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A characterization of group vertex magic trees of diameter up to 5

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Abstract

Let $\mathcal{A}$ be an abelian group. An $\mathcal{A}$-vertex magic labeling of a graph G is a mapping from the vertex set of G to the set of all non-identity elements of $\mathcal{A}$ if there exists μ in $\mathcal{A}$ such that for any vertex v of G, the sum of labels of all the neighbors of v is μ. A graph G is $\mathcal{A}$-vertex magic if G admits such a labeling. Moreover, if G is $\mathcal{A}$-vertex magic for any abelian group $\mathcal{A}$, then G is group vertex magic. In this article, we characterize $\mathcal{A}$-vertex magic trees of diameter at most 5 for any finite abelian group $\mathcal{A}$. We prove that $\mathcal{A}$-vertex magic graphs do not possess any forbidden structures, and finally we give certain techniques to construct larger $\mathcal{A}$-vertex magic graphs from the existing ones.

1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order of the vertex set $|V|$ and the size of the edge set $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology, we refer to Chartrand and Lesniak [1].

For any vertex v of G, the set $N(v) = \{u \in V : uv \in E\}$ is called the open neighborhood of v, and $|N(v)| = d(v)$ is the degree of v. A vertex v with $d(v) = 1$ is a *pendant vertex* and the unique vertex adjacent to v is a *support vertex*. If a vertex v is adjacent to two or more pendant vertices, then v is a *strong support vertex*. Also, a vertex v is a *weak support vertex* if there is a unique pendant vertex adjacent to v. The distance $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. The diameter of G is defined by $\text{diam}(G) = \max\{d(u, v) : u, v \in V\}$. The eccentricity of a vertex v is the distance from v to the vertex which is farthest from v. The center $C(G)$ of a graph G is the set of vertices with minimum eccentricity. For a tree T,

$$C(T) = \begin{cases} \{v_c\} & \text{if } \text{diam}(T) \text{ is even} \\ \{v_{c_1}, v_{c_2}\} & \text{if } \text{diam}(T) \text{ is odd.} \end{cases}$$

Also, if $|C(T)| = 2$, then the two central vertices v_{c_1} and v_{c_2} are adjacent in T. A bi-star $B_{r,s}$ is a tree with the vertex set $\{u, v, u_i, v_j : 1 \le i \le r, 1 \le j \le s\}$ and the edge set $\{uv, uu_i, vv_j : 1 \le i \le r, 1 \le j \le s\}$. Note that $B_{r,s}$ has two strong support vertices u and v, when $r \ge 2$, $s \ge 2$. Let G and H be two graphs of orders m and n, respectively. The corona product $G \odot H$ is the graph obtained by taking one copy of G and m copies of H and joining each vertex from the ith copy of H with the ith vertex of G by an edge.

Throughout this article, $\mathcal{A}$ denotes an additive abelian group with identity zero, and $|\mathcal{A}|$ denotes the order of the group $\mathcal{A}$. The order of an element g of $\mathcal{A}$ is denoted by $o(g)$.

For a finite abelian group $\mathcal{A}$, we write $e(\mathcal{A}) = k$ if k is the least positive integer such that $kg = \underbrace{g + \cdots + g}_{k \text{ times}} = 0$, for all $g \neq 0$. For group theoretic terminology, we refer to Herstein [2].

A magic square is an $n \times n$ array with the elements $1, 2, \ldots, n^2$, each appearing exactly once such that the elements in any row or column or main diagonal or main back diagonal add up to the same sum. Various authors introduced labelings that generalize the idea of magic squares. For an excellent treatment of various types of magic labeling, the reader can refer to the book by Wallis and Marr [8]. Lee et al. [5] introduced, and authors like Lee et al. [4], Low and Lee [6, 7], and Shiu and Low [10], studied the concept of group-magic graphs.

Definition 1.1. Let $\mathcal{A}$ be an abelian group. A graph $G = (V, E)$ is said to be $\mathcal{A}$-magic if there exists a labeling $\ell : E \to \mathcal{A} - \{0\}$ such that the induced vertex labeling $\ell^+ : V \to \mathcal{A}$ defined by $\ell^+(v) = \sum_{uv \in E} \ell(uv)$ is a constant map.

Kamatchi et al. [3] introduced the concept of group vertex magic graphs.

Definition 1.2. A mapping $\ell : V \to \mathcal{A} - \{0\}$ is said to be an $\mathcal{A}$-vertex magic labeling of G if there exists an element μ of $\mathcal{A}$ such that $w(v) = \sum_{u \in N(v)} \ell(u) = \mu$, for any vertex v of G. A graph G that admits such a labeling is called an $\mathcal{A}$-vertex magic graph and the corresponding μ is said to be a magic constant. If G is an $\mathcal{A}$-vertex magic graph for every non-trivial abelian group $\mathcal{A}$, then G is called a group vertex magic graph.

Kamatchi et al. [3] and Sabeel et al. [9] together obtained a characterization of all $\mathcal{A}$-vertex magic trees of diameter up to 4 for $\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this article we characterize group vertex magic trees of diameter up to 5. We also give certain techniques to construct infinitely many families of group vertex magic graphs.

Observation 1.3. [3] If $P_4 = (u_1, u_2, u_3, u_4)$ is a path in G such that $d(u_1) = 1$ and $d(u_3) = 2$, then G is not $\mathcal{A}$-vertex magic for any abelian group $\mathcal{A}$. In fact, if ℓ is an $\mathcal{A}$-vertex magic labeling of G, then $\ell(u_2) = w(u_1) = \ell(u_2) + \ell(u_4) = w(u_3)$. Hence $\ell(u_4) = 0$, which is a contradiction.

2 Main results

We know that a graph G is $\mathbb{Z}_2$-vertex magic if and only if the degrees of all vertices of G are of the same parity. Therefore, we assume that $\mathcal{A}$ is a finite abelian group with at least 3 elements.

Lemma 2.1. *Let $\mathcal{A}$ be a finite abelian group with $|\mathcal{A}| \geq 3$ and let $g \in \mathcal{A}$. Then, for each $n \geq 2$, there exist $a_1, a_2, \ldots, a_n$ in $\mathcal{A} - \{0\}$ such that $g = a_1 + a_2 + \cdots + a_n$.*

Proof. The proof is by induction on n. Since $g = a_1 + (g - a_1)$, for any $a_1 \in \mathcal{A} - \{0, g\}$, the result is true for $n = 2$. Now suppose there exist $a_1, a_2, \ldots, a_k \in \mathcal{A} - \{0\}$ such that $g = a_1 + a_2 + \cdots + a_k$, where $2 \leq n \leq k$. Then $g = a_1 + a_2 + \cdots + a'_k + a_{k+1}$, where $a_{k+1} = a_k - a'_k$ and $a'_k \in \mathcal{A} - \{0, a_k\}$. Hence by induction the proof is complete. □

The next theorem gives a sufficient condition for a graph G to be an $\mathcal{A}$-vertex magic graph.

Theorem 2.2. *Let $\mathcal{A}$ be a finite abelian group with $|\mathcal{A}| \geq 3$. If G is a graph in which every non-pendant vertex is a strong support vertex, then G is $\mathcal{A}$-vertex magic.*

Proof. Let v be a vertex of G with $d(v) \geq 2$. Let $k(v)$ denote the number of pendant neighbors of v in G. Let $g \in \mathcal{A} - \{0\}$. Then by Lemma 2.1, there exist $a_1(v), a_2(v), \ldots, a_{k(v)}(v) \in \mathcal{A} - \{0\}$ such that

$$\big(1 - d(v) + k(v)\big)g = a_1(v) + a_2(v) + \cdots + a_{k(v)}(v).$$

Now consider $\ell : V \to \mathcal{A}$ given by $\ell(v) = g$ if $d(v) \geq 2$ and $\ell(u_i) = a_i(v)$, where u_i is a pendant neighbor of v, $1 \leq i \leq k(v)$.

Clearly, ℓ is an $\mathcal{A}$-vertex magic labeling of G with magic constant g. □

Corollary 2.3. *Any graph G is an induced subgraph of an $\mathcal{A}$-vertex magic graph H.*

Proof. Take $H = G \odot \overline{K}_2$, where $G \odot \overline{K}_2$ is the corona of G with $\overline{K}_2$. □

Corollary 2.4. *For any finite abelian group $\mathcal{A}$ with $|\mathcal{A}| \geq 3$, all trees of diameter 2 are $\mathcal{A}$-vertex magic.*

Proof. If T is a tree of diameter 2, then $T = K_{1,n}$ for some $n \geq 2$. Clearly, $K_{1,n}$ has a unique non-pendant vertex, which is a strong support vertex. Hence the result follows from Theorem 2.2. □

Let T be a tree with diameter 3. Then T is isomorphic to a bi-star $B_{r,s}$.

Corollary 2.5. *Let T be a tree of diameter 3. Then T is $\mathcal{A}$-vertex magic if and only if $T = B_{r,s}$, where $r \geq 2$ and $s \geq 2$.*

Proof. If $r = 1$ or $s = 1$, then it follows from Observation 1.3 that T is not $\mathcal{A}$-vertex magic. If $r \geq 2$ and $s \geq 2$, then it follows from Theorem 2.2 that T is $\mathcal{A}$-vertex magic. □

The following theorems provide a characterization for $\mathcal{A}$-magicness of trees of diameter 4, where $\mathcal{A}$ is any finite abelian group.

Theorem 2.6. *Let $\mathcal{A}$ be a finite abelian group with $|\mathcal{A}| \geq 3$ and let T be a tree of diameter 4 with the central vertex v_c. Then T is $\mathcal{A}$-vertex magic if and only if T satisfies one of the following conditions:*

(i) *Any non-pendant vertex of T is a strong support vertex.*
(ii) *v_c is a weak support vertex, $d(v_c) \not\equiv 2 \pmod{e(\mathcal{A})}$, and all other non-pendant vertices are strong support vertices.*
(iii) *v_c is not a support vertex and $\gcd(d(v_c) - 1, |\mathcal{A}|) \neq 1$.*

Proof. Let T be an $\mathcal{A}$-vertex magic tree with a labeling ℓ and magic constant g. Let $P_5 = (x_1, y_1, v_c, y_2, x_2)$ be a diametral path in T. Since T is of diameter 4, all other neighbors of y_1 and y_2 except v_c are pendant vertices. Here the subtree T_1 of T induced by the set of all non-pendant vertices of T is a star with v_c as its central vertex.

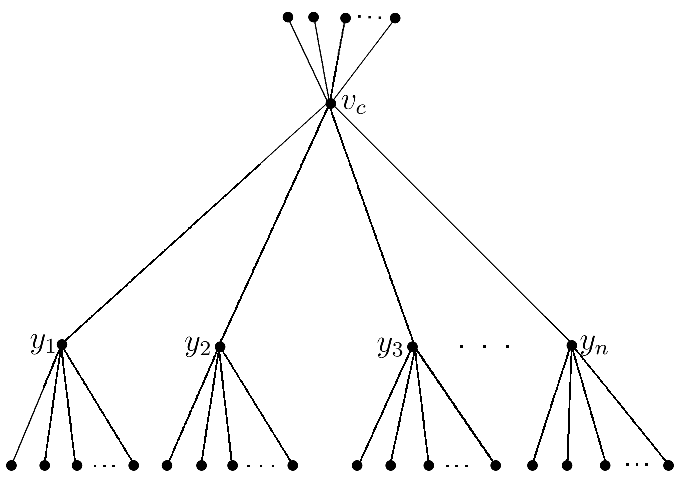

Figure 1: A typical tree of diameter 4

Let $V(T_1) = \{v_c, y_1, y_2, \cdots, y_n\}$. All vertices of $V(T) - V(T_1)$ are pendant vertices. There are three cases.

Case 1: v_c is a strong support vertex.
Clearly $\ell(v) = g$ for all $v \in V(T_1)$ and some $g \in \mathcal{A} - \{0\}$. If y_i is a weak support vertex with pendant neighbor x_i, then $g = w(y_i) = \ell(v_c) + \ell(x_i) = g + \ell(x_i)$, which is a contradiction. Hence all y_i are strong support vertices and T satisfies (i).

Case 2: v_c is a weak support vertex.
Proceeding as in case 1, we find that y_i is a strong vertex. Now $g = w(v_c) = (d(v_c) - 1)g + \ell(z)$, where z is the unique pendant neighbor of v_c. Hence $\ell(z) = (2 - d(v_c))g \neq 0$ and therefore $d(v_c) \not\equiv 2 \pmod{e(\mathcal{A})}$. Thus T satisfies (ii).

Case 3: v_c is not a support vertex.
In this case, $g = w(v_c) = d(v_c)g$. Then $(d(v_c) - 1)g = 0$. Therefore $d(v_c) - 1 \equiv 0 \pmod{o(g)}$. Also, $|\mathcal{A}| \equiv 0 \pmod{o(g)}$ implies $\gcd(d(v_c) - 1, |\mathcal{A}|) \equiv 0 \pmod{o(g)}$. Now $g \neq 0$ implies $o(g) \neq 1$. Hence $\gcd(d(v_c) - 1, |\mathcal{A}|) \neq 1$. Therefore T satisfies (iii).

Conversely, let T satisfy (i), (ii) or (iii). If T satisfies (i), it follows from Theorem 2.2 that T is $\mathcal{A}$-vertex magic. Suppose T satisfies (ii). For any non-pendant vertex $y_i \neq v_c$, let r_i denote the number of pendant vertices adjacent to y_i. Since $r_i \geq 2$, we can choose r_i elements in $\mathcal{A} - \{0\}$ such that their sum is zero. Label the pendant neighbors of y_i with these r_i elements. Now choose an element $g \in \mathcal{A}$ such that $o(g) = e(\mathcal{A})$. Assign label g to all the non-pendant vertices and the label $(2 - d(v_c))g$ for the unique pendant neighbor z of v_c. Since $d(v_c) \not\equiv 2 \pmod{e(\mathcal{A})}$, it follows that $(2 - d(v_c))g \neq 0$. Now $w(v_c) = (d(v_c) - 1)g + (2 - d(v_c))g = g$ and $w(v) = g$ for all the vertices v of T. Thus T is $\mathcal{A}$-vertex magic.

Assume that T satisfies (iii). Let $\gcd(d(v_c)-1,|\mathcal{A}|)=m>1$ and p be a prime divisor of m. By Cauchy's theorem, $\mathcal{A}$ has an element g of order p. Hence $o(g)$ divides $d(v_c)-1$ and $d(v_c)g=g$. Assign g as label to all the non-pendant vertices $y_i \neq v_c$, assign an element $g' \in \mathcal{A}-\{0,g\}$ to v_c and label the pendant neighbors of y_i such that that their label sum is $g-g'$. This completes an $\mathcal{A}$-vertex magic labeling of T. Thus T is an $\mathcal{A}$-vertex magic graph. □

Thus we have characterized all $\mathcal{A}$-vertex magic trees of diameter 4 for any finite abelian group $\mathcal{A}$. Now we proceed to characterize all $\mathcal{A}$-vertex magic trees of diameter 5.

Theorem 2.7. *Let $\mathcal{A}$ be a finite abelian group with $|\mathcal{A}|\geq 3$ and let T be a tree of diameter 5 such that neither of the central vertices v_{c_1} and v_{c_2} are support vertices. Then T is $\mathcal{A}$-vertex magic if and only if the following conditions are satisfied:*

(i) $d(v_{c_i}) \not\equiv 2 \pmod{e(\mathcal{A})}$.

(ii) *If there exists a weak support vertex u such that $v_{c_i} \notin N(u)$, then* $d(v_{c_i}) \not\equiv 1 \pmod{e(\mathcal{A})}$.

Proof. Suppose T is $\mathcal{A}$-vertex magic with a labeling ℓ and magic constant g. Clearly $g\neq 0$ and $\ell(v)=g$ for all the support vertices v. Then $g=w(v_{c_1})=(d(v_{c_1})-1)g+\ell(v_{c_2})$. Hence $\ell(v_{c_2})=(2-d(v_{c_1}))g$. Since $\ell(v_{c_2})\neq 0$, $o(g)$ does not divide $d(v_{c_1})-2$. On the other hand $o(g)$ divides $e(\mathcal{A})$ and hence $d(v_{c_1}) \not\equiv 2 \pmod{e(\mathcal{A})}$. Similarly $d(v_{c_2}) \not\equiv 2 \pmod{e(\mathcal{A})}$. Therefore T satisfies (i).

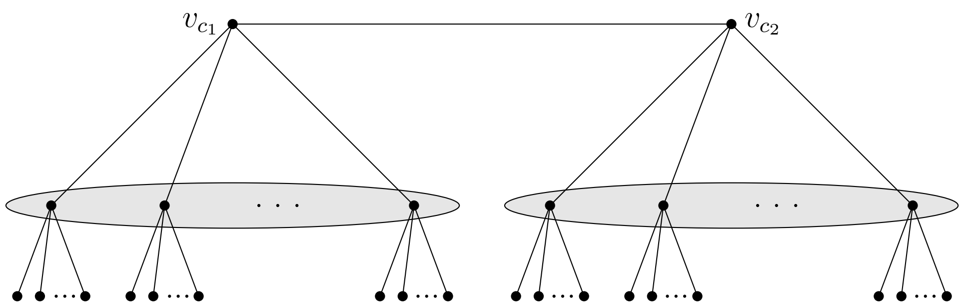

Figure 2: A typical tree of diameter 5 with neither v_{c_1} nor v_{c_2} supports

Suppose u is weak support vertex adjacent to v_{c_1}. Let $N(u)=\{y,v_{c_1}\}$. Now $g=w(v_{c_2})=(d(v_{c_2})-1)g+\ell(v_{c_1})$. Also $g=w(u)=\ell(v_{c_1})+\ell(y)$, and hence $(d(v_{c_2})-1)g=\ell(y)$. Since $\ell(y)\neq 0$, we have $d(v_{c_2})\not\equiv 1 \pmod{e(\mathcal{A})}$. Therefore T satisfies (ii).

Conversely, suppose T satisfies (i) and (ii). Choose $g\in\mathcal{A}$ such that $o(g)=e(\mathcal{A})$. Let $g'=(d(v_{c_1})-1)g$ and $g''=(d(v_{c_2})-1)g$. Since $d(v_{c_i})\not\equiv 2 \pmod{e(\mathcal{A})}$, $g-g'$ and $g-g''$ are non-zero. Now define $\ell(v_{c_1})=g-g''$, $\ell(v_{c_2})=g-g'$, and $\ell(v)=g$ for any support vertex v. If u is a support vertex adjacent to v_{c_1}, label the pendant neighbors of u in such a way that their label sum is g''. If u is a support vertex adjacent to v_{c_2}, label the pendant neighbors of u in such a way that their label sum is g'. Clearly ℓ is an $\mathcal{A}$-vertex magic labeling of T with magic the constant g. □

Theorem 2.8. *Let $\mathcal{A}$ be any finite abelian group with $|\mathcal{A}| \geq 3$ and let T be a tree of diameter 5 such that both its central vertices v_{c_1} and v_{c_2} are support vertices. Then T is $\mathcal{A}$-vertex magic if and only if T satisfies the following conditions:*

(i) *T has no vertex of degree 2.*

(ii) *If v_{c_i} is a weak support vertex, then $d(v_{c_i}) \not\equiv 2 \pmod{e(\mathcal{A})}$.*

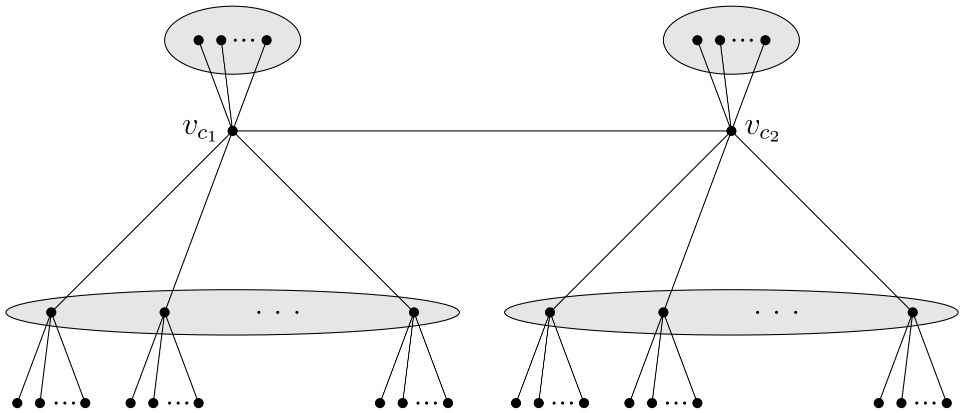

Figure 3: A typical tree of diameter 5 with both v_{c_1} and v_{c_2} supports

Proof. Suppose T is $\mathcal{A}$-vertex magic with a labeling ℓ and a magic constant g. Since v_{c_1} and v_{c_2} are support vertices, $\ell(v_{c_1}) = g = \ell(v_{c_2})$, and both v_{c_1} and v_{c_2} have degrees more than 2. Suppose T has a vertex u with $d(u) = 2$. Then $N(u) = \{v_{c_i}, y\}$ for some i and a pendant vertex y. Now $g = w(u) = \ell(v_{c_i}) + \ell(y) = g + \ell(y)$, and so $\ell(y) = 0$, which is a contradiction. Hence T has no vertex of degree 2. Thus T satisfies (i).

Suppose v_{c_1} is a weak support vertex with the unique pendant neighbor z. Then $g = w(v_{c_1}) = (d(v_{c_1}) - 1)g + \ell(z)$. Hence $(2 - d(v_{c_1}))g = \ell(z) \neq 0$, and so $d(v_{c_1}) \not\equiv 2 \pmod{e(\mathcal{A})}$. Therefore T satisfies (ii).

Conversely, suppose T satisfies (i) and (ii). Choose $g \in \mathcal{A}$ such that $o(g) = e(\mathcal{A})$. Let r_i denote the number of pendant neighbors adjacent to v_{c_i} and $g' = (d(v_{c_1}) - r_1)g$ and $g'' = (d(v_{c_2}) - r_2)g$. Label all the support vertices of T by g. Label the pendant neighbors of v_{c_1} such that their label sum is $g - g'$. Similarly label the pendant neighbors of v_{c_2} such that their label sum is $g - g''$. If u is a support vertex adjacent to v_{c_i}, then label all the pendant neighbors of u such that their label sum is 0. □

Theorem 2.9. *Let $\mathcal{A}$ be a finite abelian group such that $|\mathcal{A}| \geq 4$ and let T be a tree of diameter 5 such that v_{c_1} is a support vertex and the other central vertex v_{c_2} is not a support vertex. Then T is $\mathcal{A}$-vertex magic if and only if T satisfies the following conditions:*

(i) $\gcd(d(v_{c_2}) - 1, |\mathcal{A}|) \neq 1$*;*

(ii) *v_{c_1} is not adjacent to a weak support vertex.*

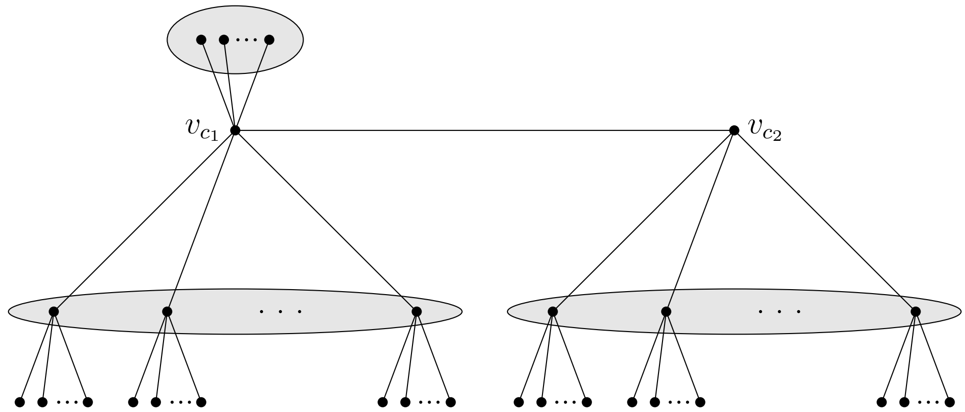

Figure 4: A typical tree of diameter 5 with v_{c_1} a support and v_{c_2} not a support

Proof. Suppose T is $\mathcal{A}$-vertex magic with a labeling ℓ and magic constant g. Any vertex $v \neq v_{c_2}$ with $d(v) \geq 2$ is a support vertex and hence $\ell(v) = g$.

If v_{c_1} is adjacent to a weak support vertex u and if y is the pendant vertex adjacent to u, then $g = w(u) = \ell(y) + \ell(v_{c_1}) = \ell(y) + g$. Hence $\ell(y) = 0$, which is a contradiction. Thus T satisfies (ii).

Now $g = w(v_{c_2}) = d(v_{c_2})g$ and so $(d(v_{c_2}) - 1)g = 0$. Therefore $o(g)$ divides $d(v_{c_2}) - 1$. Also $o(g)$ divides $|\mathcal{A}|$ and hence $o(g)$ divides $\gcd(d(v_{c_2}) - 1, |\mathcal{A}|)$. Since $g \neq 0$, $o(g) \neq 1$, and hence $\gcd(d(v_{c_2}) - 1, |\mathcal{A}|) \neq 1$. Thus T satisfies (i).

Conversely, suppose T satisfies (i) and (ii). There are two cases.

Case 1. Suppose that $|\mathcal{A}| \geq 5$.
Let $\gcd(d(v_c) - 1, |\mathcal{A}|) = m$ and let p be a prime divisor of m. By using Cauchy's theorem, $\mathcal{A}$ has an element g of order p. Let $\ell(v) = g$ for each support vertex v. Let r be the number of pendants adjacent to v_{c_1} and $g' = (d(v_{c_1}) - r)g$ and choose $h \in \mathcal{A} - \{0, g, g'\}$. If v_{c_1} is a weak support vertex with pendant neighbor z, and if $2g - (h + g') = 0$, then choose an $h' \in \mathcal{A} - \{0, g, g', h\}$, which is possible since $|\mathcal{A}| \geq 5$. Define $\ell(v_{c_2}) = h'$ and $\ell(z) = 2g - (g' + h')$. Otherwise, label $\ell(v_{c_2}) = h$ and label the r pendant neighbors of v_{c_1} such that their label sum is $2g - (h + g')$. Then $w(v_{c_1}) = g = w(v_{c_2})$.

Now, if u is a support vertex adjacent to v_{c_1}, then label pendant neighbors of u such that their label sum is 0. If v is a support vertex adjacent to v_{c_2} then label the pendant neighbors of v such that their label sum is $g - \ell(v_{c_2})$. This gives an $\mathcal{A}$-vertex magic labeling of T.

Case 2. Suppose that $|\mathcal{A}| = 4$.
Then $\mathcal{A} = \mathbb{Z}_4$ or $\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Suppose $\mathcal{A} = \mathbb{Z}_4 = \{0, 1, 2, 3\}$. Define $\ell(v) = 1$ for all support vertices. Let $k \equiv (d(v_{c_1}) - r - 1) \pmod 4$, where r is the number of pendant neighbors of v_{c_1}.

If $r = 1$ and $k = 3$, then define $\ell(v_{c_2}) = 3 = \ell(z)$, where z is the unique pendant vertex adjacent to v_{c_1}. Otherwise, define $\ell(v_{c_2}) = 2$ and label pendant vertices of v_{c_1} such that their label sum is $-(k + 1)$.

Now, if u is a support vertex adjacent to v_{c_1}, then label pendant neighbors of u such that their label sum is 0. If v is a support vertex adjacent to v_{c_2} then label the pendant neighbors of v such that their label sum is $1 - \ell(v_{c_2})$. Hence T is $\mathbb{Z}_4$-vertex magic.

If $\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$, then define $\ell(v) = (1, 1)$, for all support vertices and $\ell(v_{c_2}) = (1, 0)$. Label all the pendant neighbors of v_{c_1} in such a way that their label sum is $(0, 1)$ or $(1, 0)$ respectively, accordingly as the number of non-pendant neighbors of v_{c_1} is odd or even. Hence $w(v_{c_1}) = w(v_{c_2}) = (1, 1)$. The remaining pendant vertices can be labeled such that $w(v) = (1, 1)$, for each support vertex v. Hence T is $\mathcal{A}$-vertex magic. □

Theorem 2.10. *Let T be a tree of diameter 5 such that v_{c_1} is a support vertex and the other central vertex v_{c_2} is not a support vertex. Then T is $\mathbb{Z}_3$-vertex magic if and only if T satisfies the following conditions:*

(i) $d(v_{c_2}) \equiv 1 \pmod 3$.
(ii) v_{c_1} *is not adjacent to a weak support vertex.*
(iii) *If v_{c_1} is a weak support vertex and $d(v_{c_1}) \equiv 1 \pmod 3$, then v_{c_2} is not adjacent to a weak support vertex other than v_{c_1}.*

Proof. Suppose T is $\mathbb{Z}_3$-vertex magic with a labeling ℓ and magic constant $g \neq 0$. Since $w(v_{c_2}) = d(v_{c_2})g = g$ and $o(g) = 3$, we have $d(v_{c_2}) \equiv 1 \pmod 3$.

The proof for T satisfying (ii) is analogous to the proof of Theorem 2.9.

Suppose z is the unique pendant neighbor of v_{c_1} and $d(v_{c_1}) \equiv 1 \pmod 3$. Then $g = w(v_{c_1}) = 2g + \ell(v_{c_2}) + \ell(z)$. Since $\ell(v_{c_2}), \ell(z) \neq 0$, $\ell(v_{c_2}) = \ell(z) = g$. Since $\ell(v_{c_2}) = g$, the vertex v_{c_2} cannot be adjacent to a weak support vertex other than v_{c_1}. Hence T satisfies (iii).

Conversely, suppose T satisfies (i), (ii) and (iii). Define a labeling ℓ such that $\ell(v) = 1$ for any support vertex v. If x is a support vertex adjacent to v_{c_1}, label the pendant neighbors of x in such a way that their label sum is 0. Labels of remaining vertices are defined based on two cases.

Case 1: v_{c_1} is adjacent to a unique pendant z and $d(v_{c_1}) \equiv 1 \pmod 3$.
Define $\ell(v_{c_2}) = 1$, $\ell(z) = 1$, and if y is a support vertex adjacent to v_{c_2}, then label the pendant neighbors of y such that the label sum is 0.

Case 2: Otherwise.
Define $\ell(v_{c_2}) = 2$, and if y is a support vertex adjacent to v_{c_2}, label the pendant neighbors of y such that the label sum is 2. Let m be the number of support vertices adjacent to v_{c_1} and let r be the remainder when m is divided by 3. Then label the pendant neighbors of v_{c_1} such that the label sum is

$$\begin{cases} 2 & \text{if } r = 0 \\ 1 & \text{if } r = 1 \\ 0 & \text{if } r = 2. \end{cases}$$

Hence ℓ is a $\mathbb{Z}_3$-vertex magic labeling of T. □

Therefore the above theorems provide a complete characterization of $\mathcal{A}$-vertex magic trees of diameter 5 for any finite abelian group $\mathcal{A}$.

3 Technique to construct infinite classes of group vertex magic graphs

In this section we establish a technique to construct an infinite number of group vertex magic graphs. The following theorem provides a technique to construct $\mathcal{A}$-vertex magic graphs from existing ones by preserving the same magic constant in both graphs.

Theorem 3.1. *Let $t \geq 2$. Suppose G is an $\mathcal{A}$-vertex magic graph of order n with a labeling ℓ and magic constant g. If there exists an edge uv in G with $\ell(u) = \ell(v) = g$, then the graph $G^\dagger$, obtained from G by subdividing the edge uv and by attaching t pendant vertices at the new vertex x, is an $\mathcal{A}$-vertex magic graph of order $n + t + 1$ with the same magic constant g.*

Proof. Define $\ell^\dagger : V(G^\dagger) \to \mathcal{A}$ by $\ell^\dagger(x) = g$, and label the t pendant vertices adjacent to x such that its label sum is $-g$ and $\ell^\dagger(w) = \ell(w)$ for all $w \in V(G)$. Clearly $\ell^\dagger$ is an $\mathcal{A}$-vertex magic $G^\dagger$ with magic constant g. □

Figure 5: The graph G and its graph $G^\dagger$

Theorem 3.2. *Let G be a graph of order n. Let $G^\dagger$ be the graph obtained from G by subdividing an edge xy four times. Then G is $\mathcal{A}$-vertex magic if and only if $G^\dagger$ is $\mathcal{A}$-vertex magic.*

Proof. Let $P = (x, u_1, u_2, u_3, u_4, y)$ be the $x - y$ path in $G^\dagger$. Suppose $G^\dagger$ is $\mathcal{A}$-vertex magic with labeling $\ell^\dagger$ and magic constant g. Since $w(u_1) = w(u_3)$ and $w(u_4) = w(u_2)$, we get $\ell^\dagger(x) = \ell^\dagger(u_4)$ and $\ell^\dagger(y) = \ell^\dagger(u_1)$. Hence the labeling ℓ obtained by restricting $\ell^\dagger$ to $V(G)$ gives an $\mathcal{A}$-vertex magic labeling of G with the same magic constant g. Conversely, let ℓ be an $\mathcal{A}$-vertex magic labeling of G. Consider a mapping

$\ell^\dagger : V(G^\dagger) \to \mathcal{A} - \{0\}$ given by

$$\ell^\dagger(v) = \begin{cases} \ell(v) & \text{if } v \in V(G) \\ \ell(y) & \text{if } v = u_1 \\ g - \ell(x) & \text{if } v = u_2 \\ g - \ell(y) & \text{if } v = u_3 \\ \ell(x) & \text{if } v = u_4. \end{cases}$$

Clearly, $\ell^\dagger$ is an $\mathcal{A}$-vertex magic labeling of $G^\dagger$ with the magic constant g. □

4 Concluding Remarks

In this article, a characterization of $\mathcal{A}$-vertex magic trees of diameter at most 5 has been given, where $\mathcal{A}$ is any finite abelian group. Further, certain techniques to construct larger $\mathcal{A}$-vertex magic graphs from existing $\mathcal{A}$-vertex magic graphs have been provided.

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