\rangle \\
&= L(K_{k+1}) \oplus L(K_{n-k}) \oplus (K_k \Box K_{n-k-1}) \oplus A' \oplus H,
\end{aligned}
$$

where $H = \underbrace{K_{k,n-k-1} \oplus \cdots \oplus K_{k,n-k-1}}_{(n-1)\text{ copies}}$, as each of the graphs $\langle E(R_i, A_1)\rangle$ and $\langle E(Q_j, A_2)\rangle$ is isomorphic to $K_{k,n-k-1}$; see Figure 1.

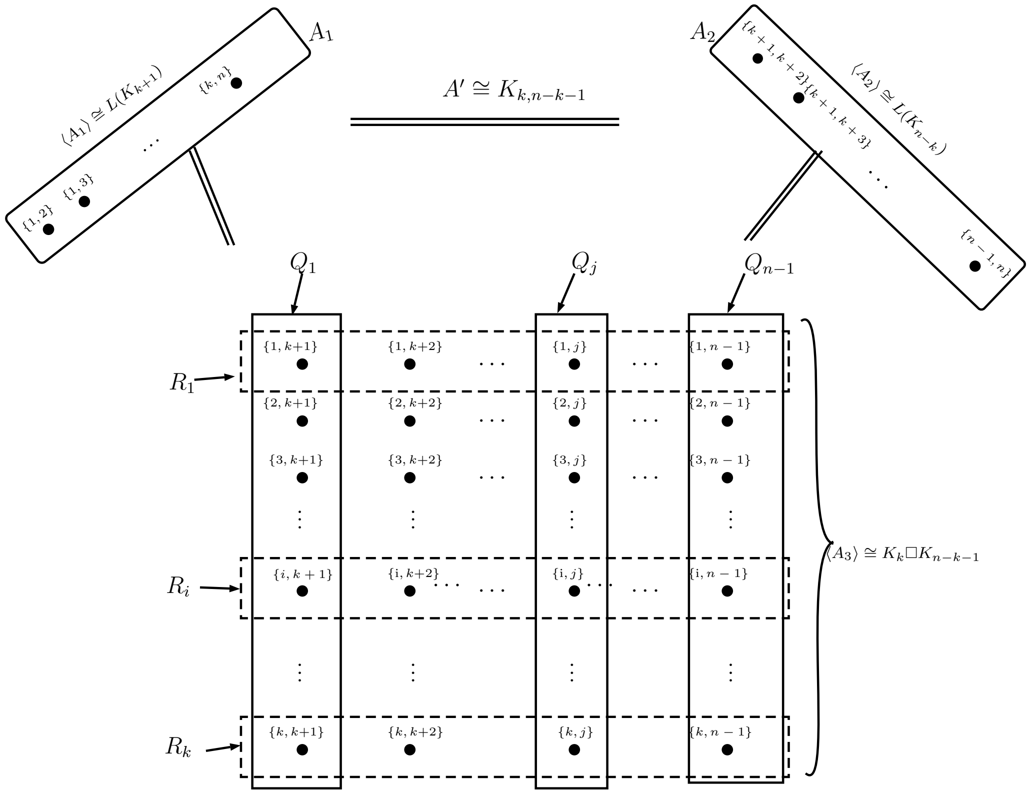

Figure 1: The graph $L(K_n) = L(K_{k+1}) \oplus L(K_{n-k}) \oplus (K_k \Box K_{n-k-1}) \oplus A' \oplus H$.

Note: This observation, in particular, the notation A' and the decomposition of $L(K_n)$, will be used extensively in the rest of the paper.

Lemma 2.2. *The graph $L(K_5)$ has a 4-cycle packing with leave L, $L \in \{C_6, B, 2C_3\}$, and B denotes the bowtie; also it has a 4-cycle covering with padding $K_2(2)$.*

Proof. Let $V(K_5) = \{1, 2, 3, 4, 5\}$. Then $V(L(K_5)) = \mathcal{P}_2(\{1, 2, 3, 4, 5\})$.

(i) A 4-cycle packing of $L(K_5)$ with leave C_6 is given by the set of 4-cycles in

$$\begin{aligned}\mathcal{F}_1 = \{&(\{1,2\},\{1,3\},\{2,3\},\{2,4\}), & (\{1,2\},\{1,4\},\{4,5\},\{1,5\}),\\ &(\{1,2\},\{2,3\},\{3,5\},\{2,5\}), & (\{1,3\},\{1,4\},\{2,4\},\{3,4\}),\\ &(\{1,4\},\{1,5\},\{3,5\},\{3,4\}), & (\{2,3\},\{2,5\},\{4,5\},\{3,4\})\}\end{aligned}$$

and the 6-cycle $(\{1,3\},\{1,5\},\{2,5\},\{2,4\},\{4,5\},\{3,5\})$.

(ii) A 4-cycle packing of $L(K_5)$ with leave B is given by the set of 4-cycles in

$$\begin{aligned}\mathcal{F}_2 = \{&(\{1,2\},\{1,5\},\{3,5\},\{2,3\}), & (\{1,2\},\{2,4\},\{2,3\},\{2,5\}),\\ &(\{1,3\},\{1,5\},\{1,4\},\{3,4\}), & (\{1,3\},\{2,3\},\{3,4\},\{3,5\}),\\ &(\{1,5\},\{2,5\},\{3,5\},\{4,5\}), & (\{2,4\},\{2,5\},\{4,5\},\{3,4\})\}\end{aligned}$$

and the two 3-cycles of the bowtie are $(\{1,2\},\{1,3\},\{1,4\})$ and $(\{1,4\},\{2,4\},\{4,5\})$.

(iii) A 4-cycle packing of $L(K_5)$ with leave $2C_3$ is given by the set of 4-cycles in

$$\begin{aligned}\mathcal{F}_3 = \{&(\{1,2\},\{2,4\},\{2,3\},\{2,5\}), & (\{1,2\},\{1,4\},\{1,3\},\{1,5\}),\\ &(\{1,3\},\{3,5\},\{2,3\},\{3,4\}), & (\{1,4\},\{2,4\},\{4,5\},\{3,4\}),\\ &(\{1,5\},\{2,5\},\{4,5\},\{3,5\}), & (\{2,4\},\{2,5\},\{3,5\},\{3,4\})\}\end{aligned}$$

and the $2C_3$ is given by the two 3-cycles $(\{1,2\},\{1,3\},\{2,3\})$ and $(\{1,4\},\{1,5\},\{4,5\})$.

(iv) A 4-cycle covering of $L(K_5)$ with padding $K_2(2)$ is described below:

Clearly, the cycles in $\mathcal{F}_1$ (described in (i) above) together with the two 4-cycles, namely, $(\{1,3\},\{1,5\},\{4,5\},\{3,5\})$ and $(\{1,5\},\{2,5\},\{2,4\},\{4,5\})\}$, yield a 4-cycle covering of $L(K_5)$ with padding $K_2(2)$ given by the edges in $\{\{1,5\}\{4,5\},\{1,5\}\{4,5\}\}$. □

Lemma 2.3. *The graph $L(K_7)$ has a 4-cycle packing with leave C_5; also it has a 4-cycle covering with padding C_3.*

Proof. Let $V(K_7) = \{1,2,\ldots,7\}$. Then $V(L(K_7)) = \mathcal{P}_2(\{1,2,\ldots,7\})$.

(i) A 4-cycle packing of $L(K_7)$ with leave C_5 is given by the set of 4-cycles in

$$\begin{aligned}\mathcal{F}_1 = \{&(\{1,2\},\{1,3\},\{1,4\},\{1,5\}), & (\{1,2\},\{1,4\},\{3,4\},\{2,4\}),\\ &(\{1,2\},\{1,6\},\{6,7\},\{1,7\}), & (\{1,2\},\{2,3\},\{2,4\},\{2,7\}),\\ &(\{1,2\},\{2,5\},\{2,3\},\{2,6\}), & (\{1,3\},\{1,5\},\{3,5\},\{2,3\}),\\ &(\{1,3\},\{1,6\},\{4,6\},\{3,6\}), & (\{1,3\},\{3,4\},\{2,3\},\{3,7\}),\\ &(\{1,3\},\{1,7\},\{3,7\},\{3,5\}), & (\{1,4\},\{4,6\},\{5,6\},\{4,5\}),\\ &(\{1,5\},\{1,6\},\{3,6\},\{5,6\}), & (\{1,5\},\{2,5\},\{2,7\},\{5,7\}),\\ &(\{1,5\},\{1,7\},\{4,7\},\{4,5\}), & (\{1,6\},\{1,7\},\{5,7\},\{5,6\}),\\ &(\{1,6\},\{2,6\},\{2,4\},\{1,4\}), & (\{2,3\},\{2,7\},\{6,7\},\{3,6\}),\\ &(\{2,4\},\{4,5\},\{5,7\},\{4,7\}), & (\{2,5\},\{5,7\},\{6,7\},\{5,6\}),\\ &(\{2,5\},\{2,4\},\{4,6\},\{4,5\}), & (\{2,6\},\{6,7\},\{3,7\},\{3,6\}),\\ &(\{2,6\},\{2,7\},\{4,7\},\{4,6\}), & (\{3,4\},\{3,5\},\{5,7\},\{3,7\}),\\ &(\{3,4\},\{4,6\},\{6,7\},\{4,7\}), & (\{3,4\},\{3,6\},\{3,5\},\{4,5\}),\\ &(\{3,5\},\{2,5\},\{2,6\},\{5,6\})\}\end{aligned}$$

and the 5-cycle $(\{1,4\},\{1,7\},\{2,7\},\{3,7\},\{4,7\})$.

(ii) A 4-cycle packing of $L(K_7)$ with padding C_3 is given by the set of 4-cycles in $\mathcal{F}_1$ (described in (i) above) together with the 4-cycles $(\{1,4\},\{1,7\},\{2,7\},\{4,7\})$ and $(\{2,7\},\{2,4\},\{4,7\},\{3,7\})$, where the padding $C_3 = (\{2,7\},\{4,7\},\{2,4\})$. □

Lemma 2.4. *The graph $L(K_{11})$ has a 4-cycle packing with leave C_3; also it has a 4-cycle covering of $L(K_{11})$ with padding C_5, F_1, F_2 or F_3.*

Proof. Let $V(K_{11}) = \{1, 2, \ldots, 11\}$. Then $V(L(K_{11})) = \mathcal{P}_2(\{1, 2, \ldots, 11\})$.

(i) First we obtain a 4-cycle packing of $L(K_{11})$ with leave C_3. We partition the vertex set of $L(K_{11})$ into three sets A_1, A_2 and A_3, where $A_1 = \mathcal{P}_2(\{1, 2, 3, 4, 11\})$, $A_2 = \mathcal{P}_2(\{5, 6, \ldots, 11\})$ and $A_3 = \{\{i, j\} \mid 1 \leq i \leq 4, 5 \leq j \leq 10\}$. The subgraphs induced by the vertices in A_1 and A_2 are isomorphic to $L(K_5)$ and $L(K_7)$, respectively. The graph $L(K_{11}) = L(K_5) \oplus L(K_7) \oplus (K_4 \Box K_6) \oplus A' \oplus H$, where $H = \underbrace{K_{4,6} \oplus K_{4,6} \oplus \cdots \oplus K_{4,6}}_{10\text{ copies}}$, by Observation 2.1, where A' is as defined in Observation 2.1. Lemmas 2.2 and 2.3 explicitly give a 4-cycle decomposition of $(L(K_5) - E(C_6))$ and $(L(K_7) - E(C_5))$, with $C_6 = (\{1, 3\}, \{1, 11\}, \{2, 11\}, \{2, 4\}, \{4, 11\}, \{3, 11\})$ and $C_5 = (\{5, 8\}, \{5, 11\}, \{6, 11\}, \{7, 11\}, \{8, 11\})$. By Theorems 1.8 and 1.7, $C_4|(K_4 \Box K_6)$ and $C_4|H$. Let H_1 be the subgraph of $L(K_{11})$ excluding the edges of the 4-cycles in the decomposition of $L(K_5) - E(C_6)$, $L(K_7) - E(C_5)$, $K_4 \Box K_6$ and H (listed above); clearly $H_1 = C_6 \oplus A' \oplus C_5$; see Figure 3 in the Appendix. A 4-cycle packing of H_1 with leave C_3 follows by Item 2 in the Appendix.

(ii) From the proof described in (i) above, we have $C_4|(L(K_{11}) - E(H_1))$. Now a 4-cycle covering of H_1 with padding C_5, F_1, F_2, or F_3 follows by the Items 3, 4, 5 and 6 in the Appendix. □

Lemma 2.5. *The graph $(K_3 \Box K_3)(2)$ admits a 4-cycle decomposition.*

Proof. Let $V(G) = \{1, 2, 3\}$ and $V(H) = \{a, b, c\}$.
A 4-cycle decomposition of $(K_3 \Box K_3)(2)$ is given by:

$$
\begin{array}{lll}
((1,a),(1,b),(2,b),(2,a)), & ((1,a),(1,c),(2,c),(2,a)), & ((1,a),(1,b),(3,b),(3,a)),\\
((1,a),(1,c),(3,c),(3,a)), & ((1,b),(1,c),(3,c),(3,b)), & ((1,b),(1,c),(2,c),(2,b)),\\
((2,a),(2,b),(3,b),(3,a)), & ((2,a),(2,c),(3,c),(3,a)), & ((2,b),(2,c),(3,c),(3,b)).
\end{array}
$$

□

Lemma 2.6. *The graphs $L(K_7)(2)$ and $L(K_{11})(2)$ admit a 4-cycle packing with leave $K_2(2)$; also they admit a 4-cycle covering with padding $K_2(2)$.*

Proof. (i) Let $V(K_7(2)) = \{1, 2, \ldots, 7\}$. Then $V(L(K_7)(2)) = \mathcal{P}_2(\{1, 2, \ldots, 7\})$. We partition the vertex set of $L(K_7)(2)$ into three sets A_1, A_2 and A_3, where $A_1 = \mathcal{P}_2(\{1, 2, 3, 7\})$, $A_2 = \mathcal{P}_2(\{4, 5, 6, 7\})$ and $A_3 = \{\{i, j\} \mid 1 \leq i \leq 3, 4 \leq j \leq 6\}$. The graph $L(K_7)(2) = L(K_4)(2) \oplus L(K_4)(2) \oplus (K_3 \Box K_3)(2) \oplus A'(2) \oplus H_2$, by Observation 2.1, where $H_2 = \underbrace{K_{3,3}(2) \oplus K_{3,3}(2) \oplus \cdots \oplus K_{3,3}(2)}_{6\text{ copies}}$, $H_2 \simeq H(2)$ and $A'(2)$ is as in Observation 2.1. The graphs $L(K_4)(2)$, $(K_3 \Box K_3)(2)$ and H_2 have 4-cycle decompositions, by Theorem 1.3, Lemma 2.5 and Item 7 in the Appendix. Thus $C_4|(L(K_7)(2) - E(H_3))$, where $A'(2) = H_3$. Now we obtain a 4-cycle packing and a 4-cycle covering of H_3 with leave L, and the padding P is $\{\{3, 7\}\{4, 7\}, \{3, 7\}\{4, 7\}\}$; see Figure 4, as given in Items 8(a) and 9 of the Appendix.

(ii) Let $V(K_{11}(2)) = \{1, 2, \ldots, 11\}$. Then $V(L(K_{11})(2)) = \mathcal{P}_2(\{1, 2, \ldots, 11\})$. We partition the vertex set of $L(K_{11})(2)$ into three sets A_1, A_2 and A_3, where $A_1 = \mathcal{P}_2(\{1, 2, 3, 4, 11\})$, $A_2 = \mathcal{P}_2(\{5, 6, 7, 8, 9, 10, 11\})$ and $A_3 = \{\{i, j\} \mid 1 \leq i \leq 4, 5 \leq j \leq 10\}$. The graph $L(K_{11})(2) = L(K_5)(2) \oplus L(K_7)(2) \oplus (K_4 \square K_6)(2) \oplus A'(2) \oplus H(2)$, by Observation 2.1, where $H(2) = \underbrace{K_{4,6}(2) \oplus K_{4,6}(2) \oplus \cdots \oplus K_{4,6}(2)}_{\text{10 copies}}$. By Theorems 1.3, 1.8 and 1.7, the graphs $L(K_5)(2)$, $(K_4 \square K_6)(2)$, $A'(2)$ and $H(2)$ have 4-cycle decompositions, where A' is as in Observation 2.1. The required packing and covering follow by Case (i) above, because $L(K_7)(2)$ has a 4-cycle packing and a 4-cycle covering with leave and padding $K_2(2)$ having the edges $\{\{7, 11\}\{8, 11\}, \{7, 11\}\{8, 11\}\}$. □

Lemma 2.7. *The graph $L(K_5)(3)$ admits a 4-cycle packing with leave $L = K_2(2)$ and a 4-cycle covering with padding $P = K_2(2)$.*

Proof. (i) A 4-cycle packing of $L(K_5)(3)$ with leave $K_2(2)$ is given by:

$(\{1,2\},\{1,3\},\{1,4\},\{1,5\})$, $(\{1,2\},\{2,3\},\{2,4\},\{2,5\})$, $(\{1,2\},\{1,3\},\{3,4\},\{1,4\})$,
$(\{1,2\},\{2,4\},\{4,5\},\{2,5\})$, $(\{1,2\},\{1,4\},\{4,5\},\{2,5\})$, $(\{1,2\},\{1,5\},\{3,5\},\{2,3\})$,
$(\{1,2\},\{1,3\},\{2,3\},\{2,4\})$, $(\{1,2\},\{1,4\},\{3,4\},\{2,3\})$, $(\{1,2\},\{1,5\},\{4,5\},\{2,4\})$,
$(\{1,3\},\{2,3\},\{3,4\},\{3,5\})$, $(\{1,3\},\{1,4\},\{1,5\},\{3,5\})$, $(\{1,3\},\{1,5\},\{4,5\},\{3,4\})$,
$(\{1,3\},\{1,5\},\{3,5\},\{2,3\})$, $(\{1,3\},\{1,5\},\{2,5\},\{3,5\})$, $(\{1,3\},\{1,4\},\{2,4\},\{3,4\})$,
$(\{1,4\},\{2,4\},\{3,4\},\{4,5\})$, $(\{1,4\},\{3,4\},\{3,5\},\{4,5\})$, $(\{1,4\},\{1,5\},\{2,5\},\{2,4\})$,
$(\{1,5\},\{2,5\},\{3,5\},\{4,5\})$, $(\{2,3\},\{2,4\},\{2,5\},\{3,5\})$, $(\{2,4\},\{3,4\},\{3,5\},\{4,5\})$,
$(\{2,3\},\{2,5\},\{4,5\},\{3,4\})$,

and the leave $K_2(2)$ is given by $L = \{\{2,3\}\{2,5\}, \{2,3\}\{2,5\}\}$.

(ii) The graph $L(K_5)(3) = L(K_5) \oplus L(K_5)(2)$. By Theorem 1.3 and Lemma 2.2, the graph $L(K_5)(2)$ has a 4-cycle decomposition and $L(K_5)$ has a 4-cycle covering with padding $K_2(2)$. □

Lemma 2.8. *Each of the graphs $L(K_7)(3)$, $L(K_7)(5)$ and $L(K_{11})(3)$ admits a 4-cycle packing and a 4-cycle covering with every possible leave and padding. The leaves L and paddings P are as follows:*

- (i) *for $L(K_7)(3)$, the leave $L = C_3$ and the padding P, $P \in \{C_5, F_1, F_2, F_3\}$;*
- (ii) *for $L(K_7)(5)$, the leave $L \in \{C_5, F_1, F_2, F_3\}$ and the padding $P = C_3$;*
- (iii) *for $L(K_{11})(3)$, the leave $L \in \{C_5, F_1, F_2, F_3\}$ and the padding $P = C_3$.*

Proof. (i) A 4-cycle packing and a 4-cycle covering of $L(K_7)(3)$ with leave C_3 and padding C_5, F_1, F_2, or F_3 are given below.

The graph $L(K_7)(3) = L(K_7) \oplus L(K_7)(2)$. By Lemma 2.3 and the proof of Lemma 2.6, $C_4|(L(K_7) - E(C_5))$, where $C_5 = (\{1,4\},\{1,7\},\{2,7\},\{3,7\},\{4,7\})$ and $C_4|(L(K_7)(2) - E(H_3))$; see Figure 4 in the Appendix. Let the graph $H_4 = C_5 \oplus H_3$; see Figure 5. A 4-cycle packing and a 4-cycle covering of H_4 with leave C_3 and padding C_5, F_1, F_2, or F_3 are given in Items 10, 11, 12, 13 and 14 of the Appendix.

(ii) A 4-cycle packing and a 4-cycle covering of $L(K_7)(5)$ with leave C_5, F_1, F_2, or F_3, and padding C_3, are given below.

The graph $L(K_7)(5) = L(K_7) \oplus L(K_7)(4)$. By Theorem 1.3, $C_4|L(K_7)(4)$ and by Lemma 2.3, we get a 4-cycle packing and a 4-cycle covering with leave C_5 and padding C_3. The graph $L(K_7)(5) = L(K_7)(2) \oplus L(K_7)(3)$. From the proof of Lemma 2.6 and Case (i) above, the graphs $L(K_7)(2)$ and $L(K_7)(3)$ have a 4-cycle packing with leave $K_2(2)$ and leave C_3 (given in Items 8 and 10 of the Appendix), respectively. From the leaves $K_2(2)$ and C_3, the union of leave $K_2(2)$ in Item 8(a) and leave C_3 in Item 10(a) gives the leave F_1; the union of leave $K_2(2)$ in Item 8(b) and leave C_3 in Item 10(b) gives the leave F_2; the union of leave $K_2(2)$ in Item 8(a) and leave C_3 in Item 10(c) gives the leave F_3.

(iii) A 4-cycle packing and a 4-cycle covering of $L(K_{11})(3)$ with leave C_5, F_1, F_2, or F_3, and padding C_3 are given below.

The graph $L(K_{11})(3) = L(K_{11}) \oplus L(K_{11})(2)$. From the proof of Lemmas 2.4 and 2.6, we have $C_4|(L(K_{11}) - E(H_1))$ and $C_4|(L(K_{11})(2) - E(K_2(2)))$, where $E(K_2(2)) = \{\{7, 11\}\{8, 11\}, \{7, 11\}\{8, 11\}\}$. Define the graph $H_5 = H_1 \oplus K_2(2)$; see Figure 6. Now a 4-cycle packing and a 4-cycle covering of H_5 with leave C_5, F_1, F_2, or F_3, and padding C_3, follows by Items 15, 16, 17, 18 and 19 of the Appendix. □

Lemma 2.9. *For $n \geq 4$, the graph $L(K_n)$ has a 4-cycle packing and a 4-cycle covering with every possible leave and padding. The leaves L and paddings P are as follows:*
(i) if $n \equiv 3 \pmod 8$, then the leave $L = C_3$ and padding $P \in \{C_5, F_1, F_2, F_3\}$;
(ii) if $n \equiv 5 \pmod 8$, then the leave $L \in \{C_6, B, 2C_3\}$ and padding $P = K_2(2)$;
(iii) if $n \equiv 7 \pmod 8$, then the leave $L = C_5$ and padding $P = C_3$.

Proof. (i) $n \equiv 3 \pmod 8$: Let $n = 8k+3, k \geq 1$. If $k = 1$, then the result follows by Lemma 2.4. Now consider $k \geq 2$. The graph $L(K_{8k+3}) = L(K_{11}) \oplus L(K_{8(k-1)+1}) \oplus (K_{10} \Box K_{8(k-1)}) \oplus H$, by Observation 2.1, where H is as defined in Observation 2.1, namely, $H = A' \oplus \underbrace{K_{10,8(k-1)} \oplus K_{10,8(k-1)} \oplus \cdots \oplus K_{10,8(k-1)}}_{(8k+2) \text{ copies}}$. By Theorems 1.3, 1.8 and 1.7, $C_4|L(K_{8(k-1)+1})$, $C_4|(K_{10} \Box K_{8(k-1)})$ and $C_4|H$. Now the required packing and covering follow by Lemma 2.4.

(ii) $n \equiv 5 \pmod 8$: Let $n = 8k+5, k \geq 0$. For $k = 0$, the graph $L(K_5)$ has a 4-cycle packing and a 4-cycle covering, by Lemma 2.2. Now we consider $k \geq 1$. The graph $L(K_{8k+5}) = L(K_5) \oplus L(K_{8k+1}) \oplus (K_4 \Box K_{8k}) \oplus H$, by Observation 2.1, where $H = A' \oplus \underbrace{K_{4,8k} \oplus K_{4,8k} \oplus \cdots \oplus K_{4,8k}}_{(8k+4) \text{ copies}}$. Now the result follows by Lemma 2.2 and Theorems 1.3, 1.8 and 1.7.

(iii) $n \equiv 7 \pmod 8$: Let $n = 8k+7, k \geq 0$. Because of Lemma 2.3, we consider $k \geq 1$. The graph $L(K_{8k+7}) = L(K_7) \oplus L(K_{8k+1}) \oplus (K_6 \Box K_{8k}) \oplus H$, by Observation 2.1, where $H = A' \oplus \underbrace{K_{6,8k} \oplus K_{6,8k} \oplus \cdots \oplus K_{6,8k}}_{(8k+6) \text{ copies}}$. The result now follows by Lemma 2.3 and Theorems 1.3, 1.8 and 1.7. □

Lemma 2.10. *For $n \geq 4$, the graph $L(K_n)(2)$ has a 4-cycle packing and a 4-cycle covering with every possible leave and padding. The leaves and paddings are as follows:*

(i) $L = (K_2)(2)$ *and* $P = K_2(2)$ *if* $n \equiv 3 \pmod 8$;

(ii) $L = \emptyset$ *and* $P = \emptyset$ *if* $n \equiv 5 \pmod 8$;

(iii) $L = (K_2)(2)$ *and* $P = K_2(2)$ *if* $n \equiv 7 \pmod 8$.

Proof. From the proof of Lemma 2.9, it is enough to show that each of the graphs $L(K_5)(2)$, $L(K_7)(2)$ and $L(K_{11})(2)$ admits a 4-cycle packing and a 4-cycle covering with every possible leave and padding and the result follows by Theorem 1.3 and Lemma 2.6. □

Lemma 2.11. *For $n \geq 4$, the graph $L(K_n)(3)$ has a 4-cycle packing and a 4-cycle covering with every possible leave and padding. The leaves and paddings are as follows:*

(i) $L \in \{C_5, F_1, F_2, F_3\}$ *and* $P = C_3$ *if* $n \equiv 3 \pmod 8$;

(ii) $L = K_2(2)$ *and* $P = K_2(2)$ *if* $n \equiv 5 \pmod 8$;

(iii) $L = C_3$ *and* $P \in \{C_5, F_1, F_2, F_3\}$ *if* $n \equiv 7 \pmod 8$.

Proof. As in the proof of Lemma 2.9, it is enough to show that each of the graphs $L(K_5)(3)$, $L(K_7)(3)$ and $L(K_{11})(3)$ has a 4-cycle packing and a 4-cycle covering with every possible leave and padding, and the result follows by Lemmas 2.7 and 2.8. □

Lemma 2.12. *For $n \geq 4$, the graph $L(K_n)(5)$ has a 4-cycle packing and a 4-cycle covering with every possible leave and padding. The leaves and paddings are as follows:*

(i) $L = C_3$ *and* $P \in \{C_5, F_1, F_2, F_3\}$ *if* $n \equiv 3 \pmod 8$;

(ii) $L = K_2(2)$ *and* $P = K_2(2)$ *if* $n \equiv 5 \pmod 8$;

(iii) $L \in \{C_5, F_1, F_2, F_3\}$ *and* $P = C_3$ *if* $n \equiv 7 \pmod 8$.

Proof. (i) $n \equiv 3 \pmod 8$: Let $n = 8k + 3$, $k \geq 1$. The graph $L(K_{8k+3})(5) = L(K_{8k+3}) \oplus L(K_{8k+3})(4)$, and the result follows by Lemma 2.9 and Theorem 1.3.

(ii) $n \equiv 5 \pmod 8$: Let $n = 8k+5$, $k \geq 0$. The graph $L(K_{8k+5})(5) = L(K_{8k+5})(2) \oplus L(K_{8k+5})(3)$, and the result follows by Theorem 1.3 and Lemma 2.11.

(iii) $n \equiv 7 \pmod 8$: Let $n = 8k + 7$, $k \geq 0$. The graph $L(K_{8k+7})(5) = L(K_7)(5) \oplus L(K_{8k+1})(5) \oplus (K_6 \square K_{8k})(5) \oplus A'(5) \oplus H(5)$, by Observation 2.1, where $H(5) = \underbrace{K_{6,8k}(5) \oplus K_{6,8k}(5) \oplus \cdots \oplus K_{6,8k}(5)}_{(8k+6)\text{ copies}}$. The result now follows by Lemmas 1.5 and 2.8 and Theorems 1.3, 1.8 and 1.7. □

Proof of Theorem 1.1. By Lemmas 2.9, 2.10, 2.11 and 2.12, the proof follows for $\lambda \in \{1, 2, 3, 5\}$. First, we consider the proof for $\lambda \equiv 0, 2, 3 \pmod 4$. Let $\lambda = 4k + i$, $k \geq 1$, $i \in \{0, 2, 3\}$. For $i = 0$, the proof follows by Theorem 1.3. For $i \in \{2, 3\}$, let $L(K_n)(\lambda) = L(K_n)(i) \oplus L(K_n)(4k)$. Now the required maximum packing and minimum covering with 4-cycles follows by Lemma 2.10 and 2.11 and Theorem 1.3. Finally, for $\lambda \equiv 1 \pmod 4 > 5$, the graph $L(K_n)(4k+1) = L(K_n)(5) \oplus L(K_n)(4k-4)$ and the result follows by Lemma 2.12 and Theorem 1.3. □

3 Existence of a maximum packing and a minimum covering of $L(K_n)(\lambda)$ with 6-cycles

In this section, we prove the existence of a 6-cycle packing and a 6-cycle covering of $L(K_n)(\lambda)$ with every possible leave and padding.

Observation 3.1. For a graph G, $S_1(G)$ denotes the graph that arises out of the subdivision of each edge of G exactly once; $S_1(G)$ is the *first subdivision graph* of G. Let $G^\star$ be the graph obtained from G by adding to each edge $e = uv$ of G a new vertex $\{u, v\}$ such that the vertex $\{u, v\}$ is adjacent to both the vertices u and v, and $\{u, v\}$ is a vertex of degree two in $G^\star$; see Figure 2. If we delete all the edges of G in $G^\star$, then the resulting graph is isomorphic to $S_1(G)$, the first subdivision graph of G, and hence $G^\star = G \oplus S_1(G)$.

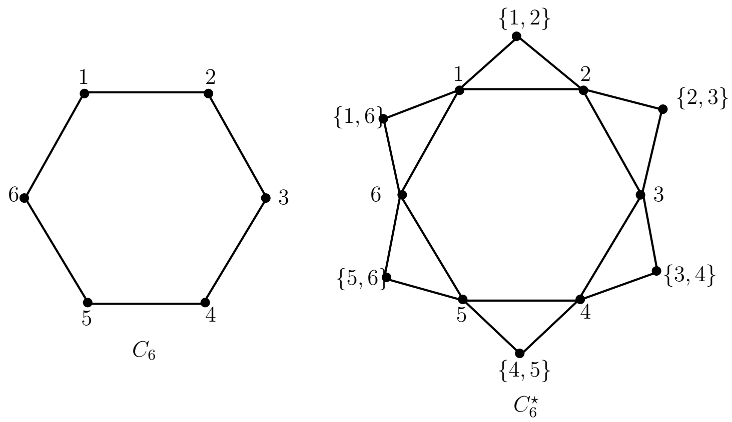

Figure 2: The graph C_6 and $C_6^\star$.

Let $V(K_{n+1}) = \{1, 2, \ldots, n+1\}$. Then $V(L(K_{n+1})) = \mathcal{P}_2(\{1, 2, \ldots, n+1\})$. We partition the vertex set of $L(K_{n+1})$ into two sets A_1 and A_2, where $A_1 = \mathcal{P}_2(\{1, 2, \ldots, n\})$ and $A_2 = \bigcup_{i=1}^{n}\{i, n+1\}$. The subgraph of $L(K_{n+1})$ induced by A_1 (respectively, A_2) is isomorphic to $L(K_n)$ (respectively, K_n). Clearly, $E(A_1, A_2)$, in $L(K_{n+1})$, is $\{\{i,j\}\{i,n+1\}, \{i,j\}\{j,n+1\}\}$, $1 \le i < j \le n$; note that each two-element subset represents a vertex in the line graph. Then $L(K_{n+1}) = \langle A_1 \rangle \oplus \langle A_2 \rangle \oplus \langle E(A_1, A_2) \rangle = L(K_n) \oplus K_n^\star$.

Lemma 3.2. *Each of the graphs $L(K_7)$, $L(K_{11})$ and $L(K_{15})$ admits a 6-cycle packing and a 6-cycle covering with leave C_3 and padding C_3.*

Proof. (*i*) Let $V(K_7) = \{1, 2, \ldots, 7\}$. Let

$$\mathcal{C} = \{(1,2,3,4,6,5), (1,6,2,5,3,7), (1,3,6,7,2,4), (4,5,7)\}$$

be a decomposition of K_7 into three copies of C_6 and a C_3. Clearly, the graph $L(K_7) - E(L(\mathcal{C})) = \underbrace{(K_6 - I) \oplus (K_6 - I) \oplus \cdots \oplus (K_6 - I)}_{7 \text{ copies}}$, where I is a perfect matching of K_6. As $C_6 \mid (K_6 - I)$, a 6-cycle packing of $L(K_7)$ with leave $C_3 = (\{4,5\}, \{5,7\}, \{4,7\})$ exists. Now the graph $L(K_7) = L(K_6) \oplus K_6^\star$, by Observation 3.1, and a required 6-cycle covering follows by Corollary 1.6 and Item 20 of the Appendix.

(ii) Let $\mathcal{P}_2(\{1,2,\ldots,10,11\}) = V(L(K_{11}))$. We partition the vertex set of $L(K_{11})$ into three sets A_1, A_2 and A_3, where $A_1 = \mathcal{P}_2(\{1,2,3,4,11\})$, $A_2 = \mathcal{P}_2(\{5,6,7,8,9,10,11\})$ and $A_3 = \{\{i,j\} \mid 1 \le i \le 4, 5 \le j \le 10\}$. The subgraphs induced by A_1 and A_2 are $L(K_5)$ and $L(K_7)$, respectively. The graph $L(K_{11}) = L(K_5) \oplus L(K_7) \oplus (K_4 \Box K_6) \oplus H$, by Observation 2.1, where $H = A' \oplus \underbrace{K_{4,6} \oplus K_{4,6} \oplus \cdots \oplus K_{4,6}}_{10 \text{ copies}}$. By Corollary 1.6 and Theorems 1.9 and 1.7, the graphs $L(K_5)$, $K_4 \Box K_6$ and H admit 6-cycle decompositions. Then a required 6-cycle packing and a 6-cycle covering of $L(K_{11})$ with leave C_3 and padding C_3 exist by Case (i) above.

(iii) Let $\mathcal{P}_2(\{1,2,\ldots,14,15\}) = V(L(K_{15}))$. We partition the vertex set of $L(K_{15})$ into three sets $A_1 = \mathcal{P}_2(\{1,2,3,4,5,6,15\})$, $A_2 = \mathcal{P}_2(\{7,8,\ldots,14,15\})$ and $A_3 = \{\{i,j\} \mid 1 \le i \le 6, 7 \le j \le 14\}$. The graph $L(K_{15}) = L(K_7) \oplus L(K_9) \oplus (K_6 \Box K_8) \oplus H$, by Observation 2.1, where $H = A' \oplus \underbrace{K_{6,8} \oplus K_{6,8} \oplus \cdots \oplus K_{6,8}}_{15 \text{ copies}}$. A required 6-cycle packing and a 6-cycle covering of $L(K_{15})$ with $L = P = C_3$ follows by Corollary 1.6 and Theorems 1.9, 1.7 and Case (i) above. □

Lemma 3.3. *The graph $K_6^\star(2)$ admits a 6-cycle decomposition.*

Proof. Let $V(K_6) = \{1,2,\ldots,6\}$. The 6-cycles are

$(1,4,\{4,5\},5,\{5,6\},6)$,	$(1,2,3,\{3,5\},5,\{1,5\})$,	$(1,\{1,4\},4,3,6,\{1,6\})$,
$(1,\{1,2\},2,\{2,5\},5,6)$,	$(2,\{2,3\},3,\{3,4\},4,5)$,	$(1,2,\{2,3\},3,4,\{1,4\})$,
$(1,\{1,2\},2,\{2,6\},6,5)$,	$(1,\{1,3\},3,\{3,5\},5,\{1,5\})$,	$(2,4,\{4,5\},5,\{5,6\},6)$,
$(1,5,2,6,3,\{1,3\})$,	$(1,3,5,\{2,5\},2,4)$,	$(2,\{2,4\},4,6,\{3,6\},3)$,
$(2,\{2,4\},4,\{4,6\},6,\{2,6\})$,	$(3,\{3,6\},6,\{4,6\},4,5)$,	$(1,3,\{3,4\},4,6,\{1,6\})$.

□

Lemma 3.4. *Each of the graphs $L(K_7)(2)$, $L(K_{11})(2)$ and $L(K_{15})(2)$ admits a 6-cycle decomposition.*

Proof. (i) The graph $L(K_7)(2) = L(K_6)(2) \oplus K_6^\star(2)$, by Observation 3.1, and $C_6 | L(K_6)(2)$ and $C_6 | K_6^\star(2)$, by Corollary 1.6 and Lemma 3.3.

(ii) Let $V(L(K_{11})(2)) = \mathcal{P}_2(\{1,2,\ldots,10,11\})$. We partition the vertex set of $L(K_{11})(2)$ into three sets $A_1 = \mathcal{P}_2(\{1,2,3,4,11\})$, $A_2 = \mathcal{P}_2(\{5,6,7,8,9,10,11\})$ and $A_3 = \{\{i,j\} | 1 \le i \le 4, 5 \le j \le 10\}$. The graph $L(K_{11})(2) = L(K_5)(2) \oplus L(K_7)(2) \oplus (K_4 \Box K_6)(2) \oplus H(2)$, by Observation 2.1, where

$$H(2) = A'(2) \oplus \underbrace{K_{4,6}(2) \oplus K_{4,6}(2) \oplus \cdots \oplus K_{4,6}(2)}_{10 \text{ copies}}.$$

Hence a required decomposition follows by Corollary 1.6 and Theorems 1.9 and 1.7 and Case (i) above.

(iii) Let $V(L(K_{15})(2)) = \mathcal{P}_2(\{1,2,\ldots,14,15\})$. We partition the vertex set of $L(K_{15})(2)$ into three sets $A_1 = \mathcal{P}_2(\{1,2,3,4,5,6,15\})$, $A_2 = \mathcal{P}_2(\{7,8,\ldots,14,15\})$ and $A_3 = \{\{i,j\} | 1 \le i \le 6, 7 \le j \le 14\}$. The graph $L(K_{15})(2) = L(K_7)(2) \oplus L(K_9)(2) \oplus (K_6 \Box K_8)(2) \oplus H(2)$, by Observation 2.1, where

$$H(2) = A'(2) \oplus \underbrace{K_{6,8}(2) \oplus K_{6,8}(2) \oplus \cdots \oplus K_{6,8}(2)}_{14\,\text{copies}}.$$

Now the result follows by Case (i) above, Corollary 1.6 and Theorems 1.9 and 1.7. □

Lemma 3.5. *For $n \equiv 3 \pmod 4, n \ge 4$, the graph $L(K_n)$ admits a 6-cycle packing with leave C_3 and a 6-cycle covering with padding C_3.*

Proof. We consider the following three cases.

Case 1. $n \equiv 3 \pmod{12}$. Let $n = 12k+3$, $k \ge 1$. For $k = 1$, the result follows by Lemma 3.2. So we consider $k \ge 2$. The graph $L(K_{12k+3}) = L(K_{15}) \oplus L(K_{12k-11}) \oplus (K_{14} \Box K_{12(k-1)}) \oplus H$, where $H = A' \oplus \underbrace{K_{14,12(k-1)} \oplus \cdots \oplus K_{14,12(k-1)}}_{(12k+2)\,\text{copies}}$, by Observation 2.1. Thus a 6-cycle packing and a 6-cycle covering follow by Lemma 3.2, Corollary 1.6 and Theorems 1.9 and 1.7.

Case 2. $n \equiv 7 \pmod{12}$. Let $n = 12k+7$, $k \ge 0$. For $k = 0$, the graph $L(K_7)$ has a 6-cycle packing and a 6-cycle covering, by Lemma 3.2. Next we consider $k \ge 1$. The graph $L(K_{12k+7}) = L(K_7) \oplus L(K_{12k+1}) \oplus (K_6 \Box K_{12k}) \oplus H$. Here, $H = A' \oplus \underbrace{K_{6,12k} \oplus \cdots \oplus K_{6,12k}}_{(12k+6)\,\text{copies}}$, by Observation 2.1. Hence by Lemma 3.2, Corollary 1.6, Theorems 1.9 and 1.7, a required 6-cycle packing and a 6-cycle covering follow.

Case 3. $n \equiv 11 \pmod{12}$. Let $n = 12k+11$, $k \ge 0$. Because of Lemma 3.2, we consider $k \ge 1$. The graph $L(K_{12k+11}) = L(K_{11}) \oplus L(K_{12k+1}) \oplus (K_{10} \Box K_{12k}) \oplus H$. Now $H = A' \oplus \underbrace{K_{10,12k} \oplus \cdots \oplus K_{10,12k}}_{(12k+10)\,\text{copies}}$, by Observation 2.1. Now a 6-cycle packing and a 6-cycle covering follow by Lemma 3.2, Corollary 1.6 and Theorems 1.9 and 1.7. □

Lemma 3.6. *For $n \equiv 3 \pmod 4$, $n \ge 4$, the graph $L(K_n)(2)$ has a 6-cycle decomposition.*

Proof. From the proof of Lemma 3.5, it is enough to show that each of the graphs in $\{L(K_7)(2), L(K_{11})(2), L(K_{15})(2)\}$ admits a 6-cycle decomposition. Now a required decomposition follows by Lemma 3.4. □

Proof of Theorem 1.2.
Case 1. First we consider $\lambda \equiv 0 \pmod 2$, and let $\lambda = 2k'$, $k' \ge 1$. The graph $L(K_n)(2k') = L(K_n)(2) \oplus L(K_n)(2) \oplus \cdots \oplus L(K_n)(2)$, and a 6-cycle decomposition

follows by applying Corollary 1.6 if $n \not\equiv 3 \pmod 4$, and applying Lemma 3.6 if $n \equiv 3 \pmod 4$.

Case 2. Next, $\lambda \equiv 1 \pmod 2$, and let $\lambda = 2k'+1$, $k' \geq 0$. The graph $L(K_n)(2k'+1) = L(K_n) \oplus L(K_n)(2k')$. We obtain a 6-cycle packing and 6-cycle covering of $L(K_n)(\lambda)$ by applying Corollary 1.6, and Lemmas 3.5 and 3.6. □

4 Appendix

1. The subgraphs H_1 of $L(K_{11})$, H_3 of $L(K_7)(2)$, H_4 of $L(K_7)(3)$ and H_5 of $L(K_{11})(3)$ are shown below:

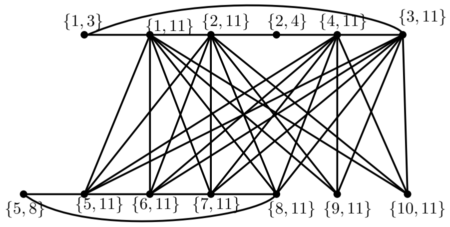

Figure 3: The subgraph H_1 of $L(K_{11})$.

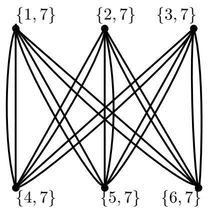

Figure 4: The subgraph H_3 of $L(K_7)(2)$.

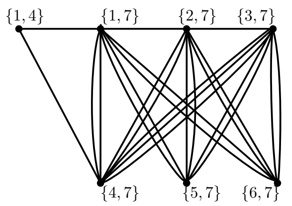

Figure 5: The subgraph H_4 of $L(K_7)(3)$.

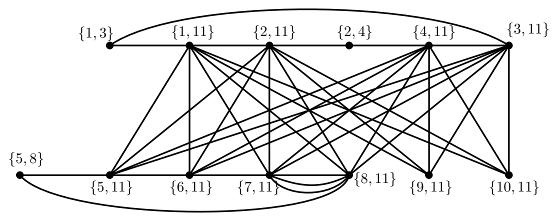

Figure 6: The subgraph H_5 of $L(K_{11})(3)$.

2. The subgraph H_1 of $L(K_{11})$ has a 4-cycle packing with leave C_3.
```
({1,3},{1,11},{7,11},{3,11}), ({1,11},{5,11},{6,11},{2,11}),
({1,11},{6,11},{7,11},{8,11}), ({1,11},{9,11},{3,11},{10,11}),
({2,11},{5,11},{4,11},{7,11}), ({2,11},{8,11},{4,11},{9,11}),
({2,11},{2,4},{4,11},{10,11}), ({5,8},{5,11},{3,11},{8,11}) and the leave
L=({4,11},{3,11},{6,11}).
```

3. The subgraph H_1 of $L(K_{11})$ has a 4-cycle covering with padding C_5.
```
({1,3},{1,11},{10,11},{3,11}), ({1,11},{5,11},{5,8},{8,11}),
({1,11},{2,11},{7,11},{6,11}), ({1,11},{7,11},{4,11},{9,11}),
({2,11},{6,11},{3,11},{9,11}), ({2,11},{2,4},{4,11},{10,11}),
({2,11},{2,4},{4,11},{5,11}), ({2,11},{7,11},{3,11},{8,11}),
({4,11},{3,11},{7,11},{8,11}), ({4,11},{3,11},{5,11},{6,11}) and the padding
P=({2,11},{2,4},{4,11},{3,11},{7,11}).
```

4. The subgraph H_1 of $L(K_{11})$ has a 4-cycle covering with padding F_1.
```
({1,3},{1,11},{10,11},{3,11}), ({1,11},{5,11},{5,8},{8,11}),
({1,11},{2,11},{7,11},{6,11}), ({1,11},{7,11},{4,11},{9,11}),
({2,11},{6,11},{3,11},{9,11}), ({2,11},{2,4},{4,11},{8,11}),
```

({2,11},{4,11},{6,11},{7,11}), ({2,11},{5,11},{4,11},{10,11}),
({3,11},{5,11},{6,11},{7,11}), ({3,11},{8,11},{7,11},{4,11}) and the padding
P={{2,11}{4,11},{4,11}{7,11},{7,11}{2,11},{6,11}{7,11}{6,11}{7,11}}.

5. The subgraph H_1 of $L(K_{11})$ has a 4-cycle covering with padding F_2.
({1,3},{1,11},{10,11},{3,11}), ({1,11},{5,11},{5,8},{8,11}),
({1,11},{2,11},{7,11},{6,11}), ({1,11},{7,11},{4,11},{9,11}),
({2,11},{6,11},{3,11},{9,11}), ({2,11},{4,11},{6,11},{5,11}),
({2,11},{7,11},{3,11},{10,11}), ({2,11},{2,4},{4,11},{8,11}),
({4,11},{5,11},{3,11},{10,11}), ({4,11},{3,11},{8,11},{7,11}) and the padding
P={{2,11}{4,11},{4,11}{7,11},{7,11}{2,11},{3,11}{10,11},{3,11}{10,11}}.

6. The subgraph H_1 of $L(K_{11})$ has a 4-cycle covering with padding F_3.
({1,3},{1,11},{10,11},{3,11}), ({1,11},{5,11},{5,8},{8,11}),
({1,11},{2,11},{7,11},{6,11}), ({1,11},{7,11},{4,11},{9,11}),
({2,11},{6,11},{3,11},{9,11}), ({2,11},{4,11},{6,11},{5,11}),
({2,11},{7,11},{4,11},{10,11}), ({2,11},{2,4},{4,11},{8,11}),
({4,11},{5,11},{3,11},{7,11}), ({4,11},{3,11},{8,11},{7,11}) and the padding
P={{2,11}{4,11},{4,11}{7,11},{7,11}{2,11},{4,11}{7,11},{4,11}{7,11}}.

7. The subgraph H_2 of $L(K_7)(2)$ has a 4-cycle decomposition.
({1,3},{3,5},{3,7},{3,4}), ({2,3},{3,6},{3,7},{3,5}), ({1,3},{1,6},{1,7},{1,4}),
({1,3},{1,5},{1,7},{1,6}), ({1,2},{1,4},{1,7},{1,5}), ({1,2},{2,4},{2,7},{2,5}),
({2,3},{2,4},{2,7},{2,6}), ({2,3},{2,5},{2,7},{2,6}), ({1,3},{3,4},{3,7},{3,6}),
({1,2},{1,6},{4,6},{2,6}), ({1,3},{1,4},{4,6},{3,6}), ({2,3},{3,4},{4,5},{2,4}),
({2,3},{3,4},{4,6},{3,6}), ({1,6},{4,6},{2,6},{5,6}), ({1,5},{4,5},{3,5},{5,6}),
({2,5},{5,6},{3,5},{4,5}), ({1,5},{4,5},{2,5},{5,6}), ({2,3},{2,5},{5,7},{3,5}),
({1,3},{1,5},{5,7},{3,5}), ({1,2},{1,6},{6,7},{2,6}), ({1,4},{4,7},{3,4},{4,5}),
({1,4},{4,7},{2,4},{4,5}), ({1,6},{5,6},{3,6},{6,7}), ({2,4},{4,7},{3,4},{4,6}),
({1,2},{1,4},{4,6},{2,4}), ({1,2},{1,5},{5,7},{2,5}), ({2,6},{5,6},{3,6},{6,7}).

8. Two choices of 4-cycle packing with leave $K_2(2)$ from the graph H_3 of $L(K_7)(2)$.
   (a) The subgraph H_3 of $L(K_7)(2)$ has a 4-cycle packing with leave $K_2(2)$.
   ({1,7},{4,7},{2,7},{5,7}), ({1,7},{5,7},{3,7},{6,7}),
   ({2,7},{5,7},{3,7},{6,7}), ({1,7},{4,7},{2,7},{6,7})
   and the leave L={{3,7}{4,7},{3,7}{4,7}}.

   (b) The subgraph H_3 of $L(K_7)(2)$ has a 4-cycle packing with leave $K_2(2)$.
   ({1,7},{5,7},{3,7},{6,7}), ({2,7},{4,7},{3,7},{6,7}),
   ({1,7},{5,7},{2,7},{6,7}), ({2,7},{4,7},{3,7},{5,7})
   and the leave L={{1,7}{4,7},{1,7}{4,7}}.

9. The subgraph H_3 of $L(K_7)(2)$ has a 4-cycle covering with padding $K_2(2)$.
({1,7},{4,7},{3,7},{5,7}), ({1,7},{4,7},{3,7},{6,7}),
({1,7},{5,7},{2,7},{6,7}), ({2,7},{4,7},{3,7},{5,7}),
({2,7},{6,7},{3,7},{4,7}) and the padding P={{3,7}{4,7},{3,7}{4,7}}.

10. Three choices of 4-cycle packing with leave C_3 from the graph H_4 of $L(K_7)(3)$.
   (a) The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle packing with leave C_3.
   ({1,7},{2,7},{3,7},{4,7}), ({1,7},{5,7},{3,7},{6,7}),
   ({2,7},{4,7},{3,7},{6,7}), ({1,7},{5,7},{2,7},{6,7}),
   ({2,7},{4,7},{3,7},{5,7}) and the leave L=({1,4},{1,7},{4,7}).

(b) The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle packing with leave C_3.
({1,4},{1,7},{2,7},{4,7}), ({1,7},{4,7},{3,7},{6,7}),
({1,7},{4,7},{3,7},{5,7}), ({1,7},{5,7},{2,7},{6,7}),
({2,7},{4,7},{3,7},{6,7}) and the leave L=({2,7},{3,7},{5,7}).

(c) The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle packing with leave C_3.
({1,4},{1,7},{2,7},{4,7}), ({1,7},{4,7},{3,7},{6,7}),
({1,7},{4,7},{3,7},{5,7}), ({1,7},{5,7},{2,7},{6,7}),
({2,7},{5,7},{3,7},{6,7}) and the leave L=({2,7},{3,7},{4,7}).

11. The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle covering with padding C_5.
({1,4},{1,7},{2,7},{4,7}), ({1,7},{4,7},{3,7},{5,7}),
({1,7},{2,7},{3,7},{4,7}), ({1,7},{5,7},{2,7},{6,7}),
({2,7},{5,7},{3,7},{6,7}), ({1,7},{5,7},{3,7},{6,7}),
({2,7},{4,7},{3,7},{6,7}) and the padding
P=({1,7},{2,7},{6,7},{3,7},{5,7}).

12. The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle covering with padding F_1.
({1,4},{1,7},{2,7},{4,7}), ({1,7},{2,7},{3,7},{5,7}),
({1,7},{6,7},{3,7},{4,7}), ({1,7},{5,7},{2,7},{6,7}),
({1,7},{6,7},{3,7},{4,7}), ({1,7},{3,7},{2,7},{6,7}),
({2,7},{4,7},{3,7},{5,7}) and the padding
P={1,7}{2,7},{2,7}{3,7},{3,7}{1,7},{1,7}{6,7},{1,7}{6,7}}.

13. The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle covering with padding F_2.
({1,4},{1,7},{3,7},{4,7}), ({1,7},{4,7},{2,7},{5,7}),
({1,7},{2,7},{3,7},{6,7}), ({1,7},{5,7},{3,7},{6,7}),
({1,7},{2,7},{6,7},{4,7}), ({2,7},{4,7},{3,7},{5,7}),
({2,7},{3,7},{4,7},{6,7}) and the padding
P={{1,7}{2,7},{2,7}{3,7},{3,7}{1,7},{4,7}{6,7},{4,7}{6,7}}.

14. The subgraph H_4 of $L(K_7)(3)$ has a 4-cycle covering with padding F_3.
({1,4},{1,7},{3,7},{4,7}), ({1,7},{2,7},{3,7},{6,7}),
({1,7},{3,7},{5,7},{2,7}), ({1,7},{4,7},{2,7},{3,7}),
({1,7},{4,7},{3,7},{5,7}), ({1,7},{5,7},{2,7},{6,7}),
({2,7},{6,7},{3,7},{4,7}) and the padding
P={{1,7}{2,7},{2,7}{3,7},{3,7}{1,7},{3,7}{1,7},{3,7}{1,7}}.

15. The subgraph H_5 of $L(K_{11})(3)$ has a 4-cycle packing with leave C_5.
({1,11},{5,11},{5,8},{8,11}), ({1,11},{2,11},{5,11},{6,11}),
({1,11},{9,11},{2,11},{10,11}), ({2,11},{2,4},{4,11},{7,11}),
({2,11},{6,11},{7,11},{8,11}), ({4,11},{3,11},{7,11},{8,11})
({4,11},{5,11},{3,11},{6,11}), ({4,11},{9,11},{3,11},{10,11}) and the leave
L=({1,3},{1,11},{7,11},{8,11},{3,11}).

16. The subgraph H_5 of $L(K_{11})(3)$ has a 4-cycle packing with leave F_1.
({1,3},{1,11},{10,11},{3,11}), ({1,11},{2,11},{5,11},{6,11}),
({1,11},{5,11},{5,8},{8,11}), ({1,11},{7,11},{4,11},{9,11}),
({2,11},{2,4},{4,11},{10,11}), ({2,11},{8,11},{3,11},{9,11}),
({4,11},{3,11},{7,11},{8,11}), ({4,11},{5,11},{3,11},{6,11}) and the leave
L={{2,11}{6,11},{6,11}{7,11},{7,11}{2,11},{7,11}{8,11},{7,11}{8,11}}.

17. The subgraph H_5 of $L(K_{11})(3)$ has a 4-cycle packing with leave F_2.
```
({1,3},{1,11},{10,11},{3,11}), ({1,11},{2,11},{6,11},{7,11}),
({1,11},{8,11},{3,11},{9,11}), ({2,11},{2,4},{4,11},{7,11}),
({2,11},{5,11},{5,8},{8,11}), ({2,11},{9,11},{4,11},{10,11}),
({4,11},{3,11},{7,11},{8,11}), ({4,11},{5,11},{3,11},{6,11}) and the leave
L={{1,11}{5,11},{5,11}{6,11},{6,11}{1,11},{7,11}{8,11},{7,11}{8,11}}.
```

18. The subgraph H_5 of $L(K_{11})(3)$ has a 4-cycle packing with leave F_3.
```
({1,3},{1,11},{9,11},{3,11}), ({1,11},{2,11},{5,11},{6,11}),
({1,11},{5,11},{3,11},{10,11}), ({2,11},{2,4},{4,11},{10,11}),
({2,11},{6,11},{4,11},{9,11}), ({2,11},{7,11},{3,11},{8,11}),
({4,11},{3,11},{6,11},{7,11}), ({4,11},{5,11},{5,8},{8,11}) and the leave
L={{1,11}{8,11},{8,11}{7,11},{7,11}{1,11},{7,11}{8,11},{7,11}{8,11}}.
```

19. The subgraph H_5 of $L(K_{11})(3)$ has a 4-cycle covering with padding C_3.
```
({1,3},{1,11},{10,11},{3,11}), ({1,11},{2,11},{5,11},{6,11}),
({1,11},{3,11},{8,11},{7,11}), ({1,11},{5,11},{5,8},{8,11}),
({1,11},{8,11},{3,11},{9,11}), ({2,11},{2,4},{4,11},{7,11})
({2,11},{6,11},{7,11},{8,11}), ({2,11},{9,11},{4,11},{10,11})
({4,11},{3,11},{7,11},{8,11}), ({4,11},{5,11},{3,11},{6,11}) and the padding
P=({1,11},{3,11},{8,11}).
```

20. The graph $K_6^\star$ has a 6-cycle covering with padding C_3.
```
(1,{1,4},4,3,{3,5},5), (1,{1,2},2,{2,3},3,6), (1,{1,3},3,5,{4,5},4),
(2,{2,5},5,{5,6},6,{2,6}), (1,3,{3,4},4,5,{1,5}), (1,{1,6},6,{3,6},3,2),
(2,{2,4},4,{4,6},6,5), (2,4,{4,5},5,3,6) and the padding
P=(2,3,6).
```

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