

# Reconstruction of small graphs and digraphs

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## Abstract

We describe computer searches that prove the graph reconstruction conjecture for graphs with up to 13 vertices and some limited classes on larger sizes. We also investigate the reconstructibility of tournaments up to 13 vertices, digraphs up to 9 vertices, and posets up to 13 points. In all cases, our results also apply to the set reconstruction problem that uses the isomorph-reduced deck.

## 1 Introduction

The late Frank Harary visited the University of Melbourne in 1976, when I was a Masters student there. I mentioned to him that I had proved the 9-point graphs to be reconstructible, using a catalogue of graphs obtained on magnetic tape from Canada [2]. Harary replied, “Send it to the Journal of Graph Theory. I accept it!”. Impressed by having a paper I hadn’t written yet accepted for a journal that hadn’t started publishing yet, I quickly wrote it up and did as Harary suggested [11].

We recount the standard definitions. Except when we say otherwise, our graphs are simple, undirected and labelled, and our digraphs are simple and unlabelled. We use the standard sloppy terminology that an *unlabelled graph* is the isomorphism type of a graph, or a labelled graph with the labels hidden. For an unlabelled graph  $G$  with vertex  $v$ , the *card*  $G-v$  is the unlabelled graph obtained from  $G$  by removing  $v$ . The *full deck* of  $G$  (usually called just the deck) is the multiset of its cards. The celebrated **Kelly-Ulam reconstruction conjecture** says that  $G$  can be uniquely determined from its full deck if  $G$  has at least 3 vertices.

We will focus on a stronger version of the conjecture due to Harary [5], since it is no more onerous for the computer. Define  $\mathcal{RD}(G)$ , called the *reduced deck* of  $G$ , to be the *set* of cards of  $G$ . That is,  $\mathcal{RD}(G)$  tells us which unlabelled graphs appear in the full deck, but not how many of each there are. The **set reconstruction conjecture** is that  $G$  is uniquely determined by  $\mathcal{RD}(G)$  if it has at least 4 vertices.

Many surveys of the reconstruction conjecture have been written; see Lauri [8] for a fairly recent one.

Twenty years after investigating the 9-point graphs, I extended the search to 11 vertices [12]. Since the number of graphs on 11 vertices is 1,018,997,864, a completely new computational method was required. Now more than another twenty years have passed, so it is time for a further extension and that is the purpose of this project. Despite having more and faster computers, it is sobering that the number of graphs on 13 vertices is 50,502,031,367,952. Our method will be similar to [12] but with some improvements to make the task less onerous. The weaker edge-reconstruction conjecture, which we otherwise will not consider, was meanwhile checked up to 12 vertices by Stolee [23].

## 2 The method

If  $X$  is a structure built from  $\{1, 2, \dots, n\}$ , and  $\phi \in S_n$  where  $S_n$  is the symmetric group on  $\{1, 2, \dots, n\}$ , then  $X^\phi$  is obtained from  $X$  by replacing each  $i$  by  $i^\phi$ . For example, if  $G$  is a graph with vertices  $\{1, 2, \dots, n\}$ , then the graph  $G^\phi$  has an edge  $i^\phi j^\phi$  for each edge  $ij$  of  $G$ . The automorphism group of  $G$  is  $\text{Aut}(G) = \{\phi \in S_n \mid G^\phi = G\}$  where “=” denotes equality not isomorphism.

Let  $\mathcal{C}$  be a non-empty class of labelled graphs that is closed under isomorphism and taking induced subgraphs, and let  $\mathcal{C}_n$  be the subset of  $\mathcal{C}$  containing those with  $n$  vertices. Clearly  $\mathcal{C}_1 = \{K_1\}$ . We will assume that the vertices of  $G \in \mathcal{C}_n$  are  $\{1, 2, \dots, n\}$ . The special case that  $\mathcal{C}$  contains all graphs will be denoted by  $\mathcal{G}$  and similarly  $\mathcal{G}_n$ .

For  $G \in \mathcal{G}$  and  $W \subseteq V(G)$ , let  $G[W]v$  denote the graph obtained from  $G$  by appending a new vertex  $v$  and joining it to each of the vertices in  $W$ . Define  $\preceq$  to be a preorder (reflexive, transitive order) on labelled graphs, invariant under isomorphism. (An example would be that  $G_1 \preceq G_2$  iff  $G_1$  has at most as many edges as  $G_2$ .) Consider the following possible properties of a function  $m : \mathcal{G} \rightarrow 2^{\mathbb{N}}$ .

- (A) For each  $H \in \mathcal{G}$ ,  $m(H)$  is an orbit of  $\text{Aut}(H)$ .
- (A') For each  $H \in \mathcal{G}$ ,  $m(H)$  is the union of all the orbits of  $\text{Aut}(H)$  such that the cards  $H-v$  for  $v \in m(H)$  are maximal under  $\preceq$  amongst all cards of  $H$ .
- (B) For each  $H \in \mathcal{G}_n$  and  $\phi \in S_n$ ,  $m(H^\phi) = m(H)^\phi$ .

Now consider the algorithm shown in Figure 1. When we form  $H := G[W]v$  in the inner loop, we say that  $G$  is a *parent* of  $H$  and  $H$  is a *child* of  $G$ . We have the following theorem.

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algorithm generate( $G$  : labelled graph;  $n$  : integer)
  if  $|V(G)| = n$  then
    output  $G$ 
  else
    for each orbit  $A$  of the action of  $\text{Aut}(G)$  on  $2^{V(G)}$  do
      select any  $W \in A$  and form  $H := G[W]v$ 
      if  $H \in \mathcal{G}$  and  $v \in m(H)$  then
        generate( $H, n$ )
      endif
    endfor
  endif
end generate
    
```

Figure 1: Generation algorithm

**Theorem 2.1.** *Let  $\mathcal{C}$  be a non-empty class of labelled graphs that is closed under isomorphism and taking induced subgraphs. Then:*

- (a) *If  $m : \mathcal{G} \rightarrow 2^{\mathbb{N}}$  satisfies (A) and (B), then calling  $\text{generate}(K_1, n)$  will cause output of exactly one member of each isomorphism class of  $\mathcal{C}_n$ .*
- (b) *Suppose  $m : \mathcal{G} \rightarrow 2^{\mathbb{N}}$  satisfies (A) and (B) for  $|V(H)| < n$  and (A') for  $|V(H)| = n$ . Let  $G_1, G_2$  be non-isomorphic members of  $\mathcal{C}_n$  with the same reduced deck. Then calling  $\text{generate}(K_1, n)$  will cause  $G_1$  and  $G_2$  to be output as children of the same non-empty set of parents.*

*Proof.* Part (a) is proved in [13]. This is the *canonical construction path* method which has been widely adopted for isomorph-free generation.

In part (b), it is no longer true that  $G_1$  and  $G_2$  will be (up to isomorphism) output only once. However, as we will show, both will be output at least once, and from the same set of parents. Let  $G_1-v$  be a card of  $G_1$  maximal under  $\preceq$ . Since  $G_2$  has the same reduced deck as  $G_1$ , there is a card  $G_2-w$  maximal under  $\preceq$  and isomorphic to  $G_1-v$ . By part (a), the call  $\text{generate}(G', n-1)$  is made for some  $G'$  isomorphic to  $G_1-v$  and  $G_2-w$ . During that call we construct (up to to isomorphism) all 1-vertex extensions of  $G'$  that lie in  $\mathcal{C}_n$ , so in particular some  $H_1 = G'[W_1]v$  isomorphic to  $G_1$  and  $H_2 = G'[W_2]w$  isomorphic to  $G_2$ . Since they pass the tests  $v \in m(H_1)$  and  $w \in m(H_2)$ , the calls  $\text{generate}(H_1, n)$  and  $\text{generate}(H_2, n)$  are both made, causing  $H_1$  and  $H_2$  to be output. □

The great advantage of this method is that most parents only have a small number of children even if the total number of graphs is huge. So detailed comparison of reduced decks can be carried out in small batches without the need to store many graphs at once.

Our code is based on the implementation `geng` of algorithm `generate` in the author’s package `nauty` [15]. For  $\preceq$  we used a hash code based on the number of edges

and triangles in the cards. For large sizes, most graphs have trivial automorphism groups and the hash code distinguishes between cards quite well on average, so the total number of graphs constructed is not much greater than the number of isomorphism classes. After collecting the children of each parent, we compute an invariant of the reduced decks based on the degree sequences of the cards, and then reject any child which is unique. For those remaining, we do a complete isomorphism check of the cards with the most edges, and for any still not distinguished a complete isomorphism check of all the cards.

As an example, there are 1,018,997,864 graphs with 11 vertices. The testing program made 1,131,624,582, an increase of only 11%. The time for testing was only 2.4 times the generation time.

Theorem 2.1 refers to detection of non-reconstructible graphs within a class  $\mathcal{C}$ , so it is important to know when membership of the class is determined by the reduced deck. This eliminates the possibility that a graph in  $\mathcal{C}$  has the same reduced deck as a graph not in  $\mathcal{C}$ .

**Lemma 2.2.** *Let  $G_1$  and  $G_2$  be graphs on  $n \geq 4$  vertices with the same reduced decks. Then the following are true.*

- (a)  $G_1$  and  $G_2$  have the same minimum and maximum degrees.
- (b) For  $3 \leq k < n$ , either both or neither  $G_1$  and  $G_2$  contain a  $k$ -cycle.
- (c) Either both or neither  $G_1$  and  $G_2$  are bipartite.

*Proof.* Part (a) was proved by Manvel [9, 10]. Part (b) is obvious as the cycles of length less than  $n$  are those appearing in the cards. For part (c), note that a non-bipartite graph  $G$  with  $n$  vertices either has an odd cycle of length less than  $n$  or  $G$  is an  $n$ -cycle. The latter situation is uniquely characterised by the reduced deck being a single path.  $\square$

### 3 Results

**Theorem 3.1.** *For at least 4 vertices, all graphs in the following classes are reconstructible from their reduced decks (and therefore reconstructible).*

- (a) Graphs with at most 13 vertices.
- (b) Graphs with no triangles and at most 16 vertices.
- (c) Graphs of girth at least 5 and at most 20 vertices.
- (d) Graphs with no 4-cycles and at most 19 vertices.
- (e) Bipartite graphs with at most 17 vertices.
- (f) Bipartite graphs of girth at least 6 and at most 24 vertices.
- (g) Graphs with maximum degree at most 3 and at most 22 vertices.
- (h) Graphs with degrees in the range  $[\delta, \Delta]$  and at most  $n$  vertices, where  $(\delta, \Delta; n)$  is  $(0, 5; 14)$ ,  $(5, 6; 14)$ ,  $(6, 7; 14)$ ,  $(0, 4; 15)$ ,  $(4, 5; 15)$  or  $(3, 4; 16)$ .

This theorem required testing of more than  $6 \times 10^{13}$  graphs and took about 1.5 years on Intel cpus running at approximately 3 GHz.

## 4 Directed graphs

The reconstruction problem is defined for directed graphs in the same way as for graphs, but in this case many counterexamples are known. Particular attention has been paid to the case of tournaments. Obviously, for 3 or more vertices, the reduced deck is enough to determine if a digraph is a tournament. To the best of our knowledge, no previous work has been done on reconstruction of digraphs from reduced decks.

Harary and Palmer [6] stated the problem and gave tournament counterexamples with 3 or 4 vertices, while Beineke and Parker [3] gave one tournament counterexample with 5 vertices and three with 6 vertices. Two pairs of non-reconstructible tournaments of order 8 were found by Stockmeyer in 1975 [18].

The real breakthrough came in 1977 when Stockmeyer published constructions of non-reconstructible tournaments on all orders  $2^t + 1$  or  $2^t + 2$  for  $t \geq 2$  [19] (see also Kocay [7]). Stockmeyer later extended this result to all orders  $2^s + 2^t$  for  $0 \leq s < t$  and included families of non-tournament digraphs [20]. Namely, for each such order there are six pairs of non-reconstructible digraphs, including a pair of tournaments. Applying the same construction when  $s = t$  gives three pairs of digraphs but no tournaments. That gap was filled by Stockmeyer with a pair of non-reconstructible tournaments for each order  $2^t$  for  $t \geq 2$  [21].

In 1988, Stockmeyer presented some additional small non-reconstructible digraphs, including an extra pair of tournaments of order 6 that everyone had hitherto overlooked [22].

The infinite families and sporadic examples we have now mentioned make up the complete set of digraphs currently known to be not reconstructible from their full decks. Finding more would of course be very interesting. We next describe the searches that have been made.

- (a) Beineke and Parker [3] searched all the tournaments up to order 6 by hand but missed one non-reconstructible pair.
- (b) Stockmeyer [18] tested all tournaments with 7 vertices (finding none) and 8 vertices (finding two pairs).
- (c) Abrosimov and Dolov [1] tested all tournaments with up to 12 vertices, finding only Stockmeyer's examples.
- (d) Kocay (unpublished, 2018) tested all tournaments up to 10 vertices, and some families of digraphs, finding only Stockmeyer's examples.
- (e) For the current project, we tested all tournaments up to 13 vertices, all digraphs up to 8 vertices, and all digraphs on 9 vertices which have no 2-cycles. We also tested all semi-regular tournaments (those whose out-degrees are 6 or 7) on 14

vertices. For reconstruction from full decks, we found no new examples. For reduced decks, see below.

- (f) Stockmeyer’s tournaments of order  $2^t + 1$  are self-complementary. We checked that the 872,687,552 self-complementary tournaments on 14 vertices have unique reduced decks. Note that this does not preclude the possibility that a self-complementary tournament has the same reduced deck as one that is not self-complementary.

Searches (a)–(d) used the full deck, while (e)–(f) used the reduced deck. There are 48,542,114,686,912 tournaments on 13 vertices, 276,013,571,133 semi-regular tournaments on 14 vertices, 1,793,359,192,848 digraphs on 8 vertices, and 415,939,243,032 2-cycle-free digraphs on 9 vertices. The searches described in (e) took about 4 years on Intel cpus at approximately 3 GHz.

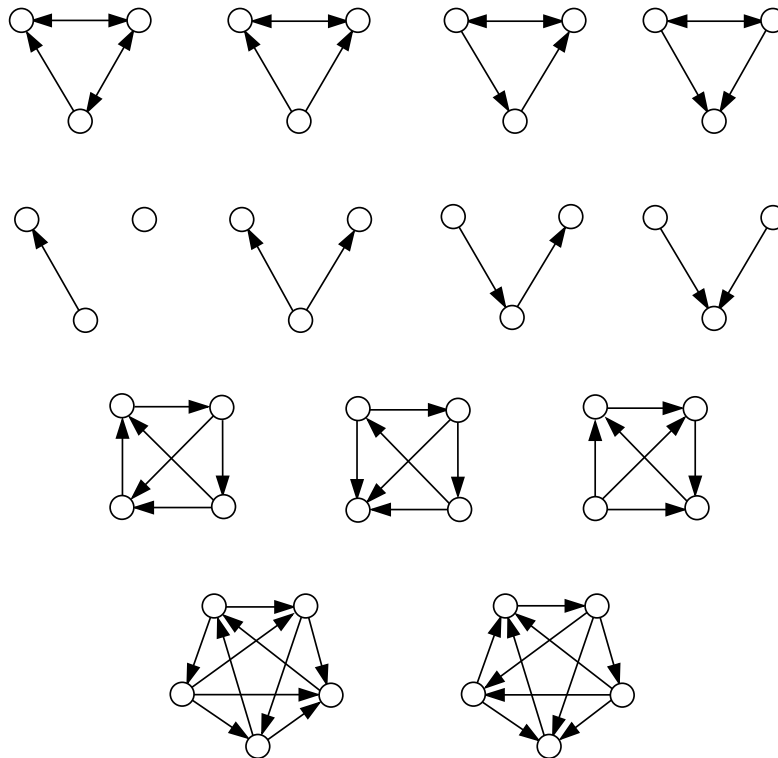


Figure 2: Four sets of digraphs with the same reduced deck.

For convenience we summarize the known digraphs for three or more vertices that are not reconstructible . When we refer to a “pair”, “triple”, etc., we mean a set of non-isomorphic digraphs with the same deck. We use “oriented graph” to mean a non-tournament digraph with no 2-cycles.

a	0	0	1	0	0	1	0	b	0	1	1	0	0	0	0
b	1	0	1	0	0	0	0	d	0	0	1	0	0	0	1
c	0	0	0	1	0	1	1	c	0	0	0	0	0	1	1
b	1	1	0	0	1	0	0	b	1	1	1	0	0	0	0
b	1	1	1	0	0	0	0	b	1	1	1	1	0	0	0
d	0	1	0	1	1	0	1	a	1	1	0	1	1	0	0
b	1	1	0	1	1	0	0	c	1	0	0	1	1	1	0
d	0	0	0	0	1	1	0	f	0	0	0	1	0	1	0
b	1	0	1	0	1	0	0	e	1	0	1	0	0	0	0
e	1	0	0	1	0	1	0	f	1	0	0	0	1	1	0
d	1	1	0	0	0	1	0	g	0	1	1	0	0	0	1
f	0	0	1	1	0	0	1	b	1	1	0	1	0	0	0
g	0	1	0	0	1	0	1	g	0	1	0	1	1	0	1
e	1	1	1	1	0	0	0	d	1	1	1	0	1	0	0
h	0	0	1	0	0	0	0	l	0	1	0	0	0	0	0
h	1	0	0	1	0	0	0	i	0	0	1	1	0	0	0
i	0	1	0	1	1	0	0	j	1	0	0	0	1	0	0
j	1	0	0	0	1	1	0	j	1	0	1	0	0	1	0
k	1	1	0	0	0	0	1	j	1	1	0	1	0	1	0
l	1	1	1	0	1	0	0	k	1	1	1	0	0	0	1
l	1	1	1	1	0	1	0	h	1	1	1	1	1	0	0

Figure 3: Three pairs of tournaments with the same reduced deck. The letters indicate card type.

**Reconstruction from full deck:**

- 3 vertices: One pair of tournaments, one triple of oriented graphs, one pair and one triple of digraphs with one 2-cycle.
- 4 vertices: One pair of tournaments, two pairs of oriented graphs, two pairs of digraphs with one 2-cycle, three pairs of digraphs with two 2-cycles.
- 5 vertices: One pair of tournaments, three pairs of oriented graphs, two pairs of digraphs with one 2-cycle, three pairs of digraphs with two 2-cycles.
- 6 vertices: Four pairs of tournaments, two pairs of oriented graphs, three pairs of digraphs with four 2-cycles.
- 7 vertices: All digraphs are reconstructible .
- 8 vertices: Two pairs of tournaments, one pair of oriented graphs, two pairs of digraphs with eight 2-cycles.
- 9 or more vertices: From this point on, only the infinite families found by Stockmeyer are known. For tournaments this is the full set up to 13 vertices. For oriented graphs it is the full set on 9 vertices.

**Reconstruction from reduced deck (excluding those above):**

- 3 vertices: For each of the two triples of digraphs not reconstructible from their full decks, there is an extra digraph having the same reduced deck.
- 4 vertices: There is a tournament having the same reduced deck as the two tournaments with the same full deck.
- 5 vertices: A pair of tournaments with the same reduced deck.
- 6 vertices: No further examples.
- 7 vertices: Three pairs of tournaments.
- 8 or more vertices: From this point on, no digraphs are known that are reconstructible from their full decks but not from their reduced decks. The search is complete for all the classes mentioned in (e) above.

The known sets of digraphs with the same reduced deck but not the same full deck are shown in Figures 2 and 3. All of the digraphs mentioned here can be found at [14].

**5 Partially-ordered sets**

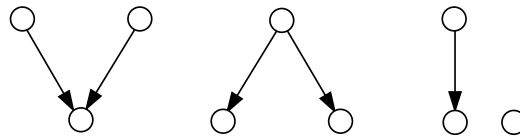


Figure 4: Three posets with the same reduced deck.

Graph reconstruction problems are special cases of reconstruction problems for binary relations. See Rampon [16] for a survey. In this paper, the only non-graph reconstruction problem we will mention is for partially-ordered sets (posets). Note that removing a point from a poset is the same as removing a vertex from the corresponding transitive digraph, but not the same as removing a vertex from the Hasse diagram. For the early history of the problem see [17].

The reader can check that both posets on 2 points have the same full deck and the first two posets in Figure 4 have the same full deck. All three posets in Figure 4 have the same reduced deck. No further examples of non-reconstructible posets are known. For 4–13 points, we have checked that every poset has a unique reduced deck (amongst posets). The number of posets with 13 points is 33,823,827,452. This was a quick computation of about 2 weeks that used the generator described in [4] to make all the reduced decks directly. The method of Section 2 would enable the computation to continue up to 15 points, but we will leave this for another time and another place (those who knew Paul Erdős will understand this allusion).



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