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Odd colourings, conflict-free colourings and strong colouring numbers

Robert Hickingbotham*

School of Mathematics
Monash University
Melbourne, Australia
robert.hickingbotham@monash.edu

Abstract

The odd chromatic number and the conflict-free chromatic number are new graph parameters introduced by Petruševski and Škrekovski (2022) and Fabrici, Lužar, Rindošová and Soták (2023) respectively. In this paper, we show that graphs with bounded 2-strong colouring number have bounded odd chromatic number and bounded conflict-free chromatic number. This implies that graph classes with bounded expansion have bounded odd chromatic number and bounded conflict-free chromatic number which is one of the broadest known classes to have these properties. As an example, it follows by known results that the odd chromatic number and the conflict-free chromatic number of k-planar graphs is $O(k)$, which improves a recent result of Dujmović, Morin and Odak (2022).

1 Introduction

All graphs in this paper are finite, simple, and undirected. For $m, n \in \mathbb{Z}$ with $m \leqslant n$, let $[m, n] := \{m, m+1, \dots, n\}$ and $[n] := [1, n]$. Let G be a graph. A *(vertex) c-colouring* of G is any function $\psi : V(G) \to C$ where $|C| \leqslant c$. If $\psi(u) \neq \psi(v)$ for all $uv \in E(G)$, then ψ is a *proper colouring.* If $N(v) := \{w \in V(G) : vw \in E(G)\}$ is the neighbourhood of a vertex v, then ψ is an *odd colouring* if for each $v \in V(G)$ with $|N(v)| > 0$, there exists a colour $\alpha \in C$ such that $|\{w \in N(v) : \psi(w) = \alpha\}|$ is odd. Similarly, ψ is a *conflict-free colouring* of G if for each $v \in V(G)$ with $|N(v)| > 0$, there exists a colour $\alpha \in C$ such that $|\{w \in N(v) : \psi(w) = \alpha\}| = 1$. The *(proper) odd chromatic number* $\chi_o(G)$ of G is the minimum integer c such that G has a (proper) odd c-colouring. Likewise, the *(proper) conflict-free chromatic number* $\chi_{pcf}(G)$ of G is the minimum integer c such that G has a (proper) conflict-free c-colouring. Clearly $\chi_o(G) \leqslant \chi_{pcf}(G)$ since a conflict-free colouring is an odd colouring.

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Motivated by connections to hypergraph colouring, the odd chromatic number and the conflict-free chromatic number were recently introduced by Petruševski and Škrekovski [20] and Fabrici, Lužar, Rindošová, and Soták [10] respectively. These parameters have gained significant traction with a particular focus on determining a tight upper bound for planar graphs. Petruševski and Škrekovski [20] showed that the odd chromatic number of planar graphs is at most 9 and conjectured that their odd chromatic number is at most 5. Petr and Portier [19] improved this upper bound to 8. For conflict-free colourings, Fabrici et al. [10] proved a matching upper bound of 8 for planar graphs. For proper minor-closed classes, a result of Cranston, Lafferty, and Song [5] implies that the odd chromatic number of K_t-minor free graphs is $O(t\sqrt{\log t})$. For non-minor closed classes, Cranston et al. [5] showed that the odd chromatic number of 1-planar graphs is at most 23 (A graph G is *k-planar* if it has an embedding in the plane such that each edge is involved in at most k crossings). Dujmović, Morin, and Odak [6] proved a more general upper bound of $O(k^5)$ for the odd chromatic number of k-planar graphs. See [1, 2, 4] for other results concerning these new graph parameters.

In this note, we bound the conflict-free chromatic number of a graph by its 2-strong colouring number. For a graph G, a total order $\preceq$ of $V(G)$, a vertex $v \in V(G)$, and an integer $s \geqslant 1$, let $R(G, \preceq, v, s)$ be the set of vertices $w \in V(G)$ for which there is a path $v = w_0, w_1, \ldots, w_{s'} = w$ of length $s' \in [0, s]$ such that $w \preceq v$ and $v \prec w_i$ for all $i \in [s-1]$. For a graph G and integer $s \geqslant 1$, the *s-strong colouring number* $\mathrm{scol}_s(G)$ is the minimum integer c such that there is a total order $\preceq$ of $V(G)$ with $|R(G, \preceq, v, s)| \leqslant c$ for every vertex v of G.

Colouring numbers provide upper bounds on several graph parameters of interest. First note that $\mathrm{scol}_1(G)$ equals the degeneracy of G plus 1, implying $\chi(G) \leqslant \mathrm{scol}_1(G)$. A proper graph colouring is *acyclic* if the union of any two colour classes induces a forest; that is, every cycle is assigned at least three colours. The *acyclic chromatic number* $\chi_{\mathrm{a}}(G)$ of a graph G is the minimum integer c such that G has an acyclic c-colouring. Kierstead and Yang [13] proved that $\chi_{\mathrm{a}}(G) \leqslant \mathrm{scol}_2(G)$ for every graph G. Other parameters that can be bounded by strong colouring numbers include weak colouring numbers [24], game chromatic number [12, 13], Ramsey numbers [3], oriented chromatic number [14], arrangeability [3], and boxicity [9].

Our key contribution is the following:

Theorem 1. *For every graph G, $\chi_{pcf}(G) \leqslant 2\,\mathrm{scol}_2(G) - 1$.*

Note that Theorem 1 is best possible in the sense that the conflict-free chromatic number is not bounded by the 1-strong colouring number [2]. Before proving Theorem 1, we highlight several noteworthy consequences.

First, Theorem 1 implies that graph classes with bounded expansion have bounded conflict-free chromatic number and bounded odd chromatic number. Let G be a graph and $r \geqslant 0$ be an integer. A graph H is an *r-shallow minor* of G if H can be obtained from a subgraph of G by contracting disjoint subgraphs each with radius at most r. Let $G \triangledown r$ be the set of all r-shallow-minors of G. For an integer $r \geqslant 0$ and graph G, let $\nabla_r(G) := \max\{|E(H)|/|V(H)| \colon H \in G \triangledown r\}$. A hereditary graph class

$\mathcal{G}$ has *bounded expansion* with *expansion function* $f_{\mathcal{G}} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ if $\nabla_r(G) \leqslant f_{\mathcal{G}}(r)$ for every $r \geqslant 0$ and graph $G \in \mathcal{G}$. Bounded expansion is a robust measure of sparsity with many characterisations [16, 17, 24]. Examples of graph classes with bounded expansion includes classes that have bounded maximum degree [17], bounded stack number [18], bounded queue-number [18], bounded nonrepetitive chromatic number [18], or strongly sublinear separators [8], as well as proper-minor closed classes [17]. See the book by Nešetřil and Ossona de Mendez [16] for further background on bounded expansion and strong colouring numbers.

Zhu [24] showed that a graph class $\mathcal{G}$ has bounded expansion if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{scol}_s(G) \leqslant f(s)$ for every graph $G \in \mathcal{G}$. In particular, his results imply that $\operatorname{scol}_2(G) \leqslant 8(\nabla_1(G))^3 + 1$ for every graph G. Thus, we have the following consequence of Theorem 1.

Corollary 2. *For every graph G, $\chi_{pcf}(G) \leqslant 16(\nabla_1(G))^3 + 1$.*

Thus Theorem 1 implies that each of the aforementioned graph classes have bounded conflict-free chromatic number and bounded odd chromatic number.

Second, Theorem 1 implies a stronger bound for the odd chromatic number and the conflict-free chromatic number of k-planar graphs. Van den Heuvel and Wood [22, 23] showed that $\operatorname{scol}_2(G) \leqslant 30(k+1)$ for every k-planar graph G. Thus we have the following consequence of Theorem 1:

Theorem 3. *For every k-planar graph G, $\chi_{pcf}(G) \leqslant 60k + 59$.*

Theorem 3 is the first known upper bound for the conflict-free chromatic number of k-planar graphs. For the odd chromatic number, the previous best known upper bound for k-planar graphs was $\chi_o(G) \in O(k^5)$ due to Dujmović et al. [6].

Finally, Theorem 1 gives the first known upper bound for the conflict-free chromatic number of K_t-minor free graphs. Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and Siebertz [21] showed that $\operatorname{scol}_2(G) \leqslant 5\binom{t-1}{2}$ for every K_t-minor free graph G. Thus Theorem 1 implies the following:

Theorem 4. *For every K_t-minor free graph G, $\chi_{pcf}(G) \leqslant 5(t-1)(t-2) - 1$.*

See [7, 11, 21, 22] for other graph classes to which Theorem 1 applies.

2 Proof

Proof of Theorem 1. We may assume that G has no isolated vertices. Let $\preceq$ be the ordering $(v_1, \dots, v_n)$ of $V(G)$ where $|R(G, v_i, \preceq, 2)| \leqslant \operatorname{scol}_2(G)$ for every vertex v_i of G. For each vertex $v_i \in V(G)$, let $N^-(v_i) := R(G, v_i, \preceq, 1) \setminus \{v_i\}$ be the *left neighbours* of v_i, and let $v_j \in N(v_i)$ where $j = \min\{\ell \in [n] : v_\ell \in N(v_i)\}$ be the *leftmost neighbour* of v_i. Let $\pi(v_i)$ denote the leftmost neighbour of v_i.

We now specify the conflict-free colouring $\psi : V(G) \to [2\operatorname{scol}_2(G)+1]$ by colouring the vertices left to right. For $i = 1$, let $\psi(v_1) = 1$. Now suppose $i > 1$ and that

$v_1, \dots, v_{i-1}$ are coloured. Let $X_i := \{\psi(v_j) : v_j \in R(G, u, \preceq, 2) \setminus \{v_i\}\}$ and $Y_i := \{\psi(\pi(v_j)) : v_j \in N^-(v_i)\}$. Observe that $|X_i| \leqslant |R(G, u, \preceq, 2) \setminus \{v_i\}| \leqslant \mathrm{scol}_2(G) - 1$ and $|Y_i| \leqslant |R(G, v_i, \preceq, 1) \setminus \{v_i\}| \leqslant \mathrm{scol}_2(G) - 1$ and so $|X_i \cup Y_i| \leqslant 2\,\mathrm{scol}_2(G) - 2$. As such, there exists some colour $\alpha \in [2\,\mathrm{scol}_2(G) - 1] \setminus (X_i \cup Y_i)$. Let $\psi(v_i) := \alpha$.

Now ψ is proper as each vertex receives a different colour to its left neighbours. We now show that it is conflict-free. Let $v_i \in V(G)$ and let $v_j = \pi(v_i)$. We claim that $\psi(v_j) \neq \psi(v_\ell)$ for every $v_\ell \in N(v_i) \setminus \{v_j\}$. Since v_j is the leftmost neighbour of v_i, $j < \ell$. If $\ell < i$, then $v_j \in R(G, \preceq, v_\ell, 2)$ (by the path v_ℓ, v_i, v_j) and so $\psi(v_j) \in X_\ell$. Otherwise $i < \ell$ so $v_i \in N^-(v_\ell)$ and thus $\psi(v_j) \in Y_\ell$. As such, $\psi(v_j) \in X_\ell \cup Y_\ell$ and hence $\psi(v_j) \neq \psi(v_\ell)$, as required. □

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