AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **84(1)** (2022), Pages 178–186

A characterisation of the planes meeting a hyperbolic quadric of $\mathrm{PG}(3, q)$ in a conic

Bikramaditya Sahu

Department of Mathematics
National Institute of Technology
Rourkela – 769008, Odisha
India
sahuba@nitrkl.ac.in

Abstract

In this article, a combinatorial characterisation of the family of planes of $\mathrm{PG}(3, q)$ which meet a hyperbolic quadric in an irreducible conic, using their intersection properties with the points and lines of $\mathrm{PG}(3, q)$, is given.

1 Introduction

Let $\mathrm{PG}(n, q)$ denote the n-dimensional Desarguesian projective space defined over a finite field of order q, where q is a prime power. The non-singular quadrics of $\mathrm{PG}(n, q)$ are very interesting objects with many combinatorial properties. One of the important properties of quadrics in $\mathrm{PG}(n, q)$ is that subspaces can only meet a quadric in certain ways. Therefore, we may form families of subspaces that all meet a particular quadric in the same way and then we can give a characterisation of that family.

The characterisation of classical polar spaces by means of intersection properties with the subspaces of the original projective space can be traced back to the 1950's seminal work by Segre and Talini and their schools. This was continued by Beutelspacher, Thas and their collaborators in the 1970s and 1980s. In 1991, Tallini and Ferri gave a characterisation of the parabolic quadric of $\mathrm{PG}(4, q)$ by its intersection properties with planes and hyperplanes [8]. Characterisations of non-singular quadrics and non-singular Hermitian varieties by intersection numbers with hyperplanes and spaces of codimension 2 is given in [6]. This generalizes the result of Tallini and Ferri. One can refer to [5] for a detailed study in the past two decades.

A characterisation of the family of planes meeting a non-degenerate quadric in $\mathrm{PG}(4, q)$ is given in [3]. In a series of two recent papers [1, 2], characterisations of elliptic hyperplanes and hyperbolic hyperplanes in $\mathrm{PG}(4, q)$ are given. A characterisation of the family of external lines to a hyperbolic quadric in $\mathrm{PG}(3, q)$ is given in [7]

ISSN: 2202-3518

for all q (also see [10] for a different characterisation in terms of a point-subset of the Klein quadric in $\mathrm{PG}(5,q)$). In a series of two recent papers [4, 12], a characterisation of secant lines to a hyperbolic quadric is given for all odd q. In this paper, we give a characterisation of the family of planes meeting a hyperbolic quadric in $\mathrm{PG}(3,q)$ in an irreducible conic.

Let $\mathcal{Q}$ be a hyperbolic quadric in $\mathrm{PG}(3,q)$, that is, a non-degenerate quadric of Witt index two. One can refer to [9, Section 15.3.III] for the basic properties of the points, lines and planes of $\mathrm{PG}(3,q)$ with respect to $\mathcal{Q}$. The quadric $\mathcal{Q}$ consists of $(q+1)^2$ points and $2(q+1)$ lines. Every plane of $\mathrm{PG}(3,q)$ meets $\mathcal{Q}$ in an irreducible conic or in two intersecting generators. In the first case we call the plane a *secant* plane, otherwise, a *tangent* plane. There are q^3-q secant planes and $(q+1)^2$ tangent planes. Each point of $\mathrm{PG}(3,q)\setminus\mathcal{Q}$ is contained in q^2 secant planes and each point of $\mathcal{Q}$ is contained in q^2-q secant planes. Each line of $\mathrm{PG}(3,q)$ is contained in 0, $q-1$, q or $q+1$ secant planes.

In this paper, we prove the following characterisation theorem.

Theorem 1.1. *Let Σ be a non empty family of planes of $\mathrm{PG}(3,q)$, for which the following properties are satisfied:*

(P1) Every point of $\mathrm{PG}(3,q)$ is contained in q^2-q or q^2 planes of Σ.

(P2) Every line of $\mathrm{PG}(3,q)$ is contained in 0, $q-1$, q or $q+1$ planes of Σ.

Then Σ is the set of all planes of $\mathrm{PG}(3,q)$ meeting a hyperbolic quadric in an irreducible conic.

2 Preliminaries

Let Σ be a nonempty set of planes of $\mathrm{PG}(3,q)$ for which the properties (P1) and (P2) stated in Theorem 1.1 hold. A point of $\mathrm{PG}(3,q)$ is said to be *black* or *white* according as it is contained in q^2-q or q^2 planes of Σ. Let b and w, respectively, denote the number of black points and the number of white points in $\mathrm{PG}(3,q)$. We have

$$b+w=q^3+q^2+q+1. \tag{1}$$

Counting in two ways the point-plane incident pairs,

$$\{(x,\pi)\mid x \text{ is a point in } \mathrm{PG}(3,q),\ \pi\in\Sigma \text{ and } x\in\pi\},$$

we get

$$b(q^2-q)+wq^2=|\Sigma|(q^2+q+1). \tag{2}$$

From Equations (1) and (2), by eliminating w, we get

$$b=q(q^3+q^2+q+1)-\frac{|\Sigma|}{q}(q^2+q+1). \tag{3}$$

Again, counting in two ways the incident triples,

$$\{(x, \pi, \sigma) \mid x \text{ is a point in PG}(3, q),\ \pi, \sigma \in \Sigma, \pi \neq \sigma \text{ and } x \in \pi \cap \sigma\},$$

we get

$$b(q^2 - q)(q^2 - q - 1) + wq^2(q^2 - 1) = |\Sigma|(|\Sigma| - 1)(q + 1). \tag{4}$$

Dividing both sides of Equation (4) by 2 the following equality is obtained.

$$bq(q - 1)/2(q^2 - q - 1) + wq^2(q - 1)(q + 1)/2 = |\Sigma|(|\Sigma| - 1)(q + 1)/2. \tag{5}$$

Lemma 2.1. *(i) If q is even, then $q + 1$ divides b.*

(ii) If q is odd, then $(q + 1)/2$ divides b.

Proof. Since $q^2 - q - 1 = (q + 1)(q - 2) + 1$, we have $q + 1$ is coprime to $q^2 - q - 1$. Also $q + 1$ is coprime to q being consecutive integers. If q is even, then $q + 1$ and $q - 1$ are both odd and their difference is two and so they are coprime to each other. If q is odd, then $(q + 1)/2$ and $(q - 1)/2$ are coprime being consecutive integers (also $(q + 1)/2$ and q are coprime). Now (i) and (ii) follow from Equations (4) and (5), respectively. □

Lemma 2.2. *The number of black points in a plane of Σ is independent of the choice of plane.*

Proof. Let π be a plane in Σ. Let b_π and w_π, respectively, denote the number of black and white points in π. Then $w_\pi = q^2 + q + 1 - b_\pi$.

By counting the incident point-plane pairs of the following set,

$$\{(x, \sigma) \mid x \text{ is a point in PG}(3, q),\ \sigma \in \Sigma\ , \sigma \neq \pi \text{ and } x \in \pi \cap \sigma\},$$

we get

$$b_\pi(q^2 - q - 1) + w_\pi(q^2 - 1) = (|\Sigma| - 1)(q + 1).$$

Since $q + 1$ is coprime to $q^2 - q - 1$ (see the proof of Lemma 2.1), it follows from the above equation that $q + 1$ divides b_π. Let $b_\pi = (q + 1)r_\pi$ for some $0 \leq r_\pi \leq q$.

Since $w_\pi = q^2 + q + 1 - b_\pi = q^2 + q + 1 - (q + 1)r_\pi$, substituting w_π into the preceding displayed equation, we get

$$q^3 - qr_\pi = |\Sigma|.$$

Since $|\Sigma|$ is a fixed number, $r_\pi := r$ is fixed. Hence $b_\pi = (q+1)r$, which is independent of π. □

From the proof of Lemma 2.2, we have the following:

$$q^3 - qr = |\Sigma|; \tag{6}$$

for some $0 \leq r \leq q$.

Henceforth the integer r is such that $0 \leq r \leq q$ and r satisfies Equation (6).

Corollary 2.3. *The number of black points in a plane in* Σ *is* $(q+1)r$.

Lemma 2.4. *Any plane of* $\mathrm{PG}(3,q)$ *not in* Σ *contains* $q+(q+1)r$ *black points.*

Proof. We prove the lemma with a similar argument as that of Lemma 2.2. Let π be a plane of $\mathrm{PG}(3,q)$ not in Σ. Let b_π and w_π denote the number of black and white points, respectively, in π. Then $w_\pi = q^2+q+1-b_\pi$.

By counting the incident point-plane pairs of the following set,

$$\{(x,\sigma) \mid x \text{ is a point in } \mathrm{PG}(3,q),\ \sigma \notin \Sigma\ , \sigma \neq \pi \text{ and } x \in \pi \cap \sigma\},$$

we get

$$b_\pi\left((q^2+q+1)-(q^2-q)-1\right) + w_\pi\left((q^2+q+1)-q^2-1)\right) = \\ (q^3+q^2+q+1-|\Sigma|-1)(q+1),$$

that is,

$$2qb_\pi + qw_\pi = (q^3+q^2+q+1-|\Sigma|-1)(q+1).$$

From Equation (6), $|\Sigma| = q^3r - qr$. The above equation simplifies to

$$2b_\pi + w_\pi = (q+1+r)(q+1).$$

Since $w_\pi = q^2+q+1-b_\pi$, we get $b_\pi = q+(q+1)r$. □

Corollary 2.5. $1 \leq r \leq q-1$.

Proof. Note that $0 \leq r \leq q$. Suppose $r=0$. By Equation (6), we get $|\Sigma| = q^3$. From Equation (3), we get that $b=q$. This leads to a contradiction to Lemma 2.1.

Now suppose that $r=q$. Then by Lemma 2.4, any plane of $\mathrm{PG}(3,q)$ not in Σ contains q^2+2q black points, which is a contradiction. Hence $1 \leq r \leq q-1$. □

Lemma 2.6. *We have* $r=1$, $b=(q+1)^2$ *and* $|\Sigma| = q^3-q$. *In particular, every plane in* Σ *contains* $q+1$ *black points.*

Proof. Note that, by Corollary 2.5, $1 \leq r \leq q-1$. If $q=2$, then $r=1$. Assume first that q even, $q \geq 4$.

By Lemma 2.1(i) and Equation (4), it follows that $q(q-1)$ divides $|\Sigma|(|\Sigma|-1)$. Since $|\Sigma| = q^3-qr$ (Equation (6)), we have $q(q-1)$ divides $(q^3-qr)(q^3-qr-1)$, i.e, $q-1$ divides (q^2-r) or (q^3-qr-1).

Note that $q^2-r = (q^2-1)-(r-1)$ and $q^3-qr-1 = (q^3-q)+(q-1)-qr$. If $q-1$ divides (q^2-r), then $r=1$ or $r=q$. If $q-1$ divides q^3-qr-1, then $r=0$ or $r=q-1$. Since $1 \leq r \leq q-1$, we have $r=1$ or $q-1$.

Suppose $r=q-1$. Then by Equation (6), we have $|\Sigma| = q(q^2-q+1)$. From Equation (3), we obtain $b = q^3+q-1 = q^2(q+1)-(q-1)(q+1)+(q-2)$, which is a contradiction to Lemma 2.1(i) for $q \geq 4$. Hence $r=1$ for all q even.

Now assume that q is odd.

By Equation (5) and Lemma 2.1(ii), we see that $q(q-1)/2$ divides $|\Sigma|(|\Sigma|-1)$. Thus $q(q-1)/2$ divides $(q^3-qr)(q^3-qr-1)$, i.e, $(q-1)/2$ divides (q^2-r) or (q^3-qr-1).

As before $q^2-r=(q^2-1)-(r-1)$ and $q^3-qr-1=(q^3-q)+(q-1)-qr$. If $(q-1)/2$ divides (q^2-r), then $r=1$ or $r=(q+1)/2$. If $(q-1)/2$ divides q^3-qr-1, then $r=(q-1)/2$ or $r=q-1$. So, $r=1,(q-1)/2,(q+1)/2$ or $q-1$.

Suppose that $r=q-1$. As in the q even case, here also we obtain $b=q^3+q-1=(q^3+q^2)-(q^2-1)+(q-2)$. By Lemma 2.1(ii), $(q+1)/2$ divides $q-2$ i.e, $(q+1)/2$ divides $q+1-3$. This gives $q=5$. So $b=5^3+5-1=129$, $r=4$ and hence $|\Sigma|=5^3-5\cdot 4=105$. Now $w=5^3+5^2+5+1-b=27$. On the other hand, putting the values of $b=129$ and $|\Sigma|=105$ in Equation (4), we obtain $w=55/2$, which is a contradiction.

Suppose that $r=(q-1)/2$. Then by Equation (6), $|\Sigma|=q^3-q(q-1)/2$. Putting the value of $|\Sigma|$ in Equation (3), we obtain $b=(q^3+2q-1)/2=(q^3+1)/2+(q-1)$. By Lemma 2.1(ii), $(q+1)/2$ divides $q-1$ i.e, $(q+1)/2$ divides $q+1-2$. This gives $q=3$ and hence $r=(q-1)/2=1$.

Suppose now that $r=(q+1)/2$. Then by Equation (6), $|\Sigma|=q^3-q(q+1)/2$. Putting the value of $|\Sigma|$ in Equation (3), we obtain $b=(q^3+2q^2+4q+1)/2=(q^3+1)/2+q(q+1)+q$. Again by Lemma 2.1(ii), $(q+1)/2$ divides q, which is a contradiction as q and $q+1$ are coprime. Hence $r=1$ for all q odd.

Hence for all q, we have $r=1$ and $|\Sigma|=q^3-q$. Now by Equation (3), $b=(q+1)^2$ and by Corollary 2.3, every plane in Σ contains $q+1$ black points. □

Corollary 2.7. *(i) There are $(q+1)^2$ planes in* $\mathrm{PG}(3,q)$ *which are not in* Σ.

(ii) Every plane in $\mathrm{PG}(3,q)$ *not in* Σ *contains* $2q+1$ *black points.*

Proof. (i) follows from Lemma 2.6 and the fact that there are $(q^2+1)(q+1)$ planes in $\mathrm{PG}(3,q)$. Now (ii) follows from Lemma 2.4, since $r=1$. □

We call a plane of $\mathrm{PG}(3,q)$ *tangent* if it is not a plane in Σ. By Corollary 2.7, every tangent plane contains $2q+1$ black points. We need the following result to prove a couple of results.

Lemma 2.8. *Every black point is contained in* $2q+1$ *tangent planes. Every white point is contained in* $q+1$ *tangent planes*

Proof. Let x be a point of $\mathrm{PG}(3,q)$. If x is a black point, then x is contained in q^2-q planes of Σ and so it is contained in $q^2+q+1-(q^2-q)=2q+1$ tangent planes. Similarly, if x is a white point, then it is contained in $q^2+q+1-q^2=q+1$ tangent planes. □

Lemma 2.9. *Let l be a line of* $\mathrm{PG}(3,q)$*. Then the number of tangent planes containing l is equal to the number of black points contained in l.*

Proof. Let t and s, respectively, denote the number of tangent planes containing l and the number of black points contained in l. Observe also that for every plane π of $\mathrm{PG}(3,q)$, either π contains one point of l or the whole line l. By applying Lemmas 2.6, 2.9 and Corollary 2.7 together with the above observation, we count in two different ways the number of point-plane incident pairs of the following set,

$$\{(x,\pi) \mid x \in l,\ \pi \text{ is a tangent plane and } x \in \pi\}.$$

Hence

$$s(2q+1) + (q+1-s)(q+1) = ((q+1)^2 - t) \cdot 1 + t \cdot (q+1).$$

It follows that $s = t$. This proves the lemma. $\square$

Corollary 2.10. *Every line of* $\mathrm{PG}(3,q)$ *contains* $0, 1, 2$ *or* $q+1$ *black points.*

Proof. From Theorem 1.1(P2), every line is contained in $0, 1, 2$ or $q+1$ tangent planes. The proof now follows from Lemma 2.9 $\square$

3 Black lines

Let B be the set of all black points in $\mathrm{PG}(3,q)$. We call a line *black* if all its $q+1$ points are contained in B.

Lemma 3.1. *Let* π *be a tangent plane. Then the set* $\pi \cap B$ *is a union of two (intersecting) black lines.*

Proof. By Corollary 2.7(ii), we have $|\pi \cap B| = 2q+1$. Then $\pi \cap B$ is not an arc, since an arc in any plane contains at most $q+2$ points. Thus there is a line l in π such that $|l \cap B| \geq 3$. By Corollary 2.10, $l \subseteq B$.

Let x be a point on l. Since $|\pi \cap B| = 2q+1$ and $l \subseteq B$, q other points (points not on l) of $\pi \cap B$ are contained in q lines through x, different from l. If there is a line $m(\neq l)$ containing x which contains two points of $\pi \cap B \setminus \{x\}$, then $|m \cap B| \geq 3$. By Corollary 2.10, $m \subseteq B$. So, in this case $\pi \cap B = l \cup m$. On the other hand, if there are two points y and z (different from x) of B which are contained in two different lines (different from l) containing x, then the line $m' := yz$ intersects l at a point different from y and z. In particular $|m' \cap B| \geq 3$. Again by Corollary 2.10, $m' \subseteq B$. In any case, $\pi \cap B = l \cup m'$. This proves the lemma. $\square$

Lemma 3.2. *Every black point is contained in at most two black lines.*

Proof. Let x be a black point. If possible, suppose that there are three distinct black lines l, m, k each of which contains x.

Note that, by Lemma 2.9, all the $q+1$ planes containing l (similarly, m) are tangent planes. This gives $2q+2$ tangent planes through x (with repetition allowed) and there is a unique tangent plane, namely $\langle l, m\rangle$, containing both l and m. Since by Lemma 2.8, there are $2q+1$ tangent planes containing x, it follows that each of the $2q+1$ tangent planes containing x either contains l or m.

On the other hand, again by Lemma 2.9, there are $q+1$ tangent planes containing k. Out of these $q+1$ planes, the planes $\langle k, l\rangle$ and $\langle k, m\rangle$ have already been counted as planes containing x. Since $q \geq 2$, there is a tangent plane containing k (and hence containing x), different from $\langle k, l\rangle$ and $\langle k, m\rangle$. This gives a contradiction to the fact that each of the tangent planes containing x either contains l or m. Hence, there are at most two black lines containing x. □

Lemma 3.3. *Every black point is contained in precisely two black lines.*

Proof. Let x be a black point and l be a black line containing x. The existence of such a line l follows from Lemma 2.8. By Lemma 2.9, let $\pi_1, \pi_2, \ldots, \pi_{q+1}$ be the $q+1$ tangent planes containing l. For $1 \leq i \leq q+1$, by Lemma 3.1, we have $B \cap \pi_i = l \cup l_i$ for some black line l_i of π_i different from l. Let $\{p_i\} = l \cap l_i$. Lemma 3.2 implies that $p_i \neq p_j$ for $1 \leq i \neq j \leq q+1$, and so $l = \{p_1, p_2, \ldots, p_{q+1}\}$. Since $x \in l$, we have $x = p_j$ for some $1 \leq j \leq q+1$. Thus, x is contained in precisely two black lines, namely, l and l_j. □

4 Proof of Theorem 1.1

We refer to [11] for the basics on finite generalized quadrangles. Let s and t be positive integers. A *generalized quadrangle* of order (s,t) is a point-line geometry $\mathcal{X} = (P, L)$ with point set P and line set L satisfying the following three axioms:

(Q1) Every line contains $s+1$ points and every point is contained in $t+1$ lines.

(Q2) Two distinct lines have at most one point in common (equivalently, two distinct points are contained in at most one line).

(Q3) For every point-line pair $(x, l) \in P \times L$ with $x \notin l$, there exists a unique line $m \in L$ containing x and intersecting l.

Let $\mathcal{X} = (P, L)$ be a generalized quadrangle of order (s,t). Then, $|P| = (s+1)(st+1)$ and $|L| = (t+1)(st+1)$ [11, 1.2.1]. If P is a subset of the point set of some projective space $\mathrm{PG}(n,q)$, L is a set of lines of $\mathrm{PG}(n,q)$ and P is the union of all lines in L, then $\mathcal{X} = (P, L)$ is called a *projective generalized quadrangle*. The points and the lines contained in a hyperbolic quadric in $\mathrm{PG}(3,q)$ form a projective generalized quadrangle of order $(q,1)$. Conversely, any projective generalized quadrangle of order $(q,1)$ with ambient space $\mathrm{PG}(3,q)$ is a hyperbolic quadric in $\mathrm{PG}(3,q)$, this follows from [11, 4.4.8].

The following two lemmas complete the proof of Theorem 1.1.

Lemma 4.1. *The points of B together with the black lines form a hyperbolic quadric in* $\mathrm{PG}(3,q)$.

Proof. We have $|B| = b = (q+1)^2$ by Lemma 2.6. It is enough to show that the points of B together with the black lines form a projective generalized quadrangle of order $(q, 1)$.

Each black line contains $q + 1$ points of B. By Lemma 3.3, each point of B is contained in exactly two black lines. Thus axiom (Q1) is satisfied with $s = q$ and $t = 1$. Clearly, axiom (Q2) is satisfied.

We verify axiom (Q3). Let $l = \{x_1, x_2, \ldots, x_{q+1}\}$ be a black line and x be a black point not contained in l. By Lemma 3.3, let l_i be the second black line through x_i (different from l) for $1 \leq i \leq q+1$. If l_i and l_j intersect for $i \neq j$, then the tangent plane π generated by l_i and l_j contains l as well. This implies that $\pi \cap B$ contains the union of three distinct black lines (namely, l, l_i, l_j), which is not possible by Lemma 3.1. Thus the black lines $l_1, l_2, \ldots, l_{q+1}$ are pairwise disjoint. These $q+1$ black lines contain $(q+1)^2$ black points and hence their union must be equal to B. In particular, x is a point of l_j for unique $j \in \{1, 2, \ldots, q+1\}$. Then l_j is the unique black line containing x_j and intersecting l.

From the above, it follows that the points of B together with the black lines form a projective generalized quadrangle of order $(q, 1)$. This completes the proof. □

Lemma 4.2. *The set of planes in Σ are the planes meeting B in a conic.*

Proof. Let π be a plane in Σ. Note that $|\pi \cap B| = q + 1$ by Lemma 2.6. Suppose that l is a line of π containing three points of $\pi \cap B$. Then by Corollary 2.10, l is contained in $\pi \cap B$. Since $|\pi \cap B| = q + 1$, we have $l = \pi \cap B$. By Lemma 2.9, all the planes containing l are tangent, which is a contradiction to the fact that π is not a tangent plane. Hence $|l \cap B| \leq 2$. Since the line l in π was arbitrary, $\pi \cap B$ is an oval in π. Since B is a quadric by Lemma 4.1, the planes of Σ are exactly those that meet B in an irreducible conic. □

Acknowledgements

The author wants to thank the anonymous referees for their careful reading and their helpful suggestions which improved the paper.

References

[1] S. G. Barwick, A. M. W. Hui and W.-A. Jackson, Characterising elliptic solids of $Q(4, q)$, q even, *Discrete Math.* 343 (2020), 111857.

[2] S. G. Barwick, A. M. W. Hui, W.-A. Jackson and J. Schillewaert, Characterising hyperbolic hyperplanes of a non-singular quadric in $\mathrm{PG}(4, q)$, *Des. Codes Crypto.* 88 (2020), 33–39.

[3] D. K. Butler, A characterisation of the planes meeting a non-singular quadric of $\mathrm{PG}(4, q)$ in a conic, *Combinatorica* 33 (2013), 161–179.

[4] B. De Bruyn, P. Pradhan, B. K. Sahoo and B. Sahu, A characterisation of the family of secant lines to a hyperbolic quadric in PG$(3, q)$, q odd, Part II, Available online: `https://cage.ugent.be/geometry/Files/456/Secants.pdf`.

[5] F. De Clerke and N. Durante, Constructions and Characterizations of classical sets in PG(n, q), *Current Research Topics in Galois Geometry*, Nova Science Publishers, New York, 2012, 1–31.

[6] S. De Winter and J. Schillewaert, A characterization of finite polar spaces by intersection numbers, *Combinatorica* 30 (2010), 25–45.

[7] R. Di Gennaro, N. Durante and D. Olanda, A characterization of the family of lines external to a hyperbolic quadric of PG$(3, q)$, *J. Geom.* 80 (2004), 65–74.

[8] O. Ferri and G. Tallini, A characterization of non-singular quadrics in PG$(4, q)$, *Rend. Mat. Appl.* 11 (1991), 15–21.

[9] J. W. P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1985.

[10] S. Innamorati, M. Zannetti and F. Zuanni, A characterization of the lines external to a hyperbolic quadric in PG$(3, q)$, *Ars Combin.* 103 (2012), 3–11.

[11] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Second ed., EMS Series of Lectures in Math., European Mathematical Society (EMS), Zürich, 2009.

[12] P. Pradhan and B. Sahu, A characterisation of the family of secant lines to a hyperbolic quadric in PG$(3, q)$, q odd, *Discrete Math.* 343 (2020), 112044.

(Received 18 Nov 2021; revised 7 July 2022)