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# The existence of $(1, m)$-near-Skolem sequences

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**Abstract**

Let $m_1, m_2, n$ be three positive integers, $m_1 < m_2 \leq n$. An $(m_1, m_2)$-near-Skolem sequence of order $n$ with defects $m_1$ and $m_2$ is a sequence $S = (s_1, s_2, \ldots, s_{2n-4})$ of $2n - 4$ positive integers which satisfies the following two conditions: (1) For every $k \in \{1, 2, \ldots, n\} \setminus \{m_1, m_2\}$ there are exactly two elements $s_i, s_j \in S$, such that $s_i = s_j = k$; (2) If $s_i = s_j = k$ then $|j - i| = k$. In this paper, we prove that the necessary conditions are also sufficient for the existence of $(1, m)$-near-Skolem sequences.

## 1 Introduction

The concept of an $(m_1, m_2)$-near-Skolem sequence was first introduced by Shalaby [18]. Let $m_1, m_2, n$ be three positive integers, $m_1 < m_2 \leq n$. An $(m_1, m_2)$-*near-Skolem sequence* of order $n$ and defects $m_1$ and $m_2$ is a sequence $S = (s_1, s_2, \ldots, s_{2n-4})$ of $2n - 4$ positive integers which satisfies the following conditions:

(1) For every $k \in \{1, 2, \ldots, n\} \setminus \{m_1, m_2\}$ there are exactly two elements $s_i, s_j \in S$, such that $s_i = s_j = k$.

(2) If $s_i = s_j = k$ then $|j - i| = k$.

Reid and Shalaby [15] conjectured the necessary conditions for the existence of $(m_1, m_2)$-near-Skolem sequences of order $n$ are sufficient.

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**Conjecture 1.1.** ([15]) *An $(m_1, m_2)$-near-Skolem sequence of order $n$ exists if and only if one of the following is true:* (1) $n \equiv 0, 1 \pmod 4$ *and $m_1$ and $m_2$ are of the same parity;* (2) $n \equiv 2, 3 \pmod 4$ *and $m_1$ and $m_2$ are of opposite parity.*

The research of related sequences can be traced back to Skolem [21] who first proposed the concept of Skolem sequences in 1957. Since then various generalizations of Skolem sequences have been studied by many people—for example: hooked Skolem sequences [14]; Rosa sequences and hooked Rosa sequences [16]; Langford sequences and hooked Langford sequences [20]; extended Skolem sequences [1, 2] and hooked extended Skolem sequences [10]; near-Skolem sequences and hooked near-Skolem sequences [18]; near-Rosa sequences and hooked near-Rosa sequences [19]; $(p, q)$-extended Rosa sequences [11]; and $k$-extended $m$-near-Skolem sequences [3, 4]. There are numerous open problems on the existence of related generalizations of Skolem sequences [13, 17, 8].

An $m$-near-Skolem sequence of order $n$ is a sequence $S = (s_1, s_2, \ldots, s_{2n-2})$ of $2n - 2$ positive integers which satisfies the following conditions:

(1) For every $k \in \{1, 2, \ldots, n\} \setminus \{m\}$ there are exactly two elements $s_i, s_j \in S$, such that $s_i = s_j = k$.

(2) If $s_i = s_j = k$ then $|j - i| = k$.

Near-Skolem sequences can be used to construct various types of designs. For example, Billington applied near-Skolem sequences to obtain several types of designs [6, 7], Fu and Wu [9] used near-Skolem sequences to obtain a cyclic decomposition of complete graphs, and Meszka and Rosa [12] used near Skolem sequences to construct cubic graphs with at most 22 vertices which are leaves of partial triple systems.

**Theorem 1.2.** ([18]) *An $m$-near-Skolem sequence of order $n$ exists if and only if $n \equiv 0, 1 \pmod 4$ and $m$ is odd, or $n \equiv 2, 3 \pmod 4$ and $m$ is even.*

A few results have been obtained on Conjecture 1.1. Recently, we proved Conjecture 1.1 under the condition that $m_2 - m_1 \leq 2$, or $m_2 - m_1 > 2$ and $n \geq 3m_2 + 1$.

**Theorem 1.3.** ([22]) (1) *For $n \equiv 0, 1 \pmod 4$, there is an $(m_1, m_2)$-near-Skolem sequence of order $n$, where $n \geq 3m_2 + 1$ and $m_2 - m_1 \equiv 0 \pmod 2$.*
(2) *For $n \equiv 2, 3 \pmod 4$, there is an $(m_1, m_2)$-near-Skolem sequence of order $n$, where $n \geq 3m_2 + 1$ and $m_2 - m_1 \equiv 1 \pmod 2$.*

**Theorem 1.4.** ([22]) (1) *An $(m, m + 2)$-near-Skolem sequence of order $n$ exists if and only if $n \equiv 0, 1 \pmod 4$, $n \geq 4$, and $(m, n) \neq (1, 4), (1, 5)$.*
(2) *An $(m, m + 1)$-near-Skolem sequence of order $n$ exists if and only if $n \equiv 2, 3 \pmod 4$, $n \geq 3$, and $(m, n) \neq (1, 3), (1, 6)$.*

In this paper, we mainly prove that a $(1, m)$-near-Skolem sequence of order $n$ exists if and only if $n \equiv 0, 1 \pmod 4$, $n \geq 5$, $m \equiv 1 \pmod 2$, and $(m, n) \neq (3, 5)$, or $n \equiv 2, 3 \pmod 4$, $n \geq 6$, $m \equiv 0 \pmod 2$, and $(m, n) \neq (2, 6)$.

## 2 Main result

We first introduce the concept of extended near Skolem sequences. A *$k$-extended $m$-near-Skolem sequence* of order $n$ is a sequence $S = (s_1, s_2, \ldots, s_{2n-1})$ of $2n-1$ positive integers which satisfies the following three conditions:

(1) For every $l \in \{1, 2, \ldots, n\} \setminus \{m\}$ there are exactly two elements $s_i, s_j \in S$, such that $s_i = s_j = l$.

(2) If $s_i = s_j = l$ then $|j-i| = k$.

(3) $s_k = 0$.

**Theorem 2.1.** ([3, 4, 5]) *There exists an $(n-4)$-extended 1-near-Skolem sequence of order $n-2$ if and only if $n \equiv 0, 3 \pmod 4$, $n \geq 7$.*

**Theorem 2.2.** *A $(1, n-1)$-near-Skolem sequence of order $n$ exists if $n \equiv 0, 3 \pmod 4$, $n \geq 7$.*

*Proof:* By Theorem 2.1 since there exists an $(n-4)$-extended 1-near-Skolem sequence of order $n-2$, so we put $n$ in positions $n-4$ and $2n-4$ of this sequence. Then we get a $(1, n-1)$-near-Skolem sequence of order $n$. ∎

Before proving our main conclusions, we briefly describe the main methods used. Let $M = \{2, 3, \ldots, n\} \setminus \{m\}$. Now $M$ can be partitioned into three subsets $M_1 = \{D : D \in M, D - m \equiv 0 \pmod 4\}$, $M_2 = \{D : D \in M, D - m \equiv 2 \pmod 4\}$, and $M_3 = \{D : D \in M, D - m \equiv 1 \pmod 2\}$.

(1) For all the differences in $M_1$ we put the biggest one at the end of the sequence, i.e. in position $P = 2n-4$. Then move inward two positions each time and put the largest difference of the remaining differences. This continues until $m-4$ is the maximum of the remaining differences in $M_1$. Now we move inward two positions and four positions, respectively, and put 2 in both. Then continue to move inward two positions each time again and put the largest difference of the remaining differences. This discontinuous subsequence is denoted by $S_1$ in the next example.

(2) For all the differences in $M_2$ we put the biggest one at the end of the sequence, i.e. in position $P = 2n-5$. Then move inward two positions each time and put the largest difference of the remaining differences. This continuous subsequence is denoted by $S_2$ in the next example.

(3) For all the differences in $M_3$ we distinguish three subcases and use a similar method to obtain two continuous subsequences $S_3$ and $S_4$, and a discontinuous subsequence $S_5$.

**Example 1.** *A $(1, 33)$-near-Skolem sequence of order 64 and its five subsequences.*

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| D | 58 | 56 | 54 | 52 | 50 | 48 | 46 | 44 | 42 | 40 | 38 | 36 | 34 | 32 | 4 | 28 | 26 | 24 | 4 | 20 | 18 | 16 | 22 | 12 | 10 |
| P | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| D | 8 | 14 | 62 | 64 | 3 | 6 | 60 | 3 | 8 | 10 | 12 | 6 | 16 | 18 | 20 | 14 | 24 | 26 | 28 | 22 | 32 | 34 | 36 | 38 | 40 |
| P | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 |
| D | 42 | 44 | 46 | 48 | 50 | 52 | 54 | 56 | 58 | 63 | 30 | 59 | 61 | 55 | 57 | 51 | 53 | 47 | 49 | 43 | 45 | 39 | 41 | 35 | 37 |
| P | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| D | 31 | 29 | 27 | 25 | 23 | 21 | 19 | 17 | 15 | 13 | 11 | 9 | 7 | 5 | 62 | 30 | 60 | 64 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| P | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | |
| D | 19 | 21 | 23 | 25 | 27 | 29 | 31 | 2 | 35 | 2 | 39 | 37 | 43 | 41 | 47 | 45 | 51 | 49 | 55 | 53 | 59 | 57 | 63 | 61 | |

| $S_1$ | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 |
| D | | | | | | | | | | | | | 61 | | 57 | | 53 | | 49 | | 45 | | 41 | | 37 |
| P | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| D | | 29 | | 25 | | 21 | | 17 | | 13 | | 9 | | 5 | | | | | 5 | | 9 | | 13 | | 17 |
| P | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | |
| D | | 21 | | 25 | | 29 | | 2 | | 2 | | 37 | | 41 | | 45 | | 49 | | 53 | | 57 | | 61 | |

| $S_2$ | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 |
| D | | | | | | | | | | 63 | | 59 | | 55 | | 51 | | 47 | | 43 | | 39 | | 35 | |
| P | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| D | 31 | | 27 | | 23 | | 19 | | 15 | | 11 | | 7 | | | | | | | 7 | | 11 | | 15 | |
| P | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | |
| D | 19 | | 23 | | 27 | | 31 | | 35 | | 39 | | 43 | | 47 | | 51 | | 55 | | 59 | | 63 | | |

| $S_3$ | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| D | | 56 | | 52 | | 48 | | 44 | | 40 | | 36 | | 32 | | 28 | | 24 | | 20 | | 16 | | 12 | |
| P | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| D | 8 | | | | | | | | 8 | | 12 | | 16 | | 20 | | 24 | | 28 | | 32 | | 36 | | 40 |
| P | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 |
| D | | 44 | | 48 | | 52 | | 56 | | | | | | | | | | | | | | | | | |

| $S_4$ | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| D | 58 | | | | 50 | | | | 42 | | | | 34 | | | | 26 | | | | 18 | | | | 10 |
| P | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| D | | | | | | | | | | 10 | | | | 18 | | | | 26 | | | | 34 | | | |
| P | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 |
| D | 42 | | | | 50 | | | | 58 | | | | | | | | | | | | | | | | |

| $S_5$ | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *P* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* | *12* | *13* | *14* | *15* | *16* | *17* | *18* | *19* | *20* | *21* | *22* | *23* | *24* | *25* |
| *D* | | | *54* | | | | *46* | | | | *38* | | | | | | | | | | | | *22* | | |
| *P* | *26* | *27* | *28* | *29* | *30* | *31* | *32* | *33* | *34* | *35* | *36* | *37* | *38* | *39* | *40* | *41* | *42* | *43* | *44* | *45* | *46* | *47* | *48* | *49* | *50* |
| *D* | | *14* | | | | *6* | | | | | | *6* | | | | *14* | | | | *22* | | | | *38* | |
| *P* | *51* | *52* | *53* | *54* | *55* | *56* | *57* | *58* | *59* | *60* | *61* | *62* | *63* | *64* | *65* | *66* | *67* | *68* | *69* | *70* | *71* | *72* | *73* | *74* | *75* |
| *D* | | | *46* | | | | *54* | | | | | | | | | | | | | | | | | | |

**Theorem 2.3.** *A* $(1, m)$*-near-Skolem sequence of order* $n$ *exists if and only if* $n \equiv 0, 1 \pmod 4$, $n \geq 5$, $m \equiv 1 \pmod 2$, *and* $(m, n) \neq (3, 5)$, *or* $n \equiv 2, 3 \pmod 4$, $n \geq 6$, $m \equiv 0 \pmod 2$, *and* $(m, n) \neq (2, 6)$.

*Proof:* Necessity is obvious and the exceptions $n < 5$ and $(m, n) \neq (3, 5), (2, 6)$ are easily verified. For sufficiency, we distinguish four cases. In each case, the solution is given in the form of a table, where $a_i$, $b_i$ denote the first and second appearance, respectively, of the number $i$. Define $c(n, k) = n - k\lfloor \frac{n}{k} \rfloor$.

**Case 1**: $n \equiv 0 \pmod 4$, $m \equiv 1 \pmod 2$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n + c(m, 4) - 4 - 4r$ | $n - c(m, 4) + 2r$ | $2n - 4 - 2r$ | $0 \leq r \leq \lfloor \frac{n-m-5}{4} \rfloor$ |
| $m - 4 - 4r$ | $\frac{3}{2}n - 2\lceil \frac{m-1}{4} \rceil - 3 + 2r$ | $\frac{3}{2}n + 2\lfloor \frac{m-1}{4} \rfloor - 6 - 2r$ | $0 \leq r \leq \lfloor \frac{m-11}{4} \rfloor$ |
| $2$ | $\frac{3}{2}n + 2\lfloor \frac{m-1}{4} \rfloor - 4$ | $\frac{3}{2}n + 2\lfloor \frac{m-1}{4} \rfloor - 2$ | |
| $n - c(m, 4) - 4r$ | $n + c(m, 4) - 5 + 2r$ | $2n - 5 - 2r$ | $0 \leq r \leq \frac{n-12}{4}$ |

**Case 1.1**: $n \equiv 0 \pmod 8$.

For $n \geq 16$ and $m = 3$, a $(1, 3)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \geq 16$ and $5 \leq m \leq n - 1$, we split into two cases: $m \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$.

**Case 1.1.1**: $m \equiv 1 \pmod 4$, $5 \le m \le n-3$, $n \ge 16$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-8-4r$ | $2+2r$ | $n-6-2r$ | $0 \le r \le \frac{n-16}{4}$ |
| $\frac{n}{2}-10-8r$ | $\frac{n}{4}+7+4r$ | $\frac{3n}{4}-3-4r$ | $0 \le r \le \lfloor\frac{n-32}{16}\rfloor$ |
| $\frac{n}{2}-2$ | $n-3$ | $\frac{3n}{2}-5$ | |
| $7$ | $\frac{3n}{2}-8$ | $\frac{3n}{2}-1$ | |
| $5$ | $\frac{3n}{2}-7$ | $\frac{3n}{2}-2$ | |
| $4$ | $\frac{n}{4}-1$ | $\frac{n}{4}+3$ | |
| $n-6-\frac{c(n,16)}{2}-8r$ | $1+\frac{c(n,16)}{4}+4r$ | $n-5-\frac{c(n,16)}{4}-4r$ | $0 \le r \le \frac{n-16}{8}$ |
| $n-10+\frac{c(n,16)}{2}-8r$ | $3-\frac{c(n,16)}{4}+4r$ | $n-7+\frac{c(n,16)}{4}-4r$ | $0 \le r \le \lfloor\frac{n-24}{16}\rfloor$ |
| $n$ | $\frac{n}{2}-3-\frac{c(n,16)}{8}$ | $\frac{3n}{2}-3-\frac{c(n,16)}{8}$ | |
| $n-2$ | $\frac{n}{2}-4+\frac{3c(n,16)}{8}$ | $\frac{3n}{2}-6+\frac{3c(n,16)}{8}$ | |
| $n-4$ | $\frac{n}{2}-\frac{c(n,16)}{4}$ | $\frac{3n}{2}-4-\frac{c(n,16)}{4}$ | |
| $3$ | $\frac{n}{2}-2+\frac{c(n,16)}{4}$ | $\frac{n}{2}+1+\frac{c(n,16)}{4}$ | |

**Case 1.1.2**: $m \equiv 3 \pmod 4$, $7 \le m \le n-1$, $n \ge 16$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-6-4r$ | $2+2r$ | $n-4-2r$ | $0 \le r \le \frac{n-12}{4}$ |
| $n-12-8r$ | $3+4r$ | $n-9-4r$ | $0 \le r \le \frac{n-24}{8}$ |
| $n-8-8r$ | $1+4r$ | $n-7-4r$ | $0 \le r \le \lfloor\frac{n-32}{16}\rfloor$ |
| $3$ | $\frac{3n}{2}-6$ | $\frac{3n}{2}-3$ | |
| $\frac{n}{2}-8+\frac{c(n,16)}{2}-8r$ | $\frac{n}{4}-3-\frac{c(n,16)}{4}+4r$ | $\frac{3n}{4}-11+\frac{c(n,16)}{4}-4r$ | $0 \le r \le \lfloor\frac{n-24}{16}\rfloor$ |
| $n$ | $\frac{n}{2}-1-\frac{3c(n,16)}{4}$ | $\frac{3n}{2}-1-\frac{3c(n,16)}{4}$ | |
| $n-2$ | $\frac{n}{2}-5+\frac{3c(n,16)}{8}$ | $\frac{3n}{2}-7+\frac{3c(n,16)}{8}$ | |
| $n-4$ | $\frac{n}{2}-\frac{c(n,16)}{8}$ | $\frac{3n}{2}-4-\frac{c(n,16)}{8}$ | |
| $\frac{n}{2}+\frac{c(n,16)}{2}$ | $n-5$ | $\frac{3n}{2}-5+\frac{c(n,16)}{2}$ | |
| $5$ | $\frac{n}{2}-7+\frac{c(n,16)}{4}$ | $\frac{n}{2}-2+\frac{c(n,16)}{4}$ | |
| $4$ | $\frac{3n}{4}-7+\frac{c(n,16)}{4}$ | $\frac{3n}{4}-3+\frac{c(n,16)}{4}$ | |

For $n = 8$ and $m = 3$, the conclusion comes from Theorem 1.4.
For $n = 8$ and $m = 5$: 8 2 7 2 3 6 4 3 8 7 4 6
For $n = 8$ and $m = 7$, the conclusion comes from Theorem 2.2.

**Case 1.2**: $n \equiv 4 \pmod 8$.

For $n \geq 12$ and $m = 3$, a $(1,3)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \geq 20$ and $5 \leq m \leq n-1$, we split into two cases: $m \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$.

**Case 1.2.1**: $m \equiv 1 \pmod 4$, $5 \leq m \leq n-3$, $n \geq 20$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $7$ | $\frac{3(n-4)}{2}-2$ | $\frac{3(n-4)}{2}+5$ | |
| $n-6-\frac{c(n-4,16)}{4}-4r$ | $1+\frac{c(n-4,16)}{8}+2r$ | $n-5-\frac{c(n-4,16)}{8}-2r$ | $0 \leq r \leq \frac{2n-24-c(n-4,16)}{8}$ |
| $n-8-\frac{c(n-4,16)}{4}-8r$ | $2+\frac{c(n-4,16)}{8}+4r$ | $n-6-\frac{c(n-4,16)}{8}-4r$ | $0 \leq r \leq \frac{n-20}{8}$ |
| $n-12+\frac{3c(n-4,16)}{4}-8r$ | $4-\frac{3c(n-4,16)}{8}+4r$ | $n-8+\frac{3c(n-4,16)}{8}-4r$ | $0 \leq r \leq \lfloor\frac{n-28}{16}\rfloor$ |
| $\frac{n}{2}-10+\frac{c(n-4,16)}{4}-8r$ | $\frac{n-4}{4}-\frac{c(n-4,16)}{8}+4r$ | $\frac{3n-44}{4}+\frac{c(n-4,16)}{8}-4r$ | $0 \leq r \leq \lfloor\frac{n-28}{16}\rfloor$ |
| $\frac{n}{2}-2+\frac{c(n-4,16)}{4}$ | $n-3$ | $\frac{3n}{2}-5+\frac{c(n-4,16)}{4}$ | |
| $n-4$ | $\frac{n}{2}-3+\frac{c(n-4,16)}{4}$ | $\frac{3n}{2}-7+\frac{c(n-4,16)}{4}$ | |
| $n-2$ | $\frac{n}{2}-\frac{c(n-4,16)}{2}$ | $\frac{3n}{2}-2-\frac{c(n-4,16)}{2}$ | |
| $n$ | $\frac{n}{2}-4+\frac{c(n-4,16)}{4}$ | $\frac{3n}{2}-4+\frac{c(n-4,16)}{4}$ | |
| $5$ | $\frac{n}{2}-6+\frac{c(n-4,16)}{8}$ | $\frac{n}{2}-1+\frac{c(n-4,16)}{8}$ | |
| $4$ | $\frac{3n}{4}-7+\frac{c(n-4,16)}{8}$ | $\frac{3n}{4}-3+\frac{c(n-4,16)}{8}$ | |
| $3$ | $\frac{3n}{2}-6-\frac{c(n-4,16)}{8}$ | $\frac{3n}{2}-3-\frac{c(n-4,16)}{8}$ | |

**Case 1.2.2**: $m \equiv 3 \pmod 4$, $7 \leq m \leq n-1$, $n \geq 20$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-10-4r$ | $3+2r$ | $n-7-2r$ | $0 \leq r \leq \frac{n-20}{4}$ |
| $\frac{n-4}{2}-8-8r$ | $\frac{n-4}{4}+8+4r$ | $\frac{3(n-4)}{4}-4r$ | $0 \leq r \leq \lfloor\frac{n-36}{16}\rfloor$ |
| $n$ | $\frac{n-4}{2}-1$ | $\frac{3(n-4)}{2}+3$ | |
| $n-2$ | $\frac{n-4}{2}+3$ | $\frac{3(n-4)}{2}+5$ | |
| $n-4$ | $\frac{n-4}{2}+1$ | $\frac{3(n-4)}{2}+1$ | |
| $n-6$ | $1$ | $n-5$ | |
| $\frac{n-4}{2}$ | $n-4$ | $\frac{3(n-4)}{2}$ | |
| $3$ | $\frac{3(n-4)}{2}-1$ | $\frac{3(n-4)}{2}+2$ | |
| $4$ | $\frac{n-4}{4}$ | $\frac{n-4}{4}+4$ | |
| $5$ | $\frac{n-4}{2}-3$ | $\frac{n-4}{2}+2$ | |
| $n-6-\frac{c(n,16)}{2}-8r$ | $1+\frac{c(n,16)}{4}+4r$ | $n-5-\frac{c(n,16)}{4}-4r$ | $0 \leq r \leq \frac{n-20}{8}$ |
| $n-14+\frac{c(n,16)}{2}-8r$ | $5-\frac{c(n,16)}{4}+4r$ | $n-9+\frac{c(n,16)}{4}-4r$ | $0 \leq r \leq \lfloor\frac{n-28}{16}\rfloor$ |
| $6$ | $\frac{n}{2}-5+\frac{c(n,16)}{4}$ | $\frac{n}{2}+1+\frac{c(n,16)}{4}$ | |

For $n = 12$ and $5 \leq m \leq 9$:

$m = 5$ : 12 8 11 4 10 3 9 4 3 8 6 7 12 11 10 9 6 2 7 2

$m = 7$ : 12 10 11 6 3 5 9 3 8 6 5 10 12 11 4 9 8 2 4 2

$m = 9$ : 12 4 11 5 10 4 3 8 5 3 6 7 12 11 10 8 6 2 7 2

For $n = 12$ and $m = 11$, the conclusion comes from Theorem 2.2.

**Case 2**: $n \equiv 1 \pmod 4$, $m \equiv 1 \pmod 2$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n - c(m,4) + 1 - 4r$ | $n + c(m,4) - 5 + 2r$ | $2n - 4 - 2r$ | $0 \le r \le \lfloor \frac{n-m-4}{4} \rfloor$ |
| $m - 4 - 4r$ | $\frac{3}{2}n - 2\lfloor \frac{m-1}{4} \rfloor - \frac{9}{2} + 2r$ | $\frac{3}{2}n + 2\lceil \frac{m-1}{4} \rceil - \frac{15}{2} - 2r$ | $0 \le r \le \lfloor \frac{m-11}{4} \rfloor$ |
| $2$ | $\frac{3}{2}n + 2\lceil \frac{m-1}{4} \rceil - \frac{11}{2}$ | $\frac{3}{2}n + 2\lceil \frac{m-1}{4} \rceil - \frac{7}{2}$ | |
| $n + c(m,4) - 3 - 4r$ | $n - c(m,4) - 2 + 2r$ | $2n - 5 - 2r$ | $0 \le r \le \frac{n-9}{4}$ |

**Case 2.1**: $n \equiv 1 \pmod 8$.

For $n \ge 17$ and $m = 3$, a $(1,3)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \ge 25$ and $5 \le m \le n$, we split into two cases: $m \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$.

**Case 2.1.1**: $m \equiv 1 \pmod 4$, $5 \le m \le n$, $n \ge 25$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n - 7 - 4r$ | $2 + 2r$ | $n - 5 - 2r$ | $0 \le r \le \frac{n-13}{4}$ |
| $\frac{n-1}{2} - 4 - 8r$ | $\frac{n-1}{4} + 3 + 4r$ | $\frac{3(n-1)}{4} - 1 - 4r$ | $0 \le r \le \lfloor \frac{n-33}{16} \rfloor$ |
| $\frac{n-1}{2} + 4$ | $n - 6$ | $\frac{3(n-1)}{2} - 1$ | |
| $4$ | $\frac{n-1}{4} - 5$ | $\frac{n-1}{4} - 1$ | |
| $5$ | $\frac{3(n-1)}{2} - 7$ | $\frac{3(n-1)}{2} - 2$ | |
| $n - 9 - \frac{c(n-1,16)}{2} - 8r$ | $1 + \frac{c(n-1,16)}{4} + 4r$ | $n - 8 - \frac{c(n-1,16)}{4} - 4r$ | $0 \le r \le \frac{n-25}{8}$ |
| $n - 13 + \frac{c(n-1,16)}{2} - 8r$ | $3 - \frac{c(n-1,16)}{4} + 4r$ | $n - 10 + \frac{c(n-1,16)}{4} - 4r$ | $0 \le r \le \lfloor \frac{n-41}{16} \rfloor$ |
| $n - 1$ | $\frac{n-1}{2} - 3 - \frac{c(n-1,16)}{4}$ | $\frac{3(n-1)}{2} - 3 - \frac{c(n-1,16)}{4}$ | |
| $n - 3$ | $\frac{n-1}{2} - 2 + \frac{c(n-1,16)}{8}$ | $\frac{3(n-1)}{2} - 4 + \frac{c(n-1,16)}{8}$ | |
| $n - 5$ | $\frac{n-1}{2} - 1 + \frac{c(n-1,16)}{8}$ | $\frac{3(n-1)}{2} - 5 + \frac{c(n-1,16)}{8}$ | |
| $8$ | $\frac{n-1}{2} - 7 + \frac{c(n-1,16)}{2}$ | $\frac{(n-1)}{2} + 1 + \frac{c(n-1,16)}{2}$ | |
| $3$ | $\frac{n-1}{2} - \frac{c(n-1,16)}{4}$ | $\frac{n-1}{2} + 3 - \frac{c(n-1,16)}{4}$ | |

**Case 2.1.2**: $m \equiv 3 \pmod 4$, $7 \leq m \leq n-2$, $n \geq 25$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-7-\frac{c(n-1,16)}{4}-4r$ | $1+\frac{c(n-1,16)}{8}+2r$ | $n-6-\frac{c(n-1,16)}{8}-2r$ | $0 \leq r \leq \frac{2n-26-c(n-1,16)}{8}$ |
| $n-13+\frac{3c(n-1,16)}{4}-8r$ | $4-\frac{3c(n-1,16)}{8}+4r$ | $n-9+\frac{3c(n-1,16)}{8}-4r$ | $0 \leq r \leq \frac{n-25+c(n-1,16)}{8}$ |
| $n-9-\frac{c(n-1,16)}{4}-8r$ | $2+\frac{c(n-1,16)}{8}+4r$ | $n-7-\frac{c(n-1,16)}{8}-4r$ | $0 \leq r \leq \lfloor\frac{n-33}{16}\rfloor$ |
| $\frac{n-17}{2}+\frac{c(n-1,16)}{4}-8r$ | $\frac{n+23}{4}-\frac{c(n-1,16)}{8}+4r$ | $\frac{3n-11}{4}+\frac{c(n-1,16)}{8}-4r$ | $0 \leq r \leq \lfloor\frac{n-25}{16}\rfloor$ |
| $n-1$ | $\frac{n-3}{2}-\frac{c(n-1,16)}{8}$ | $\frac{3n-5}{2}-\frac{c(n-1,16)}{8}$ | |
| $n-3$ | $\frac{n-9}{2}+\frac{c(n-1,16)}{8}$ | $\frac{3n-15}{2}+\frac{c(n-1,16)}{8}$ | |
| $n-5$ | $\frac{n-1}{2}$ | $\frac{3(n-1)}{2}-4$ | |
| $\frac{n-1}{2}+\frac{c(n-1,16)}{4}$ | $n-4$ | $\frac{3n-9}{2}+\frac{c(n-1,16)}{4}$ | |
| $5$ | $\frac{n-7}{2}-\frac{c(n-1,16)}{8}$ | $\frac{n+3}{2}-\frac{c(n-1,16)}{8}$ | |
| $4$ | $\frac{n-9}{4}-\frac{c(n-1,16)}{8}$ | $\frac{n+7}{4}-\frac{c(n-1,16)}{8}$ | |
| $3$ | $\frac{3n-13}{2}-\frac{c(n-1,16)}{8}$ | $\frac{3n-7}{2}-\frac{c(n-1,16)}{8}$ | |

For $n = 9$ and $m = 3$, the conclusion comes from Theorem 1.4.
For $n = 9$ and $5 \leq m \leq 7$:

$m = 5$ : 9 6 8 3 4 7 3 6 4 9 8 2 7 2

$m = 7$ : 9 4 8 5 3 4 6 3 5 9 8 2 6 2

For $n = 9$ and $m = 9$, the conclusion comes from Theorem 1.2.
For $n = 17$ and $5 \leq m \leq 15$:

$m = 5$ : 17 15 16 9 6 14 8 13 7 12 6 11 9 10 8 7 15 17 16 14 13 12 11 10 4 2 3 2 4 3

$m = 7$ : 17 15 16 10 8 14 4 13 6 12 4 11 8 10 6 9 15 17 16 14 13 12 11 5 9 2 3 2 5 3

$m = 9$ : 17 15 16 7 10 14 6 13 5 12 7 11 6 5 10 8 15 17 16 14 13 12 11 8 4 2 3 2 4 3

$m = 11$ : 17 15 16 7 9 14 5 13 6 12 7 5 10 9 6 8 15 17 16 14 13 12 10 8 4 2 3 2 4 3

$m = 13$ : 17 15 16 6 9 14 5 7 12 6 11 5 10 9 7 8 15 17 16 14 12 11 10 8 4 2 3 2 4 3

$m = 15$ : 17 14 16 13 9 7 8 4 12 10 11 4 7 9 8 14 13 17 16 10 12 11 6 3 5 2 3 2 6 5

For $n = 17$ and $m = 17$, the conclusion comes from Theorem 1.2.

**Case 2.2**: $n \equiv 5 \pmod 8$.

For $n \geq 21$ and $m = 3, 5$, a $(1, m)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \geq 21$ and $7 \leq m \leq n$, we split into two cases: $m \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$.

**Case 2.2.1**: $m \equiv 1 \pmod 4$, $9 \le m \le n$, $n \ge 21$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-9+\frac{c(n-5,16)}{4}-4r$ | $1+\frac{c(n-5,16)}{8}+2r$ | $n-8+\frac{3c(n-5,16)}{8}-2r$ | $0 \le r \le \frac{2n-34+c(n-5,16)}{8}$ |
| $n-11+\frac{c(n-5,16)}{4}-8r$ | $4-\frac{3c(n-5,16)}{8}+4r$ | $n-7-\frac{c(n-5,16)}{8}-4r$ | $0 \le r \le \frac{n-21}{8}$ |
| $n-7-\frac{3c(n-5,16)}{4}-8r$ | $2+\frac{c(n-5,16)}{8}+4r$ | $n-5-\frac{5c(n-5,16)}{8}-4r$ | $0 \le r \le \lfloor\frac{n-37}{16}\rfloor$ |
| $\frac{n-9}{2}-\frac{c(n-5,16)}{4}-8r$ | $\frac{n-13}{4}-\frac{c(n-5,16)}{8}+4r$ | $\frac{3n-31}{4}-\frac{3c(n-5,16)}{8}-4r$ | $0 \le r \le \lfloor\frac{n-21}{16}\rfloor$ |
| $n-1$ | $\frac{n-5}{2}-2+\frac{3c(n-5,16)}{8}$ | $\frac{3(n-5)}{2}+2+\frac{3c(n-5,16)}{8}$ | |
| $n-3$ | $\frac{n-5}{2}+1-\frac{c(n-5,16)}{2}$ | $\frac{3(n-5)}{2}+3-\frac{c(n-5,16)}{2}$ | |
| $n-5$ | $\frac{n-5}{2}-1+\frac{3c(n-5,16)}{8}$ | $\frac{3(n-5)}{2}-1+\frac{3c(n-5,16)}{8}$ | |
| $\frac{n-5}{2}+6-\frac{c(n-5,16)}{4}$ | $n-6$ | $\frac{3(n-5)}{2}+5-\frac{c(n-5,16)}{4}$ | |
| $5$ | $\frac{n-5}{2}-3-\frac{c(n-5,16)}{4}$ | $\frac{n-5}{2}+2-\frac{c(n-5,16)}{4}$ | |
| $4$ | $\frac{3(n-5)}{4}-\frac{3c(n-5,16)}{8}$ | $\frac{3(n-5)}{4}+4-\frac{3c(n-5,16)}{8}$ | |
| $3$ | $\frac{3(n-5)}{2}+1$ | $\frac{3(n-5)}{2}+4$ | |

**Case 2.2.2**: $m \equiv 3 \pmod 4$, $7 \le m \le n-2$, $n \ge 21$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-9+\frac{c(n-5,16)}{4}-4r$ | $2-\frac{c(n-5,16)}{8}+2r$ | $n-7+\frac{c(n-5,16)}{8}-2r$ | $0 \le r \le \frac{2n-34+c(n-5,16)}{8}$ |
| $n-7-\frac{c(n-5,16)}{4}-8r$ | $1+\frac{c(n-5,16)}{8}+4r$ | $n-6-\frac{c(n-5,16)}{8}-4r$ | $0 \le r \le \frac{n-13-c(n-5,16)}{8}$ |
| $n-11-\frac{c(n-5,16)}{4}-8r$ | $3+\frac{c(n-5,16)}{8}+4r$ | $n-8-\frac{c(n-5,16)}{8}-4r$ | $0 \le r \le \lfloor\frac{n-37}{16}\rfloor$ |
| $\frac{n-17}{2}+\frac{c(n-5,16)}{4}-8r$ | $\frac{n+23}{4}-\frac{c(n-5,16)}{8}+4r$ | $\frac{3n-11}{4}+\frac{c(n-5,16)}{8}-4r$ | $0 \le r \le \lfloor\frac{n-29}{16}\rfloor$ |
| $n-1$ | $\frac{n-5}{2}+1-\frac{3c(n-5,16)}{8}$ | $\frac{3(n-5)}{2}+5-\frac{3c(n-5,16)}{8}$ | |
| $n-3$ | $\frac{n-5}{2}-2+\frac{c(n-5,16)}{2}$ | $\frac{3(n-5)}{2}+\frac{c(n-5,16)}{2}$ | |
| $n-5$ | $\frac{n-5}{2}+2-\frac{c(n-5,16)}{8}$ | $\frac{3(n-5)}{2}+2-\frac{c(n-5,16)}{8}$ | |
| $\frac{n-5}{2}+2+\frac{c(n-5,16)}{4}$ | $n-4$ | $\frac{3(n-5)}{2}+3+\frac{c(n-5,16)}{4}$ | |
| $5$ | $\frac{n-5}{2}-\frac{c(n-5,16)}{8}$ | $\frac{n-5}{2}+5-\frac{c(n-5,16)}{8}$ | |
| $4$ | $\frac{n-5}{4}-1-\frac{c(n-5,16)}{8}$ | $\frac{n-5}{4}+3-\frac{c(n-5,16)}{8}$ | |
| $3$ | $\frac{3n-13}{2}-\frac{c(n-5,16)}{8}$ | $\frac{3(n-5)}{2}+4-\frac{c(n-5,16)}{8}$ | |

For $n = 13$ and $m = 3$, the conclusion comes from Theorem 1.4.
For $n = 13$ and $5 \le m \le 11$:

$m = 5$ : 13 11 12 7 3 4 10 3 9 4 7 8 11 13 12 6 10 9 2 8 2 6

$m = 7$ : 13 11 12 6 4 10 5 9 4 6 8 5 11 13 12 10 9 3 8 2 3 2

$m = 9$ : 13 11 12 6 4 5 10 8 4 6 5 7 11 13 12 8 10 3 7 2 3 2

$m = 11$ : 13 10 12 6 3 5 9 3 8 6 5 10 7 13 12 9 8 4 2 7 2 4

For $n = 13$ and $m = 13$, the conclusion comes from Theorem 1.2.
For $n = 5$ and $m = 5$, the conclusion comes from Theorem 1.2.

**Case 3**: $n \equiv 2 \pmod 4$, $m \equiv 0 \pmod 2$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n+c(m,4)-2-4r$ | $n-c(m,4)-2+2r$ | $2n-4-2r$ | $0 \le r \le \lfloor \frac{n-m-4}{4} \rfloor$ |
| $m-4-4r$ | $\frac{3}{2}(n-2)-2\lceil \frac{m}{4} \rceil+2r$ | $\frac{3}{2}(n-2)+2\lfloor \frac{m}{4} \rfloor-4-2r$ | $0 \le r \le \lfloor \frac{m-12}{4} \rfloor$ |
| $2$ | $\frac{3}{2}(n-2)+2\lfloor \frac{m}{4} \rfloor-2$ | $\frac{3}{2}(n-2)+2\lfloor \frac{m}{4} \rfloor$ | |
| $n-c(m,4)-4r$ | $n+c(m,4)-5+2r$ | $2n-5-2r$ | $0 \le r \le \frac{n-10}{4}$ |

**Case 3.1**: $n \equiv 2 \pmod 8$.

For $n \ge 18$, $m = 2, 4$, a $(1, m)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \ge 18$ and $6 \le m \le n$, we split into two cases: $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$.

**Case 3.1.1**: $m \equiv 0 \pmod 4$, $8 \le m \le n-2$, $n \ge 18$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-7-4r$ | $1+2r$ | $n-6-2r$ | $0 \le r \le \frac{n-14}{4}$ |
| $\frac{n-2}{2}-7-8r$ | $\frac{n-2}{4}-2+4r$ | $\frac{3(n-2)}{4}-9-4r$ | $0 \le r \le \lfloor \frac{n-34}{16} \rfloor$ |
| $\frac{n-2}{2}+1$ | $\frac{n-2}{2}-3$ | $n-4$ | |
| $4$ | $\frac{3(n-2)}{4}-5$ | $\frac{3(n-2)}{4}-1$ | |
| $n-13+\frac{c(n-2,16)}{2}-8r$ | $4-\frac{c(n-2,16)}{4}+4r$ | $n-9+\frac{c(n-2,16)}{4}-4r$ | $0 \le r \le \frac{n-26}{8}+\frac{c(n-2,16)}{8}$ |
| $n-9-\frac{c(n-2,16)}{2}-8r$ | $2+\frac{c(n-2,16)}{4}+4r$ | $n-7-\frac{c(n-2,16)}{4}-4r$ | $0 \le r \le \lfloor \frac{n-34}{16} \rfloor$ |
| $n-1$ | $\frac{n-2}{2}-2-\frac{c(n-2,16)}{4}$ | $\frac{3(n-2)}{2}-1-\frac{c(n-2,16)}{4}$ | |
| $n-3$ | $\frac{n-2}{2}-4+\frac{c(n-2,16)}{2}$ | $\frac{3(n-2)}{2}-5+\frac{c(n-2,16)}{2}$ | |
| $n-5$ | $\frac{n-2}{2}+1-\frac{c(n-2,16)}{4}$ | $\frac{3(n-2)}{2}-2-\frac{c(n-2,16)}{4}$ | |
| $6$ | $\frac{n-2}{2}-6-\frac{c(n-2,16)}{4}$ | $\frac{n-2}{2}-\frac{c(n-2,16)}{4}$ | |
| $5$ | $\frac{3(n-2)}{2}-4-\frac{c(n-2,16)}{8}$ | $\frac{3(n-2)}{2}+1-\frac{c(n-2,16)}{8}$ | |
| $3$ | $\frac{3(n-2)}{2}-3+\frac{c(n-2,16)}{8}$ | $\frac{3(n-2)}{2}+\frac{c(n-2,16)}{8}$ | |

**Case 3.1.2**: $m \equiv 2 \pmod 4$, $6 \le m \le n$, $n \ge 18$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-9-4r$ | $2+2r$ | $n-7-2r$ | $0 \le r \le \frac{n-14}{4}$ |
| $n-7-8r$ | $1+4r$ | $n-6-4r$ | $0 \le r \le \frac{n-18}{8}$ |
| $n-11-8r$ | $3+4r$ | $n-8-4r$ | $0 \le r \le \lfloor\frac{n-26}{16}\rfloor$ |
| $n-5$ | $\frac{n-2}{2}$ | $\frac{3(n-2)}{2}-3$ | |
| $6$ | $\frac{3(n-2)}{2}-6$ | $\frac{3(n-2)}{2}$ | |
| $\frac{n-2}{2}-9-\frac{c(n-2,16)}{2}-8r$ | $\frac{n-2}{4}+7+\frac{c(n-2,16)}{4}+4r$ | $\frac{3(n-2)}{4}-2-\frac{c(n-2,16)}{4}-4r$ | $0 \le r \le \lfloor\frac{n-34}{16}\rfloor$ |
| $n-1$ | $\frac{n-2}{2}-3+\frac{c(n-2,16)}{8}$ | $\frac{3(n-2)}{2}-2+\frac{c(n-2,16)}{8}$ | |
| $n-3$ | $\frac{n-2}{2}+2-\frac{5c(n-2,16)}{8}$ | $\frac{3(n-2)}{2}+1-\frac{5c(n-2,16)}{8}$ | |
| $\frac{n-2}{2}-1-\frac{c(n-2,16)}{2}$ | $\frac{n-2}{2}-2+\frac{c(n-2,16)}{2}$ | $n-5$ | |
| $4$ | $\frac{n-2}{4}-1+\frac{c(n-2,16)}{4}$ | $\frac{n-2}{4}+3+\frac{c(n-2,16)}{4}$ | |
| $3$ | $\frac{3(n-2)}{2}-4+\frac{c(n-2,16)}{4}$ | $\frac{3(n-2)}{2}-1+\frac{c(n-2,16)}{4}$ | |

For $n = 10$ and $m = 2$, the conclusion comes from Theorem 1.4.

For $n = 10$ and $4 \le m \le 8$:

$m = 4$ : 10 5 9 6 8 3 5 7 3 6 10 9 8 2 7 2

$m = 6$ : 10 4 9 5 8 4 3 7 5 3 10 9 8 2 7 2

$m = 8$ : 10 6 9 3 4 7 3 6 4 5 10 9 7 2 5 2

For $n = 10$ and $m = 10$, the conclusion comes from Theorem 1.2.

**Case 3.2**: $n \equiv 6 \pmod 8$.

For $n \ge 22$ and $m = 2, 4, 6$, a $(1, m)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \ge 22$ and $8 \le m \le n$, we split into two cases: $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$.

**Case 3.2.1**: $m \equiv 0 \pmod 4$, $8 \le m \le n-2$, $n \ge 22$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-9-4r$ | $2+2r$ | $n-7-2r$ | $0 \le r \le \frac{n-18}{4}$ |
| $\frac{n-6}{2}-5-8r$ | $\frac{n-6}{4}-1+4r$ | $\frac{3(n-6)}{4}-6-4r$ | $0 \le r \le \lfloor \frac{n-30}{16} \rfloor$ |
| $\frac{n-6}{2}+3$ | $\frac{n-6}{2}-1$ | $n-4$ | |
| $4$ | $\frac{3(n-6)}{4}-2$ | $\frac{3(n-6)}{4}+2$ | |
| $n-7-\frac{c(n-6,16)}{2}-8r$ | $1+\frac{c(n-6,16)}{4}+4r$ | $n-6-\frac{c(n-6,16)}{4}-4r$ | $0 \le r \le \frac{n-14}{8}-\frac{c(n-6,16)}{8}$ |
| $n-11+\frac{c(n-6,16)}{2}-8r$ | $3-\frac{c(n-6,16)}{4}+4r$ | $n-8+\frac{c(n-6,16)}{4}-4r$ | $0 \le r \le \lfloor \frac{n-30}{16} \rfloor$ |
| $n-1$ | $\frac{n-6}{2}-\frac{c(n-6,16)}{4}$ | $\frac{3(n-6)}{2}+5-\frac{c(n-6,16)}{4}$ | |
| $n-3$ | $\frac{n-6}{2}-2+\frac{c(n-6,16)}{2}$ | $\frac{3(n-6)}{2}+1+\frac{c(n-6,16)}{2}$ | |
| $n-5$ | $\frac{n-6}{2}+3-\frac{c(n-6,16)}{4}$ | $\frac{3(n-6)}{2}+4-\frac{c(n-6,16)}{4}$ | |
| $6$ | $\frac{n-6}{2}-5+\frac{c(n-6,16)}{4}$ | $\frac{n-6}{2}+1+\frac{c(n-6,16)}{4}$ | |
| $5$ | $\frac{3(n-6)}{2}+2-\frac{c(n-6,16)}{8}$ | $\frac{3(n-6)}{2}+7-\frac{c(n-6,16)}{8}$ | |
| $3$ | $\frac{3(n-6)}{2}+3+\frac{c(n-6,16)}{8}$ | $\frac{3(n-6)}{2}+6+\frac{c(n-6,16)}{8}$ | |

**Case 3.2.2**: $m \equiv 2 \pmod 4$, $10 \le m \le n$, $n \ge 22$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-9-4r$ | $2+2r$ | $n-7-2r$ | $0 \le r \le \frac{n-18}{4}$ |
| $\frac{n-6}{2}+3$ | $\frac{n-6}{2}-2$ | $n-5$ | |
| $n-7-\frac{c(n-6,16)}{2}-8r$ | $1+\frac{c(n-6,16)}{4}+4r$ | $n-6-\frac{c(n-6,16)}{4}-4r$ | $0 \le r \le \frac{n-14}{8}-\frac{c(n-6,16)}{8}$ |
| $n-11+\frac{c(n-6,16)}{2}-8r$ | $3-\frac{c(n-6,16)}{4}+4r$ | $n-8+\frac{c(n-6,16)}{4}-4r$ | $0 \le r \le \lfloor \frac{n-30}{16} \rfloor$ |
| $\frac{n-6}{2}-5-8r$ | $\frac{n-6}{4}+7+4r$ | $\frac{3(n-6)}{4}+2-4r$ | $0 \le r \le \lfloor \frac{n-30}{16} \rfloor$ |
| $n-1$ | $\frac{n-6}{2}+2$ | $\frac{3(n-6)}{2}+7$ | |
| $n-3$ | $\frac{n-6}{2}$ | $\frac{3(n-6)}{2}+3$ | |
| $n-5$ | $\frac{n-6}{2}+3$ | $\frac{3(n-6)}{2}+4$ | |
| $6$ | $\frac{3(n-6)}{2}$ | $\frac{3(n-6)}{2}+6$ | |
| $5$ | $\frac{n-6}{2}+1-\frac{c(n-6,16)}{4}$ | $\frac{n-6}{2}+6-\frac{c(n-6,16)}{4}$ | |
| $4$ | $\frac{n-6}{4}-1$ | $\frac{n-6}{4}+3$ | |
| $3$ | $\frac{3(n-6)}{2}+2$ | $\frac{3(n-6)}{2}+5$ | |

For $n = 14$ and $m = 2$, the conclusion comes from Theorem 1.4.
For $n = 14$ and $4 \le m \le 12$:

$m = 4$ : 14 12 13 7 5 6 11 9 10 5 7 6 8 12 14 13 9 11 10 3 8 2 3 2

$m = 6$ : 14 12 13 7 4 11 5 10 4 9 7 5 8 12 14 13 11 10 9 3 8 2 3 2

$m = 8$ : 14 12 13 5 6 11 3 10 5 3 6 9 7 12 14 13 11 10 4 7 9 2 4 2

$m = 10$ : 14 12 13 6 4 5 11 9 4 6 5 7 8 12 14 13 9 11 7 3 8 2 3 2

$m = 12$ : 14 11 13 4 5 6 10 4 9 5 8 6 11 7 14 13 10 9 8 3 7 2 3 2

For $n = 14$ and $m = 14$, the conclusion comes from Theorem 1.2.
For $n = 6$ and $m = 4$: 6 3 5 2 3 2 6 5
For $n = 6$ and $m = 6$, the conclusion comes from Theorem 1.2.

**Case 4**: $n \equiv 3 \pmod 4$, $m \equiv 0 \pmod 2$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n + c(m, 4) - 3 - 4r$ | $n - c(m, 4) - 1 + 2r$ | $2n - 4 - 2r$ | $0 \le r \le \lfloor \frac{n-m-5}{4} \rfloor$ |
| $m - 4 - 4r$ | $\frac{3}{2}(n-3) - 2\lceil \frac{m}{4} \rceil + 2 + 2r$ | $\frac{3}{2}(n-3) + 2\lfloor \frac{m}{4} \rfloor - 2 - 2r$ | $0 \le r \le \lfloor \frac{m-12}{4} \rfloor$ |
| $2$ | $\frac{3}{2}(n-3) + 2\lfloor \frac{m}{4} \rfloor$ | $\frac{3}{2}(n-3) + 2\lfloor \frac{m}{4} \rfloor + 2$ | |
| $n - c(m, 4) - 1 - 4r$ | $n + c(m, 4) - 4 + 2r$ | $2n - 5 - 2r$ | $0 \le r \le \frac{n-11}{4}$ |

**Case 4.1**: $n \equiv 3 \pmod 8$.

For $n \ge 19$ and $m = 2, 4, 6$, a $(1, m)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \ge 19$ and $8 \le m \le n-1$, we split into two cases: $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$.

**Case 4.1.1**: $m \equiv 0 \pmod 4$, $8 \le m \le n - 3$, $n \ge 19$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n - 10 - 4r$ | $3 + 2r$ | $n - 7 - 2r$ | $0 \le r \le \frac{n-19}{4}$ |
| $\frac{n-3}{2} - 9 - 8r$ | $\frac{n-3}{4} + 8 + 4r$ | $\frac{3(n-3)}{4} - 1 - 4r$ | $0 \le r \le \lfloor \frac{n-35}{16} \rfloor$ |
| $n$ | $\frac{n-3}{2} - 3$ | $\frac{3(n-3)}{2}$ | |
| $n - 2$ | $\frac{n-3}{2} + 1$ | $\frac{3(n-3)}{2} + 2$ | |
| $n - 6$ | $\frac{n-3}{2} + 2$ | $\frac{3(n-3)}{2} - 1$ | |
| $\frac{n-3}{2} - 1$ | $\frac{n-3}{2} - 1$ | $n - 5$ | |
| $n - 4$ | $1$ | $n - 3$ | |
| $6$ | $\frac{3(n-3)}{2} - 3$ | $\frac{3(n-3)}{2} + 3$ | |
| $4$ | $\frac{n-3}{4}$ | $\frac{n-3}{4} + 4$ | |
| $3$ | $\frac{3(n-3)}{2} - 2$ | $\frac{3(n-3)}{2} + 1$ | |
| $n - 8 - \frac{c(n-3,16)}{2} - 8r$ | $2 + \frac{c(n-3,16)}{4} + 4r$ | $n - 6 - \frac{c(n-3,16)}{4} - 4r$ | $0 \le r \le \frac{n-19}{8}$ |
| $n - 12 + \frac{c(n-3,16)}{2} - 8r$ | $4 - \frac{c(n-3,16)}{4} + 4r$ | $n - 8 + \frac{c(n-3,16)}{4} - 4r$ | $0 \le r \le \lfloor \frac{n-27}{16} \rfloor$ |
| $5$ | $\frac{n-3}{2} - 2 + \frac{c(n-3,16)}{4}$ | $\frac{n-3}{2} + 3 + \frac{c(n-3,16)}{4}$ | |

**Case 4.1.2**: $m \equiv 2 \pmod 4$, $10 \le m \le n-1$, $n \ge 19$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-8-4r$ | $2+2r$ | $n-6-2r$ | $0 \le r \le \frac{n-15}{4}$ |
| $\frac{n-3}{2}-7-8r$ | $\frac{n-3}{4}-1+4r$ | $\frac{3(n-3)}{4}-8-4r$ | $0 \le r \le \lfloor \frac{n-35}{16} \rfloor$ |
| $\frac{n-3}{2}+1$ | $\frac{n-3}{2}-2$ | $n-4$ | |
| $5$ | $\frac{3(n-3)}{2}-4$ | $\frac{3(n-3)}{2}+1$ | |
| $4$ | $\frac{3(n-3)}{4}-4$ | $\frac{3(n-3)}{4}$ | |
| $3$ | $\frac{3(n-3)}{2}-1$ | $\frac{3(n-3)}{2}+2$ | |
| $n-6-\frac{c(n-3,16)}{2}-8r$ | $1+\frac{c(n-3,16)}{4}+4r$ | $n-5-\frac{c(n-3,16)}{4}-4r$ | $0 \le r \le \frac{n-19}{8}$ |
| $n-10+\frac{c(n-3,16)}{2}-8r$ | $3-\frac{c(n-3,16)}{4}+4r$ | $n-7+\frac{c(n-3,16)}{4}-4r$ | $0 \le r \le \lfloor \frac{n-27}{16} \rfloor$ |
| $n$ | $\frac{n-3}{2}-3+\frac{3c(n-3,16)}{8}$ | $\frac{3(n-3)}{2}+\frac{3c(n-3,16)}{8}$ | |
| $n-2$ | $\frac{n-3}{2}+2-\frac{5c(n-3,16)}{8}$ | $\frac{3(n-3)}{2}+3-\frac{5c(n-3,16)}{8}$ | |
| $n-4$ | $\frac{n-3}{2}-1+\frac{c(n-3,16)}{4}$ | $\frac{3(n-3)}{2}-2+\frac{c(n-3,16)}{4}$ | |
| $6$ | $\frac{n-3}{2}-5-\frac{c(n-3,16)}{4}$ | $\frac{n-3}{2}+1-\frac{c(n-3,16)}{4}$ | |

For $n = 11$ and $m = 2$, the conclusion comes from Theorem 1.4.

For $n = 11$ and $4 \le m \le 8$:

$m = 4$ : 11 5 10 6 9 3 5 8 3 6 7 11 10 9 2 8 2 7

$m = 6$ : 11 4 10 5 9 4 8 3 5 7 3 11 10 9 8 2 7 2

$m = 8$ : 11 4 10 5 9 4 3 7 5 3 6 11 10 9 7 2 6 2

For $n = 11$ and $m = 10$, the conclusion comes from Theorem 2.2.

**Case 4.2**: $n \equiv 7 \pmod 8$.

For $n \ge 15$ and $m = 2, 4$, a $(1, m)$-near-Skolem sequence of order $n$ exists by Theorem 1.3. For $n \ge 23$ and $6 \le m \le n-1$, we split into two cases: $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$.

**Case 4.2.1**: $m \equiv 0 \pmod 4$, $8 \le m \le n-3$, $n \ge 23$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-8-4r$ | $2+2r$ | $n-6-2r$ | $0 \le r \le \frac{n-15}{4}$ |
| $\frac{n-7}{2}-3-8r$ | $\frac{n-7}{4}-1+4r$ | $\frac{3(n-7)}{4}-4-4r$ | $0 \le r \le \lfloor\frac{n-31}{16}\rfloor$ |
| $\frac{n-7}{2}+5$ | $\frac{n-7}{2}-1$ | $n-3$ | |
| $5$ | $\frac{3(n-7)}{2}+4$ | $\frac{3(n-7)}{2}+9$ | |
| $4$ | $\frac{3(n-7)}{4}$ | $\frac{3(n-7)}{4}+4$ | |
| $3$ | $\frac{3(n-7)}{2}+3$ | $\frac{3(n-7)}{2}+6$ | |
| $n-6-\frac{c(n-7,16)}{2}-8r$ | $1+\frac{c(n-7,16)}{4}+4r$ | $n-5-\frac{c(n-7,16)}{4}-4r$ | $0 \le r \le \frac{n-15-c(n-7,16)}{8}$ |
| $n-10+\frac{c(n-7,16)}{2}-8r$ | $3-\frac{c(n-7,16)}{4}+4r$ | $n-7+\frac{c(n-7,16)}{4}-4r$ | $0 \le r \le \lfloor\frac{n-31}{16}\rfloor$ |
| $n$ | $\frac{n-7}{2}+\frac{c(n-7,16)}{8}$ | $\frac{3(n-7)}{2}+7+\frac{c(n-7,16)}{8}$ | |
| $n-2$ | $\frac{n-7}{2}+3-\frac{3c(n-7,16)}{8}$ | $\frac{3(n-7)}{2}+8-\frac{3c(n-7,16)}{8}$ | |
| $n-4$ | $\frac{n-7}{2}+2+\frac{c(n-7,16)}{4}$ | $\frac{3(n-7)}{2}+5+\frac{c(n-7,16)}{4}$ | |
| $6$ | $\frac{n-7}{2}-5+\frac{c(n-7,16)}{4}$ | $\frac{n-7}{2}+1+\frac{c(n-7,16)}{4}$ | |

**Case 4.2.2**: $m \equiv 2 \pmod 4$, $6 \le m \le n-1$, $n \ge 23$.

| $i$ | $a_i$ | $b_i$ | |
|---|---|---|---|
| $n-6-4r$ | $1+2r$ | $n-5-2r$ | $0 \le r \le \frac{n-15}{4}$ |
| $\frac{n-7}{2}-5-8r$ | $\frac{n-7}{4}+4r$ | $\frac{3(n-7)}{4}-5-4r$ | $0 \le r \le \lfloor\frac{n-31}{16}\rfloor$ |
| $n$ | $\frac{n-7}{2}-1$ | $\frac{3(n-7)}{2}+6$ | |
| $n-2$ | $\frac{n-7}{2}+4$ | $\frac{3(n-7)}{2}+9$ | |
| $n-4$ | $\frac{n-7}{2}+2$ | $\frac{3(n-7)}{2}+5$ | |
| $\frac{n-7}{2}+3$ | $\frac{n-7}{2}$ | $n-4$ | |
| $6$ | $\frac{3(n-7)}{2}+2$ | $\frac{3(n-7)}{2}+8$ | |
| $4$ | $\frac{3(n-7)}{4}-1$ | $\frac{3(n-7)}{4}+3$ | |
| $3$ | $\frac{3(n-7)}{2}+4$ | $\frac{3(n-7)}{2}+7$ | |
| $n-8-\frac{c(n-7,16)}{2}-8r$ | $2+\frac{c(n-7,16)}{4}+4r$ | $n-6-\frac{c(n-7,16)}{4}-4r$ | $0 \le r \le \frac{n-15-c(n-7,16)}{8}$ |
| $n-12+\frac{c(n-7,16)}{2}-8r$ | $4-\frac{c(n-7,16)}{4}+4r$ | $n-8+\frac{c(n-7,16)}{4}-4r$ | $0 \le r \le \lfloor\frac{n-31}{16}\rfloor$ |
| $5$ | $\frac{n-7}{2}-4+\frac{c(n-7,16)}{4}$ | $\frac{n-7}{2}+1+\frac{c(n-7,16)}{4}$ | |

For $n = 15$ and $6 \le m \le 12$:

$m = 6$ : 15 13 14 7 4 12 5 11 4 10 7 5 8 9 13 15 14 12 11 10 8 3 9 2 3 2

$m = 8$ : 15 13 14 6 4 12 5 11 4 6 10 5 7 9 13 15 14 12 11 7 10 3 9 2 3 2

$m = 10$ : 15 13 14 8 6 12 7 5 11 9 6 8 5 7 13 15 14 12 9 11 4 2 3 2 4 3

$m = 12$ : 15 13 14 8 5 7 11 6 10 5 9 8 7 6 13 15 14 11 10 9 4 2 3 2 4 3

For $n = 15$ and $m = 14$, the conclusion comes from Theorem 2.2.

For $n = 7$ and $m = 2$, the conclusion comes from Theorem 1.4.

For $n = 7$ and $m = 4$: 7 5 2 6 2 3 5 7 3 6

For $n = 7$ and $m = 6$, the conclusion comes from Theorem 2.2. □

## 3 Conclusions

In this paper, we have shown that the necessary conditions for the existence of a $(1, m)$-near-Skolem sequence are sufficient. It is easy to show that the existence of a $(1, m)$-near-Skolem sequence of order $n$ implies the existence of an $m$-near-Skolem sequence of order $n$. Thus we also give new constructions for $m$-near-Skolem sequences. For example, if we take a $(1, 5)$-near-Skolem sequence of order 8, $S = (8, 3, 4, 7, 3, 6, 4, 2, 8, 2, 7, 6)$, and append $(1, 1)$ to $S$, then we get a 5-near-Skolem sequence of order 8: $(1, 1, 8, 3, 4, 7, 3, 6, 4, 2, 8, 2, 7, 6)$.

It should be mentioned that there is still a long way to go before Conjecture 1.1 is solved. We hope that the results of this paper can promote the research of the Conjecture.

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