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Pattern avoiding meandric permutations

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Abstract

We study and characterize meandric permutations avoiding one or more patterns of length three, and find explicit formulae for the cardinality of each of these sets. We determine the distribution of the descent statistic for the set of meandric permutations avoiding the pattern 231. The sets of meandric permutations avoiding any other pattern of length three can be either trivially determined, or deduced from the 231 case via the symmetries of the square. In the 231 case we provide a bijection with a set of Motzkin paths that maps the statistic "number of descents of a permutation" to the statistic "number of non-horizontal steps of a path".

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1 Introduction

A meander is, roughly speaking, a system formed by the intersections of two curves in the plane, with equivalence up to homeomorphism within the plane [16]. It can be visualized as a highway which crosses a river several times [17].

Even though the first pictures of meanders can be found in Poincaré [20] in connection with the study of certain geometric transformations, and the notion of meander was used by Arnol'd [3] as a tool for analyzing differential-geometric objects, the study of meanders on their own is more recent. The problem of the enumeration of meanders has received much attention since the last decade of the twentieth century, and good bounds on the asymptotic number of these objects have been found (see [2], [5], [10], [11] and [17] and references therein); however, an exact enumeration remains elusive as the precise asymptotic behaviour (see [7] and [12]). Meanders are also studied in connection with other combinatorial, geometric, and algebraic objects (foldings of strips of stamps [18], knots [21], non-crossing partitions [9], Temperley-Lieb algebras [6], to cite only a few examples), and for their applications in other fields (e.g. boundary value problems [8]). For more information see the OEIS sequence of meandric numbers (sequence A005316 in [23]) and the survey [16].

A meander with n nodes can be naturally associated with a permutation in S_n. This association is injective and a permutation that can be obtained in this way is called *meandric*.

A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ *avoids* the pattern $\tau \in S_k$ if there are no indices $i_1, i_2, \ldots, i_k$ such that the subsequence $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is order isomorphic to τ.

The theory of permutation patterns goes back to the work of Knuth [15], who, in the 1970s, introduced the definition of pattern avoidance in connection to the stack sorting problem. The first systematic study of these objects appears in the paper by Simion and Schmidt [22]. Nowadays, the theory is very rich and widely expanded, with hundreds of papers published in the last decades [14].

In this paper, we enumerate the sets of meandric permutations avoiding any subset of S_3. To begin with, we deal with sets of permutations avoiding a single pattern τ. It turns out that in the cases $\tau = 123$ and 321 these sets contain only permutations whose structure is trivially described. The four remaining cases are related to each other by the symmetries of the square. Hence we focus our attention on the set $M_n(231)$ of meandric permutation avoiding 231. We enumerate this set with respect to the descent statistic, obtaining both a functional equation satisfied by the corresponding generating function, and an explicit formula for its coefficients. This sequence appears in [23] as the enumerating sequence of Motzkin paths without up steps in odd positions. We provide a bijection between this set of Motzkin paths and $M_n(231)$ which maps the statistic "non-horizontal steps" to the statistic "number of descents".

The paper is organized as follows. In Section 2 we recall the main definitions, present several examples and introduce the notion of diagram of a meandric permutation, a tool which will be useful in the following. In Sections 3 and 4 we characterize the sets of meandric permutations avoiding the single patterns 321 and 123. In Section 5 we tackle the enumeration of sets of meandric permutations avoiding every other single pattern of length 3, taking into account the descent statistic. In Section 6 we turn our attention to meandric permutations avoiding two or three patterns of length 3.

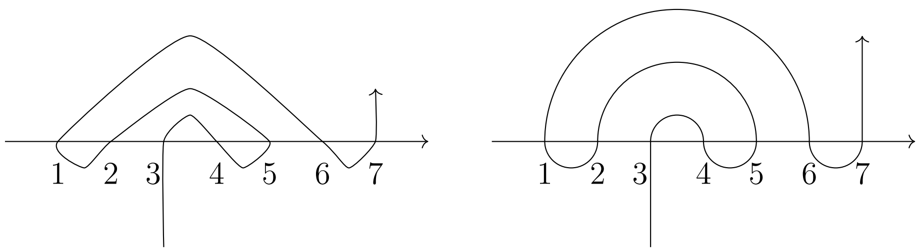
2 Main definitions

An *open meander* is a configuration consisting of an oriented simple curve and an oriented line in the plane, that cross a finite number of times and intersect only transversally. Two open meanders are equivalent if there is a homeomorphism of the plane that maps one meander to the other. In this paper we will consider only open meanders and call them simply meanders, for short.

We will always draw a meander so that the line is horizontal and oriented from left to right. The curve of a meander has two loose ends. Depending on the number of crossings, the loose ends belong to different half-planes defined by the horizontal line when the number of crossings is odd, and to the same half-plane otherwise. In the odd case we draw the curve so that the first loose end is below the line; in the even case we draw the curve so that both loose ends are below the line and the first one is on the left.

We represent the handles of the curve between two intersection points with the line by means of semicircles. We will call this representation the *canonical representation* of a meander.

Example 2.1. The following figure shows a meander and its canonical representation.



The intersections between the two curves have a natural labelling by the integers $\{1, 2, 3, \ldots, n\}$, defined by their ordering along the line.

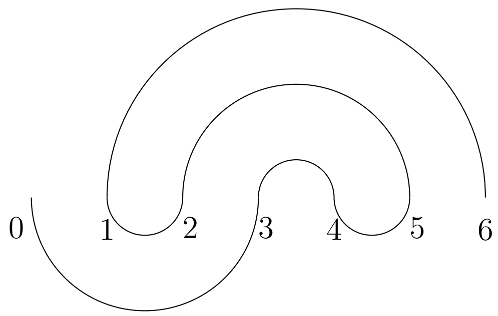
These labels also have a linear order defined by their order along the curve. When read in this order they yield a permutation in S_n. A permutation obtained in this way is said to be *meandric* (or *planar*). Not all permutations are meandric. To begin with, a meandric permutation must be *parity alternating*, namely, even integers appear in even positions and odd integers appear in odd positions (see [17]). Notice that parity alternating permutations are also studied on their own (see [13], [25] and

[26]). Moreover, if π is a meandric permutation of even length we have $\pi(1) < \pi(n)$, otherwise it would violate the orientation of the curve.

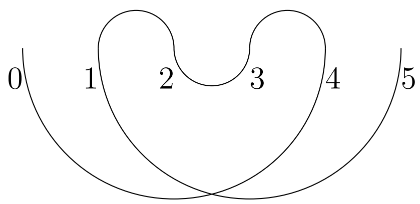
Given a permutation $\pi \in S_n$ such that $\pi(k)$ and k have the same parity for every k, we associate to π a curve in $\mathbb{R}^2$, that will be called *the diagram of* π, as follows:

- set $\pi(0) = 0$ and $\pi(n+1) = n+1$, by convention;
- draw the points $(i, 0)$ for $i = 0, \dots, n+1$;
- draw the semicircle of diameter $(\pi(2k+1), 0)$ $(\pi(2k+2), 0)$ above the $x-$axis for every $k = 0, \dots, \lfloor \frac{n-1}{2} \rfloor$ (*upper arc*);
- draw the semicircle of diameter $(\pi(2k), 0)$ $(\pi(2k+1), 0)$ below the $x-$axis for every $k = 0, \dots, \lfloor \frac{n}{2} \rfloor$, (*lower arc*).

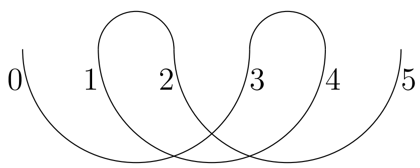
We denote by $\mathrm{diag}(\pi)$ the diagram of π. For example, the diagram of $\pi = 34521$ is



As examples of non-meandric (even if parity alternating) permutations, consider 4321 and 3412. In fact the corresponding non-self avoiding diagrams are, respectively,



and



Lemma 2.2. *The permutation* π *is meandric if and only if* $\mathrm{diag}(\pi)$ *is self-avoiding.*

Proof. The meander associated with a meandric permutation π is obtained from the diagram of π by

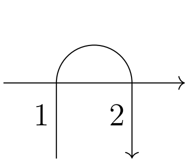
- drawing a straight line passing through the centers of the semicircles and oriented from left to right;
- replacing the semicircle of diameter $(0,0)\ (\pi(1),0)$ by an infinite half-line stemming from the point labelled $\pi(1)$ and placed below the straight line;
- replacing the semicircle of diameter $(\pi(n),0)\ (n+1,0)$ by an infinite half-line stemming from the point labelled $\pi(n)$ and placed below the straight line if n is even and above this line if n is odd.

Hence, if the diagram of π is self-avoiding, we can conclude that the corresponding curves are self-avoiding and have the correct orientation, so they form a meander.

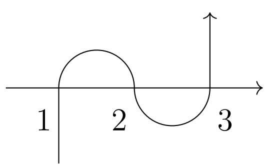
Conversely, consider a meandric permutation π of length n and the associated meander. Recall that the curve of this meander consists of $n-1$ semicircles and two vertical half-lines starting at $(\pi(1),0)$ and $(\pi(n),0)$. In order to construct the associated diagram, we simply replace the two half-lines by the semicircles γ_1 and γ_2 joining $(0,0)-(\pi(1),0)$ and $(n+1,0)-(\pi(n),0)$, respectively. First of all, γ_1 and γ_2 do not intersect each other. This is trivial when n is odd because in this case γ_1 and γ_2 are an upper arc and a lower arc. In the even case, they do not intersect because $\pi(1) < \pi(n)$, as observed above. Moreover, γ_1 and γ_2 do not intersect any arc γ_i of the meander or otherwise γ_i would have intersected on of the two half-lines. $\square$

We submit that in the literature one can find several different ways to associate a meander with a permutation. Our description follows that of Lando and Zvonkin (see [17]), while other authors associate to a meander the permutation obtained by listing the indices of the intersection points, in the same order as they are met by the curve [18]. Notice that the two permutations obtained in these two different ways are inverse of each other. In other papers, the strings associated in this way to a meander are considered as cycles of a permutation [16], [21].

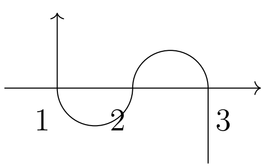
We represent here the meandric permutations of orders 2, 3, 4 and the corresponding meanders.



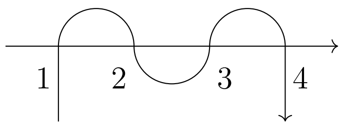
$\pi = 12$



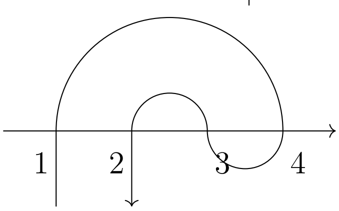
$\pi = 123$



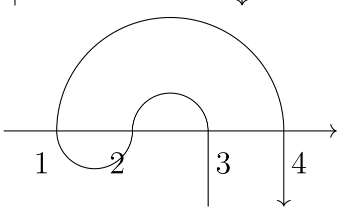
$\pi = 321$



$\pi = 1234$



$\pi = 1432$



$\pi = 3214$

Let M_n denote the set of meandric permutations of length n and $M_n(T)$ the set of meandric permutations of length n avoiding all the patterns in the set $T \subseteq \bigcup_{n\geq 0} S_n$. If T consists of a single pattern τ, we denote $M(T)$ simply by $M(\tau)$. Set

$$M := \bigcup_{n\geq 0} M_n$$

and

$$M(T) := \bigcup_{n\geq 0} M_n(T).$$

Let $\pi \in S_n$ be a permutation of length n. Write π in one line notation as $\pi = \pi_1\pi_2\cdots\pi_n$. Then the *reverse* of π is the permutation $\pi^r := \pi_n\pi_{n-1}\cdots\pi_1$, the *complement* of π is $\pi^c := (n+1-\pi_1)\,(n+1-\pi_2)\cdots(n+1-\pi_n)$ and the *reverse-complement* of π is $\pi^{rc} := (\pi^r)^c = (\pi^c)^r$.

The following result will be useful in the sequel.

Proposition 2.3. *If $\pi \in M_n$, then*

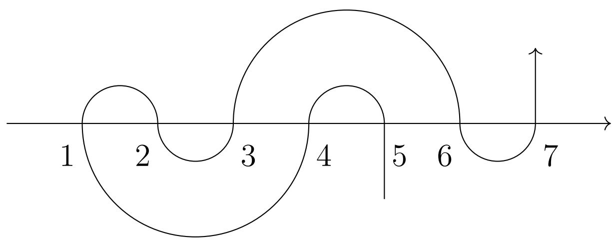
- *if n is odd, π^r and π^c are meandric permutations;*
- *if n is even, π^r and π^c are not meandric permutations;*
- *π^{rc} is a meandric permutation for every n.*

Proof. If n is odd, π^r is a meandric permutation whose diagram is obtained by flipping diag(π) along a horizontal axis, π^c is a meandric permutation whose diagram is obtained by flipping diag(π) along a vertical axis and π^{rc} is a meandric permutation whose diagram is obtained by rotating diag(π).

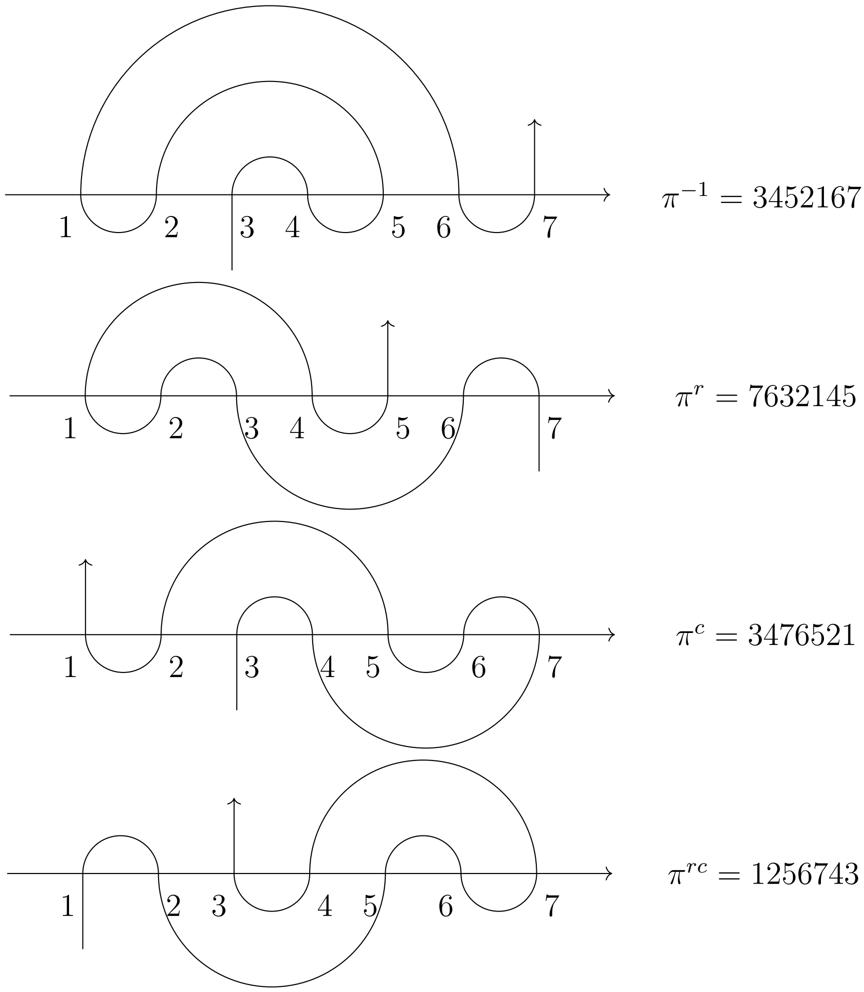
If n is even, π^r and π^c are not meandric permutations because they are not parity alternating, while π^{rc} is a meandric permutation whose diagram is obtained by flipping diag(π) along a vertical axis. □

Notice also that if a permutation π is meandric then its inverse π^{-1} is also meandric. In fact the meander associated with π^{-1} can be obtained from the meander of π by stretching the curve and, correspondingly, bending the line in a homeomorphic way.

As an example consider the permutation $\pi = 5412367$. This permutation is meandric with meander



We represent the meanders of π^{-1}, π^r, π^c, π^{rc}.



In the sequel we will make use of the notion of standardization of a permutation. Given a word w of length n whose letters are distinct integers, the *standardization* of w, indicated with $\mathrm{std}(w)$, is the unique permutation in S_n obtained from w by replacing the smallest symbol in w by the symbol 1, the second smallest symbol in w by 2, and so on. As an example $\mathrm{std}(927) = 312 \in S_3$.

3 Meandric permutations avoiding 321

There is only one meandric permutation of length n avoiding the pattern 321, for every n.

Proposition 3.1. $|M_n(321)| = 1$ *for every* n.

Proof. Let $\pi = \pi_1 \cdots \pi_n$ be a permutation in $M_n(321)$. Let $k := \pi_1$ and $h := \pi^{-1}(1)$. If $k \neq 1$ then $k \geq 3$ since it is odd. The symbols that appear between k and 1 in the one-line notation of π are greater than k, because π avoids the pattern 321. Hence the lower arc connecting the nodes 0 and k intersects the lower arc i connecting $\pi(h-1)$ and 1. This contradicts the fact that π is meandric.

As a consequence $\pi_1 = 1$. Iterating this argument we conclude that the only element of $M_n(321)$ is $1\,2 \cdots n$. □

4 Meandric permutations avoiding 123

As in the previous section, this case is almost trivial. In fact,

Proposition 4.1. *$|M_{2t}(123)| = t$ for every t and $|M_{2t+1}(123)| = 1$ for every t.*

Proof. Let π be a permutation in $M_n(123)$ and set $\pi(1) = k$, with k odd. Then $\pi^{-1}(n) = k+1$ and $\pi(i) = k - i + 1$ for every $i = 2, \ldots, k$, because π is 123-avoiding and the diagram of π is self-avoiding. For the same reason, $\pi(k+i) = n - i + 1$ for every $i = 2, \ldots, n-k$.

Now,

- if $n = 2t$ these conditions are sufficient to assure that π belongs to $M_{2t}(123)$;
- let $n = 2t+1$. If $k \neq n$ then π contains 1 and n as consecutive elements, which contradicts the fact that π is parity-alternating. Hence, the only permutation in $M_{2t+1}(123)$ is $2t+1\; 2t \cdots 2\,1$.

□

Notice that the odd case agrees with the fact that the reverse map provides a bijection between $M_{2t+1}(123)$ and $M_{2t+1}(321)$.

5 Meandric permutations avoiding 132, 213, 231 or 312

Now we consider the case of permutations avoiding exactly one among the patterns 132, 213, 231 or 312.

In these cases we enumerate restricted meandric permutations according to the descent statistic, i.e., the number of descents of a permutation. Recall that a descent of a permutation π is an index i such that $\pi_i > \pi_{i+1}$. We denote by $\operatorname{des}\pi$ the number of descents of π and by $M_{n,d}(\tau)$ the subset of $M_n(\tau)$ of permutations with d descents.

Lemma 5.1. *i) Let π be a permutation in $M_n(231)$. Then either*

- ∗ *$\pi = \sigma\, n$, where $\sigma \in M_{n-1}(231)$, or*
- ∗ *$\pi = \sigma\, n\, n-1\, \tau$, where $\sigma \in M(231)$ and τ is a permutation of odd length in $M(231)$, up to a standardization.*

ii) Let π be a permutation in $M_n(132)$. Then

- ∗ *If $n > 0$ is even, then $\pi = \sigma\, n$ where σ is any permutation in $M_{n-1}(132)$.*
- ∗ *If n is odd, then $\pi = \sigma\, n\, \tau$, where τ is any permutation of even length such that τ^r is in $M(231)$ and the standardization of σ is any permutation of even length in $M(132)$.*

Proof. Consider first meandric permutations avoiding the pattern 132. Let π be a permutation in $M_n(132)$. Since π avoids 132, we can write π as $\sigma\, n\, \tau$, where each symbol in τ is smaller than each symbol in σ and, up to a standardization, τ and σ are permutations avoiding 132.

- If $n > 0$ is even, then $\pi = \sigma\, n$ where σ is any permutation in $M_{n-1}(132)$. In fact, if $\tau \neq \emptyset$, diag(π) would not be self-avoiding, because the arcs joining $\pi(1)$ with 0 and $\pi(n)$ with $n+1$ would intersect.
- If n is odd, then $\pi = \sigma\, n\, \tau$, where τ is any permutation of even length such that τ^r is in $M(231)$ and the standardization of σ is any permutation of even length in $M(132)$.

Furthermore, a permutation π in $M_{2t}(231)$ can be written as $\sigma\, 2t\, \tau$, where the symbols of σ are smaller than the symbols of τ, σ is any permutation of odd length in $M(231)$ and the standardization of τ is any permutation of even length whose reverse is in $M(132)$.

To conclude the proof, observe that, by Proposition 2.3, the reverse map provides a bijection between $M_{2k+1,d}(231)$ and $M_{2k+1,2k-d}(132)$. □

Theorem 5.2. *We have*

$$\begin{aligned} |M_{2k+1,2d}(231)| &= |M_{2k+1,2(k-d)}(132)| = |M_{2k+2,2(k-d)}(132)| \\ &= \frac{1}{k+1}\binom{k+1}{d+1}\binom{2k+1-d}{d} \quad \text{for all } k,d \geq 0; \qquad (1) \\ |M_{2k,2d}(231)| &= \frac{1}{k+1}\binom{k+1}{d+1}\binom{2k-1-d}{d} \quad \text{for all } k \geq 1, d \geq 0; \qquad (2) \\ |M_{2k+1}(231)| &= |M_{2k+1}(132)| = |M_{2k+2}(231)| = \sum_{d=0}^{k} \frac{1}{d+1}\binom{k}{d}\binom{2k+1-d}{d} \qquad (3) \end{aligned}$$

for all $k \geq 0$, and

$$|M_{2k}(231)| = \sum_{d=0}^{k} \frac{1}{d+1}\binom{k}{d}\binom{2k-1-d}{d} \quad \text{for all } k \geq 0. \qquad (4)$$

Proof. By the previous lemma, $|M_{2k+1,d}(132)| = |M_{2k+2,d}(132)|$, for all $k, d \geq 0$.

Denote

$$A(x,y) := \sum_{\pi \in M_{2k+1}(231)} x^{|\pi|} y^{\operatorname{des} \pi} = \sum_{k,d \geq 0} |M_{2k+1,d}(231)| x^{2k+1} y^d,$$

and

$$B(x,y) := \sum_{\pi \in M_{2k}(231)} x^{|\pi|} y^{\operatorname{des} \pi} = \sum_{k,d \geq 0} |M_{2k,d}(231)| x^{2k} y^d.$$

The previous lemma allows us to write the following relations between the generating functions A and B.

$$\begin{aligned} A(x,y) &= xB(x,y) + x^2y^2A(x,y)B(x,y), \\ B(x,y) &= 1 + xA(x,y) + x^2y^2A(x,y)^2. \end{aligned}$$

Notice that from the previous equations it follows in particular that a meandric permutation avoiding 231 has an even number of descents. Straightforward computations lead to the following functional equation.

$$A(x,y) = x + (x^2 + x^2y^2)A(x,y) + 2x^3y^2A(x,y)^2 + x^4y^4A(x,y)^3.$$

Set $\tilde{A}(x,y) = (xA)(x^{\frac{1}{2}}, y^{\frac{1}{2}})$. Then

$$\tilde{A}(x,y) = x(1 + (1+y)\tilde{A}(x,y) + 2y\tilde{A}(x,y)^2 + y^2\tilde{A}(x,y)^3).$$

By the Lagrange inversion formula (see [24, Ch. 5] or [27, Sec. 5.1]) we get

$$[x^n]\tilde{A}(x,y) = \frac{1}{n}[z^{n-1}]R(z)^n$$

where $R(z) = 1 + (1+y)z + 2yz^2 + y^2z^3$.

Hence

$$[x^n]\tilde{A}(x,y) = \frac{1}{n}[z^{n-1}] \sum_{\substack{i_1,i_2,i_3,i_4 \\ i_1+i_2+i_3+i_4=n}} \binom{n}{i_1,i_2,i_3,i_4} ((1+y)z)^{i_2}(2yz^2)^{i_3}(y^2z^3)^{i_4}$$

$$= \frac{1}{n} \sum_{\substack{i_1,i_2,i_3,i_4 \\ i_1+i_2+i_3+i_4=n \\ i_2+2i_3+3i_4=n-1}} \binom{n}{i_1,i_2,i_3,i_4} (1+y)^{i_2}(2y)^{i_3}(y)^{2i_4}$$

$$= \frac{1}{n} \sum_{i_3,i_4} \binom{n}{1+i_3+2i_4, n-1-2i_3-3i_4, i_3, i_4} (1+y)^{n-1-2i_3-3i_4}(2y)^{i_3}(y)^{2i_4}.$$

Since $(1+y)^{n-1-2i_3-3i_4} = \sum_j \binom{n-1-2i_3-3i_4}{j} y^j$, we have

$$[x^ny^d]\tilde{A} = \frac{1}{n} \sum_{i_3,i_4} \binom{n}{1+i_3+2i_4, n-1-2i_3-3i_4, i_3, i_4} \binom{n-1-2i_3-3i_4}{d-i_3-2i_4} 2^{i_3}.$$

Now,

$$|M_{2k+1,2d}(231)| = [x^{2k+1}y^{2d}]A = [x^{k+1}y^d]\tilde{A} \tag{5}$$

$$= \frac{1}{k+1} \sum_{i_3,i_4} \binom{k+1}{1+i_3+2i_4, k-2i_3-3i_4, i_3, i_4} \binom{k-2i_3-3i_4}{d-i_3-2i_4} 2^{i_3}.$$

We want to show that this last expression is equal to

$$\frac{1}{k+1}\binom{k+1}{d+1}\binom{2k+1-d}{d}. \tag{6}$$

To this aim, dividing the right hand side of Equation (5) by $\frac{1}{k+1}\binom{k+1}{d+1}$, we get after trivial calculations

$$\begin{aligned}
&\sum_{i_3,i_4}\binom{d+1}{1+i_3+2i_4}\binom{k-d}{i_3,i_4,k-d-i_3-i_4}2^{i_3}\\
=\ &\sum_j\binom{d+1}{d-j}\sum_{i_4}\binom{k-d}{j-2i_4,i_4,k-d-j+i_4}2^{j-2i_4}\\
=\ &\sum_j\binom{d+1}{d-j}\binom{2k-2d}{j}\\
=\ &\binom{2k+1-d}{d}.
\end{aligned}$$

where in the first equality we applied the change of indices $j = i_3 + 2i_4$, in the second one we exploited the formula $\sum_t \binom{a}{t}\binom{a-t}{j-2t}2^{j-2t} = \binom{2a}{j}$, which is an easy consequence of Zeilberger's algorithm (see [19]), and in the third one we applied Vandermonde's formula $\sum_{t=0}^{r}\binom{a}{r-t}\binom{b}{t} = \binom{a+b}{r}$ (see [1, p. 14]). This shows that the right-hand side of Equation (5) is in fact equal to Expression (6).

This proves Formula (1).

Notice that the coefficients of $\tilde{A}(x, 1)$ were found through a slightly different application of the Lagrange Inversion formula in [4, p. 96].

Formula (2) can be obtained by a similar argument and, as a consequence, we obtain also Formulae (3) and (4). □

To conclude the enumeration of the meandric permutations avoiding a pattern of length three according to the descent distribution, note that $|M_{n,d}(312)| = |M_{n,d}(231)|$ and $|M_{n,d}(213)| = |M_{n,d}(132)|$, since the reverse-complement map is a bijection between these pairs of sets.

The sequences $\{|M_{2k}(231)|\}_{k\geq 0}$ and $\{|M_{2k+1}(231)|\}_{k\geq 0}$ appear in [23] as sequences A109081 and A106228, respectively. Both these sequences admit combinatorial interpretations in terms of Motzkin paths. This suggests the existence of a bijection between the set $M_n(231)$ and the set of Motzkin paths of length n with no up steps in odd position. In fact, these paths are enumerated by sequence A215067, which interleaves A109081 and A106228.

Recall that a *Motzkin path* of length n is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, consisting of up steps U of the form $(1, 1)$, down steps D of the form $(1, -1)$, and horizontal steps H of the form $(1, 0)$ and lying weakly above the x-axis.

Denote by $\mathfrak{M}_{n,d}$ the set of Motzkin paths of length n, with d up steps *appearing only in even position.*

Now we define recursively a bijection Ψ between the sets $M_{n,2d}(231)$ and $\mathfrak{M}_{n,d}$.

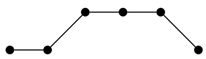
Set $\Psi(\emptyset) = \emptyset$. Let $\pi \in M_n(231)$ with $n > 0$. Then by Lemma 5.1 either $\pi = \sigma\, n$, where $\sigma \in M_{n-1}(231)$, or $\pi = \sigma\, n\, n-1\, \tau$, where $\sigma \in M(231)$ and $\mathrm{std}(\tau)$ is a permutation of odd length in $M(231)$.

- In the first case, $\Psi(\pi) = \Psi(\sigma)H$;
- in the second case, $\Psi(\pi) = \Psi(\tau)U\Psi(\sigma)D$.

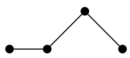
It is immediately seen that Ψ is a bijection between $M_{n,2d}(231)$ and $\mathfrak{M}_{n,d}$. In fact,

- if $\pi = \sigma\, n$, then $\mathrm{des}(\pi) = \mathrm{des}(\sigma)$ and the Motzkin paths $\Psi(\pi)$ and $\Psi(\sigma)$ have the same number of up steps;
- if $\pi = \sigma\, n\, n-1\, \tau$, then $\mathrm{des}(\pi) = \mathrm{des}(\sigma) + \mathrm{des}(\tau) + 2$ and, denoting by h and k the number of up steps in $\Psi(\tau)$ and $\Psi(\sigma)$, respectively, the Motzkin path $\Psi(\pi)$ has $h + k + 1$ up steps.

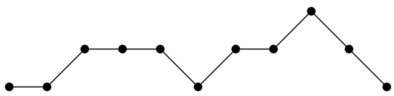
As an example, consider the permutation $\pi = 3\,2\,1\,10\,9\,4\,5\,8\,7\,6 \in M_{10,6}(231)$. Here we have $\sigma = 321$ and $\mathrm{std}(\tau) = 12543$. Since

$\Psi(\tau) =$ 

and

$\Psi(\sigma) =$ 

we have

$\Psi(\pi) =$ 

6 Meandric permutations avoiding multiple patterns of length three

In this section we consider multiple avoidances. Let T be any subset of S_3.

First of all, we consider a trivial case.

Proposition 6.1. *$|M_n(T)| = 1$ if $321 \in T$ and $123 \notin T$ for all $n \geq 0$, and $M(T) = \{1, 12\}$ if $321 \in T$ and $123 \in T$.*

Proof. In Section 3 we had that $M_n(321) = \{1\,2 \ldots n\}$. □

In Section 4 we proved that

$$M_n(123) = \begin{cases} \{n\, n-1 \cdots 1\} & \text{if } n = 2t+1, \\ \{k\, k-1\, k-2 \cdots 1\, n\, n-1 \cdots k+1 \,|\, 1 \leq k \leq n\} & \text{if } n = 2t. \end{cases}$$

Notice that all the permutations in $M_n(123)$ avoid also 312 and 231, and hence

$$M_n(123) = M_n(\{123, 312\}) = M_n(\{123, 231\}) = M_n(\{123, 231, 312\}).$$

Moreover, the only permutation in $M_n(123)$ which avoids 213 (respectively, 132) is $(n)\,(n-1)\cdots(1)$ if n is odd and $(1)\,(n)\,(n-1)\cdots(2)$ ($(n-1)\,(n-2)\cdots(1)\,(n)$, respectively) if n is even.

As a consequence we can now consider only sets T containing neither 123 nor 321.

Theorem 6.2. $|M_n(231, 312)| = F_n$, $|M_{2t+1}(132, 213)| = F_{2t+1}$, *and* $|M_{2t}(132, 213)| = 1$, *where* F_n *is the* n*-th Fibonacci number (see sequence A000045 in [23]).*

Proof. In the seminal paper [22], the authors give the following characterization of permutations in $S_n(231, 312)$.

If $\pi \in S_n(231, 312)$, then either $\pi = n\,n-1\cdots 1$, or $\pi = \tau\,n\,n-1\cdots n-k$ where τ is any permutation in $S_{n-k-1}(231, 312)$.

Notice that, on the one hand, if τ is a meandric permutation, '$\pi = \tau\,n\,n{-}1\cdots n{-}k$ is meandric if and only if k is even, or otherwise the arc $(n, n+1)$ would intersect the arc (τ_{n-k-1}, n) in $\mathrm{diag}(\pi)$. On the other hand, if τ is not meandric, neither is π.

As a consequence, if $\pi \in M_n(231, 312)$, either $\pi_n = n$, or the symbols n, $n{-}1$, $n{-}2$ appear consecutively in this order in π. Hence every permutation in $M_n(231, 312)$ can be obtained either by appending n at the end of a permutation in $M_{n-1}(231, 312)$ or by replacing the symbol $n-2$ with the word $n\,(n-1)\,(n-2)$ in a permutation of $M_{n-2}(231, 312)$, and therefore

$$|M_n(231, 312)| = |M_{n-1}(231, 312)| + |M_{n-2}(231, 312)|,$$

namely,

$$|M_n(231, 312)| = F_n.$$

If $n = 2t + 1$, the reverse map provides a bijection between $M_n(132, 213)$ and $M_n(231, 312)$, and hence

$$|M_{2t+1}(132, 213)| = F_{2t+1}.$$

Recall that a permutation π in $S_n(132, 213)$ is either the identity or of the form $\pi = (n{-}k)\,(n{-}k{+}1)\cdots(n)\tau$ where $\tau \in S_{n-k-1}(132, 213)$ (see [22]). If $n = 2t$ and τ is non-empty, the permutation $\pi = (n{-}k)\,(n{-}k{+}1)\cdots(n)\tau$ is never meandric because the arc $(0, n{-}k)$ intersects the arc $(\tau_{n-k-1}, n{+}1)$, and hence $|M_{2t}(132, 213)| = 1$. □

Theorem 6.3. $|M_n(132, 231)| = 2^{\lfloor \frac{n-1}{2} \rfloor}$, *and* $|M_n(213, 312)| = |M_n(132, 312)| = |M_n(213, 231)| = 2^{\lfloor \frac{n-1}{2} \rfloor}$.

Proof. As observed in [22], if $\pi \in S_n(132, 231)$ then $\pi_1 = n$ or $\pi_n = n$. On the one hand, if $n = 2t$, then, if $\pi_1 = n$, the arc $(0, n)$ intersects the arc $(\pi_n, n+1)$; hence, if $\pi \in M_n(132, 231)$, then $\pi = \tau\,n$ where $\tau \in M_{n-1}(132, 231)$. As a consequence,

$$|M_{2t}(132, 231)| = |M_{2t-1}(132, 231)|.$$

On the other hand, if $n = 2t + 1$, then

- if $\pi_1 = n$, then $\pi_2 = n-1$. In fact, the permutation obtained from π by removing n is in $S_{n-1}(132, 231)$, so the symbol $n-1$ is either its first element, or its last element. But $\pi_n = n-1$ is not possible, since in this case the arc (π_2, n) would intersect the arc $(n-1, n+1)$.
- If $\pi_n = n$, then $\pi_{n-1} = n-1$. In fact the permutation obtained from π by removing n is a meandric permutation of even length, and hence its first element cannot be the greatest one.

These considerations imply that a permutation π in $M_{2t+1}(132, 231)$ is either of the form $\pi = (2t+1)\,(2t)\,\tau$ or $\pi = \tau\,(2t)\,(2t+1)$, where τ is an element of $M_{2t-1}(132, 231)$; therefore $|M_{2t+1}(132, 231)| = 2|M_{2t-1}(132, 231)|$ which implies

$$|M_n(132, 231)| = 2^{\lfloor \frac{n-1}{2} \rfloor}.$$

The reverse-complement map and the inversion map allow us to prove the second statement. □

Theorem 6.4.

$$|M_n(132, 312, 213)| = \begin{cases} \frac{n+1}{2} & \textit{if } n \textit{ is odd}, \\ 1 & \textit{if } n \textit{ is even}; \end{cases}$$

$$\textit{and} \quad |M_n(132, 312, 231)| = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. A permutation π in $S_n(132, 312, 213)$ is trivially of the form $\pi = (k)\,(k+1)\cdots(n)\,(1)\,(2)\cdots(k-1)$ and it is easy to verify that such a permutation is meandric if and only if both n and k are odd, or n is even and $k = 1$. This proves the first assertion.

Lastly, consider a permutation $\pi \in M_{2t}(132, 312, 231)$. As seen before, $\pi = \tau\, n$ where $\tau \in M_{2t-1}(132, 312, 231)$, and hence

$$|M_{2t}(132, 312, 231)| = |M_{2t-1}(132, 312, 231)|.$$

Moreover, if $\pi \in M_{2t+1}(132, 312, 231)$, then either $\pi = \tau\, n$ where $\tau \in M_{2t}(132, 312, 231)$, or $\pi = (n)\,(n-1)\cdots(1)$. Hence

$$|M_{2t+1}(132, 312, 231)| = |M_{2t}(132, 312, 231)| + 1.$$

This completes the proof. □

We summarize all our results in Table 1.

$T \subseteq S_3$	$\lvert M_n(T)\rvert$
123, {123,312}, {123,231}, {123,231,312}	$\begin{cases} n/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$
$123, 213 \in T$ and $132, 321 \notin T$	1
$123, 132 \in T$ and $213, 321 \notin T$	1
$123, 132, 213 \in T$ and $321 \notin T$	$\begin{cases} 1 & \text{if } n \text{ is odd or } n = 2 \\ 0 & \text{otherwise} \end{cases}$
$321 \in T$ and $123 \notin T$	1
231, 312	$\begin{cases} \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{2k-1-i}{i} & \text{if } n = 2k \\ \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{2k+1-i}{i} & \text{if } n = 2k+1 \end{cases}$
132, 213	$\begin{cases} \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{2k+1-i}{i} & \text{if } n = 2k+2 \\ \sum_{i=0}^{k} \frac{1}{i+1}\binom{k}{i}\binom{2k+1-i}{i} & \text{if } n = 2k+1 \end{cases}$
{231, 312}	F_n
{132, 213}	$\begin{cases} F_n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$
{132,231}, {213,312}, {132,312}, {213,231}	$2^{\lfloor \frac{n-1}{2} \rfloor}$
{132, 312, 213}, {213, 231, 312}	$\begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$
{132, 312, 231}, {213, 231, 132}	$\lfloor \frac{n+1}{2} \rfloor$
{132, 312, 231, 213}	$\begin{cases} 1 & \text{if } n \text{ is even or } n = 1 \\ 2 & \text{otherwise} \end{cases}$
$123, 321 \in T$	$0 \quad \text{if } n \geq 3$

Table 1: Summary of our enumerative results.

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