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Plane triangulations without spanning 2-trees

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Abstract

A 2-tree is a graph that can be formed by starting with a triangle and iterating the operation of making a new vertex adjacent to two adjacent vertices of the existing graph. Leizhen Cai asked in 1995 whether every maximal planar graph contains a spanning 2-tree. We answer this question in the negative by constructing an infinite class of maximal planar graphs that have no spanning 2-tree.

1 Introduction

We consider the problem of whether every maximal planar graph contains a spanning 2-tree, first proposed by Leizhen Cai [6, 7] in 1995.

Definition 1.1. A *k-tree* is a graph that can be formed by starting with K_k and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph.

Note that a k-tree is a chordal graph. A more general recursive construction of k-trees is that K_k and K_{k+1} are k-trees, and any larger k-tree can be formed by identifying two k-trees on K_k or K_{k+1}.

Definition 1.2. A *spanning subgraph* of a graph has the same vertex set. A *Hamiltonian cycle* of a graph G is a spanning cycle of G. A graph with a Hamiltonian cycle is called a *Hamiltonian graph.*

A graph is *planar* if it has a drawing in the plane that has no crossings. The *regions* of a plane drawing are the maximal pieces of the plane surrounded by edges and vertices. The infinite region is the *exterior region.* The *length* of a region is the length of a walk around it. A graph is *maximal planar* if no edge can be added without making it not planar. A graph is *outerplanar* if it has a plane drawing with all vertices on the exterior region.

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Definitions of terms and notation not defined here appear in [2]. In particular, K_n and C_n are respectively the complete graph and cycle of order n, and $G + H$ is the join of graphs G and H.

Every connected graph has a spanning tree. A spanning 3-tree of a graph with order $n \geq 3$ would have size $3n - 6$, so a planar graph has a spanning 3-tree if and only if it is a 3-tree. The problem of determining whether there is a spanning 2-tree is more complicated.

Results on k-trees and related topics are surveyed in [3]. Bern [1] showed that determining whether a graph has a spanning k-tree is NP-complete when $k \geq 2$. Cai and Maffray [8] showed this is true even for planar graphs with $\Delta(G) \leq 6$ when $k = 2$. Cai found several sufficient conditions for a spanning 2-tree and showed that it is NP-complete to determine whether G has a spanning k-tree even given a spanning l-tree of G, $l < k$ [6, 7].

Any complete graph has a spanning k-tree. Bern [1] showed that it is NP-complete to find a minimum spanning 2-tree for weighted complete graphs, and found an exponential algorithm for this problem. Cai [6] showed that there is no good approximation algorithm for weighted complete graphs in general, but there is such an algorithm when they satisfy the triangle inequality. Shangin and Pardalos [17] considered various heuristics for the spanning k-tree problem.

Ding [9] found applications of spanning k-trees to linguistic grammars, the RNA 3D structure prediction problem, and learning Markov or Bayesian networks. Spanning 2-trees have applications in geodesy (geodetic surveying) [13] and logic and probability [11, 14].

2 Hamiltonian Cycles and 2-Trees

Leizhen Cai [6, 7] asked in 1995 whether every plane triangulation contains a spanning 2-tree. Cai did not conjecture an answer, but I will reframe the problem as a conjecture to simplify discussion of it.

Conjecture 2.1. Every maximal planar graph with order $n \geq 3$ contains a spanning 2-tree.

It is easy to show that some special classes of maximal planar graphs have spanning 2-trees.

Lemma 2.2. [6] *Every Hamiltonian maximal planar graph contains a spanning* 2-*tree.*

Proof. Adding the edges inside (or outside) a Hamiltonian cycle produces a spanning 2-tree. □

Denote a 4-connected maximal planar graph as a 4*MP*. For these graphs, the converse is true.

Proposition 2.3. *Every spanning 2-tree of a 4MP has a unique Hamiltonian cycle.*

Proof. A 2-tree has a Hamiltonian cycle if and only if it contains no $K_2+\overline{K}_3$ [16]. Let G be a 4MP with a spanning 2-tree T. If T is not Hamiltonian, it contains $K_2+\overline{K}_3$, so G has a separating triangle and is not 4-connected. Thus T has a Hamiltonian cycle C.

A 2-tree with order $n \geq 3$ is Hamiltonian if and only if it is outerplanar [15]. It is easily shown by induction that the cycle is unique, and T can be drawn so that C is the exterior region. □

This shows a correspondence between Hamiltonian cycles and pairs of spanning 2-trees of 4MPs.

Corollary 2.4. *Every 4MP has twice as many spanning 2-trees as Hamiltonian cycles.*

Whitney [19] showed that every 4MP is Hamiltonian. Tutte proved a stronger statement.

Theorem 2.5. [18] Every *planar 4-connected graph has a Hamiltonian cycle through any two edges of a region.*

Cai observed the following corollary to Theorem 2.5.

Corollary 2.6. [6] *Every 4MP contains a spanning 2-tree.*

There is another easy special case. Cai and Maffray [8] showed that every l-tree contains a spanning k-tree when $l \geq k \geq 1$.

Corollary 2.7. [8] *Every 3-tree contains a spanning 2-tree.*

3 Path-Tree Partitions

Denote a 4MP or K_4 as a 4-*block*. Every maximal planar graph can be formed by identifying triangles of 4-blocks. Any 4-block has a spanning 2-tree. The question is whether a spanning 2-tree for the whole graph can be pieced together from those of the 4-blocks.

A spanning 2-tree contains 0, 1, 2, or 3 edges of any given triangle. If there were some 4MP such that for every spanning 2-tree T, there is some triangle with no edge of T, that would be sufficient to disprove Conjecture 2.1. (We could simply attach another 4-block at every triangle, and a spanning 2-tree could not extend into all of them.) We will show that this is the case.

Definition 3.1. A maximal planar graph has a *linear Hamiltonian cycle* if the regions inside (or outside) the cycle share edges with at most two other such regions (that is, the dual of these regions is a path).

Conjecture 3.2. Every 4MP has a linear Hamiltonian cycle.

The discussion above implies that Conjecture 3.2 is weaker than Conjecture 2.1. We will show that Conjecture 3.2 is false, so Conjecture 2.1 is false. To study Conjecture 3.2, it is convenient to look at the dual graph.

Definition 3.3. The *Hamiltonian dual* of a planar graph with a given Hamiltonian cycle is formed by deleting any edges that cross the Hamiltonian cycle from the dual graph.

Denote the dual of a 4MP as a *4MP dual*. A 4MP dual is a 3-connected cubic planar graph with no nontrivial 3-edge cut. (A *trivial edge cut* has all edges incident with a common vertex.)

Definition 3.4. A cubic graph has a *path-tree partition* if its vertices can be partitioned into two sets so that one induces a path and the other induces a tree. A *path-path partition* and *tree-tree partition* are defined similarly. A *Yutsis graph* is a multigraph in which the vertex set can be partitioned into two parts such that each part induces a tree.

A tree-tree partition is also known as a *Yutsis decomposition*. Yutsis graphs have applications in physics, particularly the quantum theory of angular momenta [20].

A simple degree sum argument shows that the two vertex sets in a tree-tree partition must have equal size (the same number of edges must be added incident to each set to make all vertices have degree 3). Theorem 2.5 implies that every 4MP dual has a tree-tree partition.

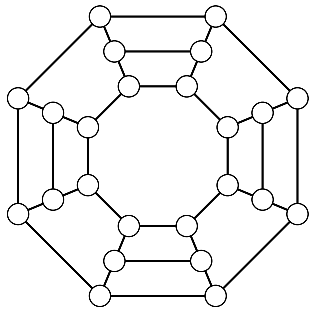
Proposition 3.5. *A 4MP has a linear Hamiltonian cycle if and only if its dual has a path-tree partition.*

Proof. Necessity: Since all vertices are on a Hamiltonian cycle, no vertex is inside it. Thus each component of the Hamiltonian dual is acyclic. Each is clearly connected, so each is a tree. They have the same order since there are the same number of regions inside and outside the Hamiltonian cycle. To have a linear Hamiltonian cycle, one of the trees must be a path.

Sufficiency: If a 4MP dual has a path-tree partition, the 4MP clearly has a linear Hamiltonian cycle. □

We can now use Proposition 3.5 to produce graphs G_k that we will show have no path-tree partition, and thereby disprove Conjecture 3.2.

Let G_k have vertices a_i, b_i, c_i, $1 \le i \le 2k$. The a_i and b_i induce $2k$-cycles, $a_i \leftrightarrow c_i$, $b_i \leftrightarrow c_i$, and $c_{2i-1} \leftrightarrow c_{2i}$ for all i, (all mod $2k$). We show G_4 below.



Theorem 3.6. *For $k \geq 4$, G_k has no path-tree partition.*

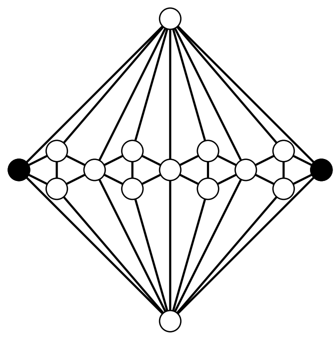
Proof. Denote the graph formed from C_6 by adding a single chord joining opposite vertices as a *brick*. Clearly G_k contains k bricks. Assume to the contrary that G_k has a path-tree partition with path P and tree T. Both P and T must contain at least one vertex from each cycle.

The ends of P are in one or two bricks, so P must pass through at least two bricks without ending. It is not possible for P and T to both pass through the same brick since the two c-vertices would be part of an induced 4-cycle of one of them. If P enters and exits a brick using two as, it must end at a b in the same brick (or vice versa).

To pass through a brick without ending there, P must enter at an a-vertex and exit at a b-vertex (or vice versa). Now P must contain (nonadjacent) a-vertices in distinct bricks with a b-vertex between them. But then the graph induced by the vertices not in P is disconnected and hence not a tree, a contradiction. □

Essentially the same construction appeared in [5], where it is used to analyze the number of triangles of certain types in Hamiltonian maximal planar graphs. Note that G_4 has order 24. In fact, a computer search conducted for [5] has shown that the smallest order of a 4MP dual with no path-tree partition is 24 (personal communication with Gunnar Brinkmann).

The dual of G_4 is a maximal planar graph with order 14. This is shown below, where the two black vertices must be identified. Adding a degree 3 vertex inside each of its 24 regions produces a maximal planar graph of order 38 with no spanning 2-tree, thereby disproving Conjecture 2.1.

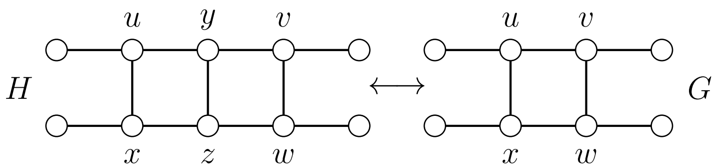


To produce infinite classes of graphs with and without path-tree partitions, we need an operation to generate cubic graphs.

Definition 3.7. Let uv and wx be edges of a cubic graph. Let *adding a handle* be the operation of subdividing uv and wx and adding a new edge yz between the new vertices. Let 4-*handling* be the operation of adding a handle between two nonadjacent edges of a region of length 4 of a 4MP dual.

Every 4MP dual can be constructed from the cube by adding handles [10, 12].

Proposition 3.8. *Let H be formed by 4-handling a 4MP dual G. If G has no path-tree partition then nor does H.*



Proof. (Contrapositive) Let $uyvwzx$ be a 6-cycle of H, and yz be a handle. Let $uvwx$ be a region of length 4 of G which is 4-handled to produce the 6-cycle. Assume H has a path-tree partition with path P and tree T. We want to show that G has a path-tree partition with path P' and tree T'.

First suppose that yz is not in P or T. Then y and z are not both in P or both in T. If uy and yv are both in P or T, let uv be in P' or T', respectively. Similarly, if wz and zx are both in P or T, let wx be in P' or T', respectively. Else do not put uv (or wx) in P' or T'. Thus P' and T' are both connected, acyclic, and have the same order in G, so we have a path-tree partition of G.

Now suppose that yz is in T. At least one of the other vertices, say x, is in T. Then u is a leaf of P. Then put ux in T' and leave v and w in the same corresponding sets. Thus P' and T' are both connected, acyclic and have the same order in G, so we have a path-tree partition of G. If we exchange the roles of P and T, the argument is similar. □

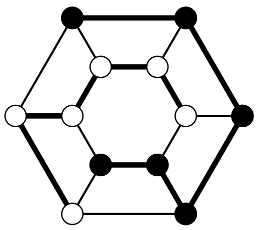
The statement of this proposition does not hold in general for regions of length more than 4. Since all graphs formed by 4-handling G_4 have no path-tree partition, we have an infinite class of counterexamples to Conjecture 2.1.

4 Path-Path Partitions

The smallest 4MPs are the double wheels $C_{n-2} + \overline{K}_2$, whose duals are the prisms $C_r \square K_2$, $r \geq 4$. Note that any prism can be generated from the cube by adding handles.

Theorem 4.1. *Any 4MP dual constructed from a prism by adding at most two handles has a path-path partition.*

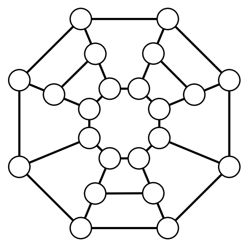
Proving this requires many tedious cases. We outline a proof and leave the details to the reader. A prism $C_r \square K_2$ has two distinct regions up to symmetry, a 4-cycle and an r-cycle. Denote the edges joining the two (chordless) r-cycles of a prism as *spokes*. A prism has a path-path partition using any two spokes and hence omits any two given edges of one of the r-cycles (see the example below).



A handle can always be added to maintain a path-path partition when both of the edges used are in a path-path partition. One way of adding a handle to a 4-cycle produces a larger prism and need not be considered further. The other way produces a path-path partition. The other way of adding a handle uses two edges of an r-cycle, which is always possible (provided they are nonconsecutive).

When two handles are added, they can be added in two separate 4-cycles (adjacent or nonadjacent), both in the same 4-cycle (two ways), both in the same r-cycle (independent or not), in two different r-cycles (multiple cases), or one in a 4-cycle and one in an r-cycle. In each case, a path-path partition is easily found.

Note that G_4 can be generated from a prism by adding four handles. Next we show that there is a 4MP dual of order 22 with no path-path partition. It can be formed by adding three handles to a prism. Call the graph below H_{22}. Let the vertices on the exterior 8-cycle be a-vertices, and the vertices on the interior 8-cycle be b-vertices. Call the two edges joining a- and b-vertices *spokes*.



Theorem 4.2. *The graph H_{22} has no path-path partition.*

Proof. Assume there is a path-path partition with paths P_1 and P_2. Now H_{22} has three bricks and two spokes. It is not possible for P_1 and P_2 to both pass through the same brick, and when one does, it must enter at an a-vertex and exit at a b-vertex (or vice versa).

If a path passes two bricks, two spokes, or one of each, it either induces a cycle or disconnects the other path, so this is not possible. If a path contains a spoke, it could have ends in two bricks, but would miss the third, so this is not possible.

Thus each path must go through one brick and have ends in the other two. Thus each path misses all edges between some consecutive pair of bricks, and this must be a different pair for each path. Thus one vertex of a spoke cannot be contained in either path, a contradiction. □

A computer search by Gunnar Brinkmann shows that 22 is the smallest order of a 4MP dual with no path-path partition (personal communication). I also hand-checked that all 4MP duals of order at most 16 have a path-path partition.

5 Spanning Maximal 2-degenerate Graphs

While Conjecture 2.1 is false, a weaker statement is true.

Definition 5.1. A graph is *k-degenerate* if its vertices can be successively deleted so that immediately prior to deletion, each has degree at most k. A graph is *maximal k-degenerate* if no edges can be added without violating this condition.

Every k-tree is maximal k-degenerate, but the converse is false when $k > 1$. We construct a maximal k-degenerate graph by starting with K_k and successively adding vertices of degree k. Unlike for a k-tree, the neighbors of a new vertex are not required to induce a clique.

Theorem 5.2. *Every maximal planar graph contains a spanning maximal 2-degenerate graph.*

Proof. This is obvious for order $n \leq 3$. Let G be maximal planar, and construct it by starting with some 4-block B_1 and iteratively adding each new 4-block B_r by identifying a triangle T_r of B_r with a triangle T_r^* of the existing graph. Let $G_1 = B_1$ and G_r be the graph after r 4-blocks have been added. We will show that for each r, G_r has a spanning maximal 2-degenerate subgraph M_r.

If B_r is a 4-block containing triangle T_r, then by Theorem 2.5 it has a spanning 2-tree that contains T_r. Any 2-tree can be constructed starting with any of its triangles. When identifying T_r and T_r^*, delete any edges of T_r from the spanning 2-tree of G_r that are not in M_{r-1}. Thus the construction of M_{r-1} can continue from T_r into G_r, producing M_r. Iterating this process proves the theorem. □

6 Conclusion

There is more work to do on this problem. In a subsequent paper with Gunnar Brinkmann [4], we construct a maximal planar graph on 29 vertices with no spanning 2-tree. We also show that for each $c > 0$ there is a maximal planar graph G with some order n so that each 2-tree that is a subgraph of G contains fewer than cn vertices. We would like to characterize exactly which maximal planar graphs have a spanning 2-tree, but that problem seems difficult.

Acknowledgments

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