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$(k + 1)$-line graphs of k-trees

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Abstract

Let G be a k-tree of order larger than $k+1$ and let $\ell_{k+1}(G)$ be its $(k+1)$-line graph. We introduce a new concept called the k-clique graph of G, and denote it by $G/[k]$. We show that $G/[k]$ is a connected block graph and $\ell_{k+1}(G)$ is isomorphic to the block graph of $G/[k]$. This provides an alternative proof for a recent result by Oliveira et al. that $\ell_{k+1}(G)$ is a connected block graph. A relation between the Wiener index of $G/[k]$ and the Wiener index of its block graph $\ell_{k+1}(G)$ is obtained as a natural generalization of the relation between the Wiener index of a tree T and the Wiener index of its line graph $L(T)$. We further show that there is a 1–1 correspondence between the set of the blocks of $\ell_{k+1}(G)$ and the set of minimal separators of G. Another new concept called the separator-k-clique graph of G, denoted by $G/[k]_S$, arises naturally with the property that $G/[k]_S$ is isomorphic to the block graph of $\ell_{k+1}(G)$. By the Szeged-Wiener Theorem, the Wiener index and the Szeged index are equal for each of the connected block graphs $G/[k]$, $\ell_{k+1}(G)$ and $G/[k]_S$.

1 Introduction

Let k be a positive integer. The concept of k-trees was first introduced by Harary and Palmer [13] as k-dimensional simplicial complexes. Beineke and Pippert [4] provided an inductive definition for k**-trees**. A k-clique is a k-tree, and a k-tree of order n can be extended to a k-tree of order $n+1$ by adding a new vertex which is adjacent to all vertices of a k-clique. Patil [20] observed that the above inductive definition of a k-tree is equivalent to a perfect elimination ordering of a k-tree. We would like to mention that standard trees are 1-trees in the k-tree notation.

A **block** of a graph is a maximal connected subgraph with more than one vertex and without cut vertices. A **block graph** is a graph whose blocks are cliques. Block graphs are a generalization of trees whose blocks are K_2's. The **block graph of a graph** G, denoted by $B(G)$, is a graph whose vertices are blocks of G and two blocks are adjacent in $B(G)$ if and only if they have a vertex in common. The block graph

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of a tree T is just its line graph $L(T)$. It was shown in [11] that a graph is a block graph if and only if it is the block graph $B(G)$ of some graph G. Block graphs are known as chordal and distance-hereditary graphs in which a shortest path between any two vertices is unique (see [12]).

The concept of k-line graphs was first introduced by Lê [16] as a generalization of line graphs. The **$(k+1)$-line graph** of a k-tree G of order larger than $k+1$, denoted by $\ell_{k+1}(G)$, is the graph whose vertices are $(k+1)$-cliques of G and two $(k+1)$-cliques are adjacent in $\ell_{k+1}(G)$ if and only if they have k vertices in common. Oliveira et al. [19] showed that $\ell_{k+1}(G)$ is a connected block graph. In [17], special types of k-trees called the **simple-clique k-trees** (briefly, SC k-tree) were characterized as k-trees whose $(k+1)$-line graphs are trees. Some well-known planar graphs such as maximal outerplanar graphs and chordal maximal planar graphs (also called Apollonian networks) are examples of SC k-trees. Sharp bounds on Wiener indices of maximal outerplanar graphs and Apollonian networks and their extremal graphs were given in [2] and [7], respectively.

Assume that G is a k-tree of order n where $n > k+1$. We first introduce a new concept called the k-clique graph of G (denoted by $G/[k]$) to show that $G/[k]$ is a connected block graph and $\ell_{k+1}(G)$ is isomorphic to the block graph of $G/[k]$. This provides an alternative proof for the result in [19] that $\ell_{k+1}(G)$ is a connected block graph. Parallel to the relation $W(T) = W(L(T)) + \binom{n}{2}$ (see [1]) between the Wiener index of a tree T of order n and the Wiener index of its line graph $L(T)$, we prove that $W(G/[k]) = k^2 W(\ell_{k+1}(G)) + \binom{1+(n-k)k}{2}$ as a relation between the Wiener index of $G/[k]$ and the Wiener index of its block graph $\ell_{k+1}(G)$ for a k-tree G of order n. Recursive formulas for the Wiener index of $\ell_{k+1}(G)$ and the Wiener index of $G/[k]$ are obtained based on their inductive constructions. We then show that there is a 1–1 correspondence between the set of the blocks of $\ell_{k+1}(G)$ and the set of minimal separators of G, that is, the set of k-cliques of G each of which is contained in at least two $(k+1)$-cliques of G. A new concept called the separator-k-clique graph of G (denoted by $G/[k]_S$) arises naturally. It turns out that $G/[k]_S$ is isomorphic to the block graph of $\ell_{k+1}(G)$. The Szeged-Wiener theorem [9] states that the Wiener index and the Szeged index of a connected graph are equal if and only if the graph is a connected block graph, which holds for each of $G/[k]$, $\ell_{k+1}(G)$ and $G/[k]_S$. This further develops our work in [6] because the Wiener index of $G/[k]$ is equivalent to the k-Wiener index of a k-tree G introduced there.

2 Preliminaries

Let G be a finite simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of G is the number of its vertices. Assume that H_1 and H_2 are two subgraphs of a graph H. Then the graph with the vertex set $V(H_1) \cap V(H_2)$ and the edge set $E(H_1) \cap E(H_2)$ is called the **intersection** of H_1 and H_2 and denoted by $H_1 \cap H_2$. Let S be a subset of $V(G)$. We use $S \cup v$ (respectively, $S \setminus v$) to represent the set obtained by adding one vertex v to S (respectively, removing one vertex v from S). We write $G[S]$ for the induced subgraph of G on the set S, and $G - S$ (respectively, $G - v$)

for the induced subgraph of G obtained by removing all vertices in S (respectively, removing one vertex v). The graph obtained from the disjoint union of a vertex v and a graph H such that v is adjacent to all vertices of H is called the **join** of v and H, and denoted by $v + H$.

Assume that G is a connected graph. Let $d_G(u, v)$ be the distance between two vertices u and v in G. The diameter of G is the maximum distance between two vertices of G. The **Wiener index** $W(G)$ of G is defined as $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v)$ [21]. The **status** $\sigma_G(u)$ of a vertex u in G is defined as $\sigma_G(u) = \sum_{v\in V(G)} d_G(u, v)$. If H is a subgraph of G satisfying $d_H(u, v) = d_G(u, v)$ for any two vertices u and v of H, then H is called an **isometric subgraph** of G. A **distance-hereditary graph** is a graph in which any connected induced subgraph is an isometric subgraph.

Lemma 2.1 *[10] Let G be a connected graph. Then*

(i) $W(G) \leq W(G - v) + \sigma_G(v)$ for any vertex v of G. The equality holds if and only if $G - v$ is an isometric subgraph of G.

(ii) $W(G) = \sum_{i\geq 1} i \cdot D_i$, where D_i is the number of unordered pairs of vertices of G with distance i in G.

Let $N_G(v)$ be the set of all vertices adjacent to a vertex v in G. A vertex v is called a **simplicial** vertex of G if $N_G(v)$ induces a clique. A **perfect elimination ordering** (briefly, **peo**) of a graph G is a bijection $\phi : \{1, 2, \ldots, n\} \to V(G)$ such that for each $1 \leq i < n$, $\phi(i) = v_i$ is a simplicial vertex of the induced subgraph $G[\{v_n, v_{n-1}, \ldots, v_i\}]$. By [20], a graph G of order n is a k-tree if and only if it has a peo $\phi = (v_1, v_2, \ldots, v_n)$ such that each v_i $(1 \leq i \leq n - k)$ is a simplicial vertex of degree k in $G[\{v_n, v_{n-1}, \ldots, v_i\}]$.

During an inductive construction of a k-tree, the first k-clique chosen is called its **base k-clique**. When a new vertex v is added, the k-clique chosen whose vertices are all adjacent to v, is called the **joint k-clique** of v and denoted by $JC(v)$, a corresponding $(k + 1)$-clique $v + JC(v)$ is generated and denoted as $\langle v\rangle$. The well-known inductive definition [4, 20] of a k-tree can be stated as follows.

Observation 2.2 *Let G be a k-tree of order n where $n > k$ and $\phi = (v_1, v_2, \ldots, v_n)$ be a peo of G. Then G can be constructed inductively with respect to ϕ as follows. Start from the base k-clique $G[\{v_n, v_{n-1}, \ldots, v_{n-k+1}\}]$, proceed by adding vertices $v_{n-k}, v_{n-k-1}, \ldots, v_1$ in order such that each of them is adjacent to all vertices of its corresponding joint k-clique $JC(v_{n-k}), JC(v_{n-k-1}), \ldots, JC(v_1)$. Then a sequence of k-trees $G_{n-k}, G_{n-k-1}, \ldots, G_1$ is generated in order. At the end, $G = G_1$ is obtained.*

It is known [4, 5] that for any k-tree of order n where $n > k$, each k-clique is contained in a $(k+1)$-clique, and the number of r-cliques is $n_r = \binom{k}{r} + (n-k)\binom{k}{r-1}$ for $r \geq 1$. In particular, $n_k = 1 + (n - k)k$, $n_{k+1} = n - k$, and $n_{k+2} = 0$. Hence, any k-tree is K_{k+2}-free, and the number of $(k + 1)$-cliques in a k-tree of order n is $n - k$. By Observation 2.2, an inductive construction can be obtained for the $(k + 1)$-line graph $\ell_{k+1}(G)$ of a k-tree G.

Corollary 2.3 *Let G be a k-tree of order n where $n > k$ and $\phi = (v_1, v_2, \ldots, v_n)$ be a peo of G. Then vertices of $\ell_{k+1}(G)$ can be represented by $\langle v_i \rangle = v_i + JC(v_i)$, where $JC(v_i)$ is the joint k-clique of v_i for $1 \leq i \leq n - k$, and generated in order $\langle v_{n-k} \rangle, \langle v_{n-k-1} \rangle, \ldots, \langle v_1 \rangle$ during an inductive construction of G in Observation 2.2.*

The concept of a k**-walk** was introduced in [5] as a generalization of a walk in a graph. An alternating sequence $\rho_0\tau_1\rho_1\tau_2\rho_2 \ldots \rho_{t-1}\tau_t\rho_t$ of k-cliques and $(k+1)$-cliques is called a k**-walk** if each $(k+1)$-clique τ_i contains two distinct k-cliques ρ_{i-1} and ρ_i for $1 \leq i \leq t$. A graph of order at least $k+1$ is called k**-linked** if any two k-cliques are joined by a k-walk, and every r-clique is contained in a k-clique for $1 \leq r < k$. A k-walk is a k**-path** if all terms of the alternating sequence are distinct. The k**-distance** between two k-cliques of a graph is the minimum number of $(k+1)$-cliques on a k-path between them. The k**-diameter** of a k-linked graph is the maximum k-distance between two k-cliques. A k-walk is a k**-circuit** if $t \geq 3$ and $\rho_t = \rho_0$, and all other terms of the sequence are distinct. A graph is k**-acyclic** if it has no k-circuits. Every k-tree of order at least $k+1$ is k-linked and k-acyclic [5].

In [6], we introduced the k**-status** of a k-clique in a k-tree and the k-Wiener index of a k-tree, and characterized the extremal graphs for the k-Wiener index of a k-tree. Let G be a k-tree of order at least $k+1$. The k**-status** of a k-clique ρ in G, denoted as $\sigma_G^{[k]}(\rho)$, is the summation of k-distances between ρ and all other k-cliques of G. The k**-Wiener index** of G, denoted as $W^{[k]}(G)$, is the summation of k-distances between every two k-cliques in G.

A **minimal separator** of a graph is an induced subgraph on a minimal set of vertices whose removal results in a graph with more components. A minimal separator on one vertex is called the **cut vertex** of the graph. A graph is k**-connected** if it has more than k vertices and the removal of any $k-1$ vertices cannot disconnect the graph. A graph is said to be **triangulated** or **chordal** if every cycle of length larger than 3 contains an edge which is not a part of the cycle but connects two vertices of the cycle. In [20], a k-tree of order at least $k+1$ was characterized as a k-connected and k-acyclic triangulated graph. Moreover, any minimal separator of a k-tree is a k-clique. It follows that a k-clique of a k-tree is a minimal separator if and only if it is contained in at least two $(k+1)$-cliques.

For a peo $\phi = (v_1, v_2, \ldots, v_n)$ of a k-tree G, the **position of a vertex** v_i is $\phi^{-1}(v_i) = i$, and the **monotone adjacency set** of v_i is the set of vertices

$$X(v_i) = \{w \in N_G(v_i) \mid \phi^{-1}(w) > \phi^{-1}(v_i)\}.$$

For $1 \leq i \leq n-k$, $|X(v_i)| = k$ and $X(v_i)$ is the set of all vertices of the joint k-clique $JC(v_i)$, and so $JC(v_i) = G[X(v_i)]$. For $n-k+1 \leq i \leq n$, $|X(v_i)| = n-i$ and $X(v_i) \subseteq \{v_n, v_{n-1}, \ldots, v_{n-k+2}\}$.

Theorem 2.4 *[18] Let G be a k-tree of order n where $n > k$ and $\phi = (v_1, v_2, \ldots, v_n)$ be a peo of G. Then for each $1 \leq i \leq n-k$, there exists a unique j satisfying $i < j \leq n-k+1$, $v_j \in X(v_i)$ and $X(v_i) \subseteq v_j \cup X(v_j)$. Moreover,*

(i) $j = min\{\phi^{-1}(w) \mid w \in X(v_i)\}$,

(ii) $|X(v_j) \setminus X(v_i)| = \begin{cases} 1, & \text{if } j \leq n-k \\ 0, & \text{if } j = n-k+1 \end{cases}$ *and* $X(v_i) \setminus X(v_j) = v_j$.

Hence, if $j \leq n-k$*, there is a unique vertex* $\beta_j \in X(v_j) \setminus X(v_i)$ *such that* $\beta_j \neq v_j$ *and* $X(v_i) = v_j \cup X(v_j) \setminus \beta_j$*; if* $j = n-k+1$*, then* $X(v_i) = \{v_n, v_{n-1}, \ldots, v_{n-k+1}\}$.

3 Main Results

A k-tree of order at most $k+1$ is either a k-clique or a $(k+1)$-clique. All k-trees considered in this section have order larger than $k+1$.

Definition 1 Let G be a k-tree of order larger than $k+1$. The **k-clique graph** of G, denoted by $G/[k]$, is a graph whose vertices are k-cliques of G, and two k-cliques are adjacent in $G/[k]$ if and only if they are contained in a common $(k+1)$-clique of G.

Lemma 3.1 *Let G be a k-tree of order larger than $k+1$. Then (i) $G/[k]$ is a connected block graph, (ii) $\ell_{k+1}(G)$ is isomorphic to $B(G/[k])$.*

Proof. By [5], any two distinct k-cliques of a k-tree G are connected by a k-path, so $G/[k]$ is connected. The set of all k-cliques contained in one $(k+1)$-clique of G induces a complete subgraph of $G/[k]$ of order $k+1$. By [20], G is K_{k+2}-free and a k-clique is a minimal separator of G if and only if it is contained in more than one $(k+1)$-clique of G. By [5], every k-tree of order at least $k+1$ is k-linked and k-acyclic, we observe that a k-clique is a minimal separator of G if and only if it is a cut vertex of $G/[k]$. It follows that all k-cliques which are vertices of a block of $G/[k]$ must be contained in one common $(k+1)$-clique of G. Hence, any block of $G/[k]$ is a complete subgraph of order $k+1$, and $G/[k]$ is a block graph.

We have shown that all vertices of a block of $G/[k]$ are the set of k-cliques contained in a $(k+1)$-clique of G. Then the set of blocks of $G/[k]$ is in a 1–1 correspondence to the set of $(k+1)$-cliques of G, which is the set of vertices of $\ell_{k+1}(G)$. Two vertices of $\ell_{k+1}(G)$ are adjacent if and only if they have a k-clique of G in common if and only if the corresponding two blocks of $G/[k]$ have one vertex in common if and only if the corresponding two blocks of $G/[k]$ are adjacent in $B(G/[k])$. Therefore, $\ell_{k+1}(G)$ is isomorphic to $B(G/[k])$. □

By Lemma 3.1, we provide an alternative proof for the following result in [19].

Corollary 3.2 *[19] Let G be a k-tree of order larger than $k+1$. Then $\ell_{k+1}(G)$ is a connected block graph.*

Proof. A graph is a block graph if and only if it is the block graph of some graph [11]. By Lemma 3.1, the conclusion follows. □

It was shown in [1] that $W(T) = W(L(T)) + \binom{n}{2}$ for any tree T of order n, where the line graph $L(T)$ of a tree T is just the block graph of T. We will generalize this result to a relation between $W(G/[k])$ and $W(\ell_{k+1}(G))$, where $\ell_{k+1}(G)$ is the block graph of $G/[k]$ for a k-tree G of order n. By definition, the distance between two vertices in the k-clique graph $G/[k]$ is the k-distance between the corresponding two k-cliques in G. Therefore, the Wiener index $W(G/[k])$ is the k-Wiener index $W^{[k]}(G)$ introduced in [6] for a k-tree G.

Theorem 3.3 *Let G be a k-tree of order n where $n > k+1$. Then*

$$W(G/[k]) = W^{[k]}(G) = k^2 \cdot W(\ell_{k+1}(G)) + \binom{1+(n-k)k}{2}.$$

Proof. Note that the diameter of $G/[k]$ is the k-diameter of G, which is at most $n-k$, the number of $(k+1)$-cliques of G. Let $1 \leq i \leq n-k-1$. Assume that μ and ν are two vertices of $\ell_{k+1}(G)$ with $d_{\ell_{k+1}(G)}(\mu,\nu) = i$. Then there is a unique path of length i between μ and ν in $\ell_{k+1}(G)$ because a shortest path between any two vertices in a block graph is unique [12], and $\ell_{k+1}(G)$ is a connected block graph by Corollary 3.2. Any vertex of $\ell_{k+1}(G)$ is a $(k+1)$-clique of G and the intersection of any two adjacent vertices in $\ell_{k+1}(G)$ is a k-clique of G. Then the unique shortest path between $\mu = \mu_0$ and $\nu = \mu_i$ in $\ell_{k+1}(G)$ can be written as an alternating sequence $(\mu = \mu_0)\rho_1\mu_1\rho_2 \ldots \mu_{i-1}\rho_i(\mu_i = \nu)$ of $(k+1)$-cliques and k-cliques of G such that for each $1 \leq j \leq i$, ρ_j is a k-clique which is the intersection of two $(k+1)$-cliques: μ_{j-1} and μ_j. The number of k-cliques contained in each $(k+1)$-clique is $k+1$. Let $\rho_\mu \neq \rho_1$ be a k-clique of G contained in $\mu = \mu_0$. Then G has k such ρ_μ's. Let $\rho_\nu \neq \rho_i$ be a k-clique of G contained in $\nu = \mu_i$. Then G has k such ρ_ν's. Recall that $G/[k]$ is a connected block graph by Lemma 3.1. Then the alternating sequence $\rho_\mu(\mu = \mu_0)\rho_1\mu_1\rho_2 \ldots \rho_i(\mu_i = \nu)\rho_\nu$ is the unique shortest path between ρ_μ and ρ_ν in $G/[k]$. So, $d_{G/[k]}(\rho_\mu, \rho_\nu) = i+1$, which is the number of $(k+1)$-cliques on the shortest path between ρ_μ and ρ_ν. It follows that for each $1 \leq i \leq n-k-1$ and any pair of vertices $\{\mu, \nu\}$ with distance i in $\ell_{k+1}(G)$, there are k^2 pairs of vertices $\{\rho_\mu, \rho_\nu\}$ with distance $i+1$ in $G/[k]$, and vice versa.

Let D'_i be the number of pairs of vertices of $\ell_{k+1}(G)$ with distance i in $\ell_{k+1}(G)$. Let D_i be the number of pairs of vertices of $G/[k]$ with distance i in $G/[k]$. We have shown that $D'_i = \frac{1}{k^2} D_{i+1}$ for $1 \leq i \leq n-k-1$. It is clear that the diameter of $\ell_{k+1}(G)$ is at most $n-k-1$ since the diameter of $G/[k]$ is at most $n-k$. By Lemma 2.1,

$$\begin{aligned} W(\ell_{k+1}(G)) &= \sum_{i=1}^{n-k-1} i \cdot D'_i = \frac{1}{k^2} \sum_{i=1}^{n-k-1} i \cdot D_{i+1} = \frac{1}{k^2} \sum_{i=2}^{n-k} (i-1) \cdot D_i \\ &= \frac{1}{k^2} \left[\sum_{i=2}^{n-k} i \cdot D_i - \sum_{i=2}^{n-k} D_i \right] = \frac{1}{k^2} \left[\sum_{i=1}^{n-k} i \cdot D_i - \sum_{i=1}^{n-k} D_i \right]. \end{aligned}$$

By Lemma 2.1, $\sum_{i=1}^{n-k} i \cdot D_i = W(G/[k])$. Note that $\sum_{i=1}^{n-k} D_i = \binom{1+(n-k)k}{2}$, which is the number of 2-element subsets of the set of k-cliques in G, and the number of k-cliques in G is $1+(n-k)k$. Hence, $W(\ell_{k+1}(G)) = \frac{1}{k^2}\left[W(G/[k]) - \binom{1+(n-k)k}{2}\right]$. It follows that

$$W(G/[k]) = W^{[k]}(G) = k^2 \cdot W(\ell_{k+1}(G)) + \binom{1+(n-k)k}{2}.$$

□

By Lemma 3.1, $G/[k]$ is a connected block graph, and the set of blocks of $G/[k]$ is in a 1–1 correspondence to the set of $(k+1)$-cliques of G. Parallel to the inductive construction of $\ell_{k+1}(G)$, an inductive construction of $G/[k]$ can also be obtained by Observation 2.2.

Corollary 3.4 *Let G be a k-tree of order n where $n > k+1$ and $\phi = (v_1, v_2, \dots, v_n)$ be a peo of G. During an inductive construction of G in Observation 2.2, a sequence of k-clique graphs $G_{n-k}/[k], G_{n-k-1}/[k], \dots, G_1/[k]$ can be generated in order. For each $n-k-1 \geq i \geq 1$, when a vertex v_i is added to the k-tree G_{i+1} to get the k-tree G_i, a block B_i whose vertices are k-cliques of G contained in $v_i + JC(v_i)$ is added to $G_{i+1}/[k]$ to get $G_i/[k]$ with the property that B_i has exactly one common vertex $JC(v_i)$ with $G_{i+1}/[k]$.*

By Observation 2.2, for $1 \leq i \leq n-k-1$, each v_i is a simplicial vertex of G_i, and so $G_{i+1} = G_i - v_i$ is an isometric subgraph of G_i. By Lemma 2.1, $W(G_i) = W(G_{i+1}) + \sigma_{G_i}(v_i)$ for $1 \leq i \leq n-k-1$. Note that $W(G_{n-k}) = \binom{k+1}{2}$ since G_{n-k} is a $(k+1)$-clique. Then $W(G) = \binom{k+1}{2} + \sum_{i=1}^{n-k-1} \sigma_{G_i}(v_i)$. Similar formulas for Wiener indices $W(\ell_{k+1}(G))$ and $W(G/[k])$ can be obtained by the inductive constructions of $\ell_{k+1}(G)$ and $G/[k]$, respectively.

Lemma 3.5 *Let G be a k-tree of order n where $n > k+1$ and $\phi = (v_1, v_2, \dots, v_n)$ be a peo of G. Assume that G_i where $n-k \geq i \geq 1$ is the sequence of k-trees generated during the inductive construction of G in Observation 2.2. Then $G_1 = G$ and*

(i) $W(\ell_{k+1}(G)) = \sum_{i=1}^{n-k-1} \sigma_{\ell_{k+1}(G_i)}(\langle v_i \rangle)$, *where $\langle v_i \rangle$ is a vertex of the $(k+1)$-line graph $\ell_{k+1}(G_i)$ of G_i for $1 \leq i \leq n-k-1$;*

(ii) $W(G/[k]) = k\binom{k+1}{2} - n\binom{k}{2} + k\left[\sum_{i=1}^{n-k-1} \sigma_{G_i/[k]}(\rho_i)\right]$, *where ρ_i is a k-clique of the k-tree G_i containing v_i for $1 \leq i \leq n-k-1$.*

Proof. (i) For $1 \leq i \leq n-k$, write $H_i = \ell_{k+1}(G_i)$. By Corollary 3.2, we observe that H_i is a block graph of order $n-i+1-k$ since G_i is a k-tree of order $n-i+1$, and $\langle v_i \rangle = v_i + JC(v_i)$ is a vertex of H_i. Then $H_{i+1} = H_i - \langle v_i \rangle$ is an isometric subgraph of H_i for $1 \leq i \leq n-k-1$. By Lemma 2.1, we have $W(H_i) = W(H_{i+1}) + \sigma_{H_i}(\langle v_i \rangle)$ for $1 \leq i \leq n-k-1$. Note that $H_1 = \ell_{k+1}(G_1)$ where $G_1 = G$. It follows that

$$\begin{aligned} W(\ell_{k+1}(G)) &= W(H_{n-k}) + \sigma_{H_{n-k-1}}(\langle v_{n-k-1} \rangle) + \ldots + \sigma_{H_1}(\langle v_1 \rangle) \\ &= \sum_{i=1}^{n-k-1} \sigma_{H_i}(\langle v_i \rangle). \end{aligned}$$

The last equality is valid because $W(H_{n-k}) = 0$ where $H_{n-k} = \ell_{k+1}(G_{n-k})$ is a one vertex graph.

(ii) Recall that the Wiener index of $G/[k]$ is the k-Wiener index of G, and the status of a vertex in $G/[k]$ is the k-status of the corresponding k-clique in G. By Theorem 4.3 in [6], $W(G/[k]) = k\left[\sum_{i=1}^{n-k} \sigma_{G_i/[k]}(\rho_i)\right] - (n-k)\binom{k}{2}$, where ρ_i is a k-clique of G_i containing v_i for $1 \le i \le n-k$. Note that $G_{n-k}/[k]$ is a $(k+1)$-clique and ρ_{n-k} is a vertex of $G_{n-k}/[k]$. Then the vertex status $\sigma_{G_{n-k}/[k]}(\rho_{n-k}) = k$. It follows that

$$\begin{aligned} W(G/[k]) &= k^2 + k\left[\sum_{i=1}^{n-k-1} \sigma_{G_i/[k]}(\rho_i)\right] - (n-k)\binom{k}{2} \\ &= k\binom{k+1}{2} - n\binom{k}{2} + k\left[\sum_{i=1}^{n-k-1} \sigma_{G_i/[k]}(\rho_i)\right]. \end{aligned}$$

□

The k-**star** of order n, denoted by S_n^k, is a k-tree obtained from a base k-clique by adding $n-k$ vertices, each of them is adjacent to all vertices of the base k-clique. The k**-th power of a path** of order n, denoted by P_n^k, is a k-tree whose vertices can be labelled as $v_1, v_2, \dots, v_n$ such that two vertices v_i and v_j are adjacent if and only if $1 \le |j-i| \le k$. In [6], we showed that the k-Wiener index of a k-tree G of order n where $n > k$ is bounded below by $2\binom{1+(n-k)k}{2} - (n-k)\binom{k+1}{2}$ and above by $k^2\binom{n-k+2}{3} - (n-k)\binom{k}{2}$. The bounds are attained when G is a k-star and a k-th power of a path, respectively. The above results for the k-Wiener index of a k-tree G also hold for the Wiener index of its k-clique graph $G/[k]$ since $W(G/[k]) = W^{[k]}(G)$. It is well-known that the Wiener indices of connected graphs of order $n-k$ are bounded below by $\binom{n-k}{2}$ and above by $\binom{n-k+1}{3}$, whose extremal graphs are a complete graph and a path of order $n-k$, respectively. Therefore, the bounds and extremal graphs for $W(\ell_{k+1}(G))$ follow immediately.

Corollary 3.6 *Let G be a k-tree of order n where $n > k+1$. Then*

(i) $2\binom{1+(n-k)k}{2} - (n-k)\binom{k+1}{2} \le W(G/[k]) \le k^2\binom{n-k+2}{3} - (n-k)\binom{k}{2}$;

(ii) $\binom{n-k}{2} \le W(\ell_{k+1}(G)) \le \binom{n-k+1}{3}$.

Moreover, the lower bounds (respectively, upper bounds) can be attained when G is S_n^k (respectively, G is P_n^k).

Parallel to the **compact code** of a k-tree defined in [18], we provide the following terminology.

Definition 2 Let G be a k-tree of order n where $n > k+1$ and $\phi = (v_1, v_2, \dots, v_n)$ be a peo of G. For $1 \le i \le n-k$, the unique j satisfying the property stated in Theorem 2.4 is called the **compact code index** of i with respect to ϕ and denoted by $c_\phi(i)$.

By Theorem 2.4 and the definition of a compact code index, if $j = c_\phi(i) \le n-k$, then $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_i)$, and so $\langle v_i \rangle$ and $\langle v_j \rangle$ are adjacent in $\ell_{k+1}(G)$.

Theorem 3.7 *Let G be a k-tree of order n where $n > k+1$ and let $\phi = (v_1, v_2, \ldots, v_n)$ be a peo of G.*

(i) Let $i < j \leq n-k$. Then $\langle v_i \rangle$ and $\langle v_j \rangle$ are adjacent in $\ell_{k+1}(G)$ if and only if $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_i)$. Moreover, $JC(v_i) = JC(v_j)$ if and only if $\langle v_i \rangle$ and $\langle v_j \rangle$ are adjacent in $\ell_{k+1}(G)$ and $j \neq c_\phi(i)$.

(ii) Let B be a block of $\ell_{k+1}(G)$ with vertices $\langle v_{i_j} \rangle = v_{i_j} + JC(v_{i_j})$, where $1 \leq j \leq b$ and $1 \leq i_1 < i_2 < \ldots < i_b \leq n-k$. Then $\bigcap_{j=1}^{b} \langle v_{i_j} \rangle = JC(v_{i_1})$. Moreover, either all $JC(v_{i_j})$ where $1 \leq j \leq b$ are the base k-clique of G with respect to ϕ, or $JC(v_{i_j o})$ are the same for $1 \leq j \leq b-1$ and different from $JC(v_{i_b})$.

Proof. (i) Assume that $i < j \leq n-k$. Note that $\langle v_i \rangle = v_i + JC(v_i)$ and $\langle v_j \rangle = v_j + JC(v_j)$. By an inductive construction of G in Observation 2.2, v_i cannot be a vertex of $JC(v_j)$ since $j > i$. So, v_i cannot be a vertex of $\langle v_i \rangle \cap \langle v_j \rangle$. Then $\langle v_i \rangle$ and $\langle v_j \rangle$ are adjacent in $\ell_{k+1}(G)$ if and only if $\langle v_i \rangle \cap \langle v_j \rangle$ is a k-clique of G if and only if $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_i)$.

If $JC(v_i) = JC(v_j)$, then $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_i)$ and $v_j \notin X(v_i)$. It follows that $j \neq c_\phi(i)$ by Theorem 2.4. On other hand, if $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_i)$ and $j \neq c_\phi(i)$, then $v_j \notin X(v_i)$. Otherwise, if $v_j \in X(v_i)$, then j satisfies the property stated in Theorem 2.4: $i < j \leq n-k$, $v_j \in X(v_i)$ and $X(v_i) \subseteq v_j \cup X(v_j)$. So, $j = c_\phi(i)$. This is a contradiction. Therefore, $v_j \notin X(v_i)$. By the assumption that $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_i)$ which is a k-clique of G, we have $\langle v_i \rangle \cap \langle v_j \rangle = JC(v_j)$ since $v_j \notin X(v_i)$. Then $JC(v_i) = JC(v_j)$.

(ii) Note that $b \geq 2$ since any block B has at least two vertices. By Corollary 2.3, $\langle v_{i_b} \rangle, \langle v_{i_{b-1}} \rangle, \ldots, \langle v_{i_1} \rangle$ are added to B in order during an inductive construction of $\ell_{k+1}(G)$ with respect to ϕ. Since $\ell_{k+1}(G)$ is a connected block graph, all vertices of B are pairwise adjacent. Then the intersection of any two vertices of B is a k-clique of G. By (i), $\bigcap_{j=1}^{b} \langle v_{i_j} \rangle = JC(v_{i_1})$ since $\langle v_{i_1} \rangle$ is the last vertex added to the block B. In particular, the intersection of any two vertices of B is $JC(v_{i_1})$.

By (i), for all $1 \leq j \leq b-1$, $\langle v_{i_b} \rangle \cap \langle v_{i_j} \rangle = JC(v_{i_j})$ since $i_j < i_b \leq n-k$. We have shown that the intersection of any two vertices of B is $JC(v_{i_1})$. Then $JC(v_{i_j}) = JC(v_{i_1})$ for all $1 \leq j \leq b-1$. It follows that $X(v_{i_j}) = X(v_{i_1})$ for all $1 \leq j \leq b-1$. By Theorem 2.4, $c_\phi(i_j) = \min\{\phi^{-1}(w) \mid w \in X(v_{i_j})\} = \min\{\phi^{-1}(w) \mid w \in X(v_{i_1})\} = c_\phi(i_1)$ for all $1 \leq j \leq b-1$. By Theorem 2.4, either $c_\phi(i_1) = n-k+1$ or $c_\phi(i_1) \leq n-k$.

If $c_\phi(i_1) = n-k+1$, then $c_\phi(i_j) = c_\phi(i_1) = n-k+1$ for all $1 \leq j \leq b-1$. Moreover, $i_b \notin X(v_{i_1})$ since $i_b \leq n-k$. Then $\langle v_{i_b} \rangle \cap \langle v_{i_1} \rangle = JC(v_{i_1})$ implies that $X(v_{i_b}) = X(v_{i_1})$ and so $c_\phi(i_b) = c_\phi(i_1) = n-k+1$. Therefore, for all $1 \leq j \leq b$, $JC(v_{i_j}) = G[\{v_n, v_{n-1}, \ldots, v_{n-k+1}\}]$, which is the base k-clique of G with respect to ϕ.

If $c_\phi(i_1) \leq n-k$, then $c_\phi(i_j) = c_\phi(i_1) \leq n-k$ for $1 \leq j \leq b-1$. Since $i_j < c_\phi(i_j) \leq n-k$ for $1 \leq j \leq b-1$, we observe that $\langle v_{i_j} \rangle$ and $\langle v_{c_\phi(i_j)} \rangle$ are adjacent with $\langle v_{i_j}\rangle \cap \langle v_{c_\phi(i_j)}\rangle = JC(v_{i_j}) = JC(v_{i_1})$ for $1 \le j \le b-1$. Then the vertex $\langle v_{c_\phi(i_j)}\rangle = \langle v_{c_\phi(i_1)}\rangle$ is also contained in the block B for $1 \le j \le b-1$. Note that $c_\phi(i_j) \notin \{i_{b-1}, \ldots, i_j, \ldots, i_1\}$ for each $1 \le j \le b-1$. Then $\langle v_{c_\phi(i_j)}\rangle \notin \{\langle v_{i_{b-1}}\rangle, \ldots, \langle v_{i_j}\rangle, \ldots, \langle v_{i_1}\rangle\}$ for each $1 \le j \le b-1$. It follows that $\langle v_{c_\phi(i_j)}\rangle$ is the vertex $\langle v_{i_b}\rangle$ of B for $1 \le j \le b-1$. Therefore, $c_\phi(i_j) = i_b$ for all $1 \le j \le b-1$. By (i), $JC(v_{i_j})$ are the same for $1 \le j \le b-1$ and different from $JC(v_{i_b})$. □

Corollary 3.8 *Let G be a k-tree of order larger than $k+1$ and $\ell_{k+1}(G)$ be its $(k+1)$-line graph. Then there is a 1–1 correspondence between the set of the blocks of $\ell_{k+1}(G)$ and the set of minimal separators of G.*

Proof. By Theorem 3.7, the intersection of all vertices in a block of $\ell_{k+1}(G)$ is a k-clique of G. So, each block of $\ell_{k+1}(G)$ corresponds to a k-clique of G which is contained in at least two $(k+1)$-cliques of G. On the other hand, if a k-clique of G is contained in at least two $(k+1)$-cliques of G, then all $(k+1)$-cliques containing the same k-clique are pairwise adjacent in $\ell_{k+1}(G)$ and form a block of $\ell_{k+1}(G)$. Recall that a k-clique of G is a minimal separator of G if and only if it is contained in at least two $(k+1)$-cliques of G. Therefore, there is a 1–1 correspondence between the set of the blocks of $\ell_{k+1}(G)$ and the set of minimal separators of G. □

Definition 3 Let G be a k-tree of order larger than $k+1$. The **separator-k-clique graph** of G, denoted by $G/[k]_S$, is a graph whose vertices are the minimal separators of G, that is, the k-cliques of G each of which is contained in at least two $(k+1)$-cliques of G, and two minimal separators of G are adjacent in $G/[k]_S$ if and only if they are contained in a common $(k+1)$-clique of G.

The cut-point graph was first defined by Harary in [11]. The **cut-point graph** of a graph G, denoted by $C(G)$, is a graph whose vertices are the cut vertices of G and two cut vertices are adjacent if and only if they are contained in a common block. It was shown in [11] that a graph is a block graph if and only if it is the block graph $B(G)$ of some graph G and $B(B(G)) = C(G)$.

Lemma 3.9 *Let G be a k-tree of order larger than $k+1$. Then both $B(\ell_{k+1}(G))$ and $C(G/[k])$ are isomorphic to $G/[k]_S$, and $G/[k]_S$ is an isometric subgraph of $G/[k]$.*

Proof. By Corollary 3.8, there is a 1–1 correspondence between the set of the blocks of $\ell_{k+1}(G)$ and the set of vertices of $G/[k]_S$. Two blocks of $\ell_{k+1}(G)$ are adjacent in $B(\ell_{k+1}(G))$ if and only if two blocks of $\ell_{k+1}(G)$ have a cut vertex $\langle v\rangle$ of $\ell_{k+1}(G)$ in common if and only if the corresponding two vertices of $G/[k]_S$ (considered as k-cliques of G) are contained in $\langle v\rangle$ (considered as $(k+1)$-cliques of G) if and only if the corresponding two vertices of $G/[k]_S$ are adjacent in $G/[k]_S$. Therefore, $B(\ell_{k+1}(G))$ is isomorphic to $G/[k]_S$. By Lemma 3.1, $\ell_{k+1}(G)$ is isomorphic to $B(G/[k])$. Then $B(B(G/[k]))$ is isomorphic to $G/[k]_S$. By [11], $B(B(G/[k])) = C(G/[k])$. It follows that $C(G/[k])$ is isomorphic to $G/[k]_S$. By the definition of a separator-k-clique graph, $G/[k]_S$ is an induced subgraph of $G/[k]$. Moreover, $G/[k]_S$ is isometric in

$G/[k]$ because $G/[k]$ is a block graph and block graphs are distance-hereditary graphs by [12]. □

Assume that G is a connected graph. Let $e = uv$ be an edge of G. A vertex w of G is said to be **closer to u than to v** in G if $d_G(w, u) < d_G(w, v)$. Let $n_e(u)$ be the number of vertices that are closer to u than to v in G, and $n_e(v)$ be the number of vertices that are closer to v than to u in G. The **Szeged index** of G is defined as $Sz(G) = \sum_{uv \in E(G)} n_e(u)n_e(v)$ [8]. The Wiener index and the Szeged index are two closely related graph invariants. It is known [15] that $W(G) \leq Sz(G)$ for any connected graph G. The Szeged-Wiener Theorem [9] states that $W(G) = Sz(G)$ if and only if G is a connected block graph; proofs are available in [3, 9, 14]. In particular, $W(G) = Sz(G)$ if G is a tree [21]. By Lemma 3.1 and Lemma 3.9, $G/[k]$, $\ell_{k+1}(G)$ and $G/[k]_S$ are connected block graphs, since a graph is a block graph if and only if it is the block graph of some graph [11]. We have the following conclusion by the Szeged-Wiener Theorem.

Corollary 3.10 *Let G be a k-tree of order larger than $k + 1$. Then*

(i) $W(G/[k]) = Sz(G/[k])$.

(ii) $W(\ell_{k+1}(G)) = Sz(\ell_{k+1}(G))$.

(iii) $W(G/[k]_S) = Sz(G/[k]_S)$.

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