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Note on the domination number of graphs with forbidden cycles of lengths not divisible by 3

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Abstract

In this note, we prove that the domination number of a graph of order n and minimum degree at least 2 that does not contain cycles of length $3r+2$, where $1 \le r \le k$, and cycles of length $3r+1$ for $1 \le r \le 2k+2$, is at most $\frac{k+2}{3k+5}n$. This improves some previous results.

1 Introduction

In this paper, $G = (V, E)$ is a simple graph with vertex set V of order n and edge set E. The *degree* of a vertex $v \in V$, denoted $\deg_G(v)$ (or simply $\deg(v)$ if no confusion arises), is the number of vertices adjacent to it. Let $\delta = \delta(G)$ denote the minimum degree of G.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex v in $V - S$ has at least one neighbor in S. The *domination* number $\gamma(G)$ equals the minimum cardinality

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of a dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. It is well-known that computing the domination number is NP-complete even for some classes of graphs such as bipartite or split graphs [1], which leads to looking for good upper and lower bounds in general graphs or in graphs with restrictions on the minimum degree or with forbidden induced subgraphs.

Ore's [5] classical upper bound on the domination number says that $\gamma(G) \leq n/2$ for graphs G with $\delta(G) \geq 1$. Restricted to graphs G with $\delta(G) \geq 2$, McCuaig and Shepherd [4] showed that $\gamma(G) \leq 2n/5$, with the exception of seven graphs. In 1996, Reed [6] made another improvement on the upper bound by showing that $\gamma(G) \leq 3n/8$ for graphs G with $\delta(G) \geq 3$. This last upper bound has been extended for graphs of minimum degree at least two with forbidden cycles. Indeed, Harant and Rautenbach [2] showed the following result, which was improved in 2011 by Henning et al. [3].

Theorem 1 ([2]). *If G is a graph of order n, minimum degree at least 2 that does not contain cycles of length 4, 5, 7, 10 or 13, then $\gamma(G) \leq 3n/8$.*

Theorem 2 ([3]). *If G is a (C_4, C_5)-free connected graph of order $n \geq 14$ with $\delta(G) \geq 2$, then $\gamma(G) \leq 3n/8$.*

In this paper we provide an upper bound on the domination number of graphs with minimum degree at least two that do not contain certain special cycles.

2 Main result

Before stating our main result, we give some definitions and notation that will be useful in the sequel. Let G be a connected graph with $\delta(G) \geq 2$, and let B be the set of vertices of G with degree at least 3. A path $P = v_1 \dots v_k$ in G is called a 2-*path* if $V(P) \subseteq V(G) - B$, that is, $\deg(v_1) = \cdots = \deg(v_k) = 2$. The 2-path P is said to be maximal if each of the endvertices of P has a neighbor in B. Let P be a (q_1, q_k)-path in G and let $q_1, q_2, \dots, q_k$ $(k \geq 2)$ be some (distinguished) vertices belonging to P, where q_{i+1} follows q_i in the ordering of vertices of the path P, and let P_i be a subpath of P between q_i and q_{i+1} but not including them such that $|V(P_i)| \equiv 2 \pmod 3$ and all vertices on P_i have degree 2, for each $i \in \{1, \dots, k-1\}$. For the sake of simplicity, we will write $P = q_1P_1q_2P_2 \dots q_k$ and call this a *special path*. Let C be a cycle in G and let $q_1, q_2, \dots, q_k$ $(k \geq 2)$ be some (distinguished) vertices belonging to C, where q_{i+1} follows q_i in the ordering of vertices of the cycle C, and let P_i be a subpath of C between q_i and q_{i+1} but not including them, such that $|V(P_i)| \equiv 2 \pmod 3$ and all vertices on P_i have degree 2, for each $i \in \{1, \dots, k\}$, where P_k connects q_k to q_1. Again, for the sake of simplicity, we will write $C = q_1P_1q_2P_2 \dots q_kP_kq_1$ and call this a *special cycle*. Note that the order of a special path is $1 \pmod 3$ while the order of a special cycle is $0 \pmod 3$. A cycle C is called a *tailed-cycle* if all vertices of C except one have degree 2. If C is a tailed-triangle and x is the unique vertex of C of degree at least 3, then x is said to be a *triangular vertex*. It is worth noting that a graph might have more tailed-triangles than triangular vertices.

Let $\mathcal{A}_k$ be the set of all cycles C of G with length $\ell(C) = 3r+2$ where $1 \le r \le k$, and let $\mathcal{B}_k$ be the set of all cycles C of G with length $\ell(C) = 3r+1$ where $1 \le r \le k$.

Now we are ready to state and prove the main result.

Theorem 3. *Let G be a graph on n vertices with $\delta(G) \ge 2$ and $\mathcal{A}_k \cup \mathcal{B}_{2k+2} = \emptyset$. Then*

$$\gamma(G) \le \frac{k+2}{3k+5} n.$$

Furthermore, the bound is sharp for an infinite family of graphs.

Proof. Suppose, to the contrary, that there is a graph G with $\delta(G) \ge 2$ and $\mathcal{A}_k \cup \mathcal{B}_{2k+2} = \emptyset$ such that $\gamma(G) > \frac{k+2}{3k+5} n$. We will assume that such a graph G was chosen such that: (i) $|V(G)| + |E(G)| = d^*(G)$ is as small as possible, and (ii) subject to (i), the sum of the number of tailed-triangles and triangular vertices is as large as possible. Clearly, G is a connected graph. Moreover, from the assumption $\gamma(G) > \frac{k+2}{3k+5} n$, we deduce that G is not a cycle (or else, if G is a cycle, then it has length $0 \pmod 3$, or has length $1 \pmod 3$ on at least $6k+10$ vertices, or has length $2 \pmod 3$ on at least $3k+5$ vertices, which leads, according to each situation, to $\gamma(C_n) \le \frac{k+2}{3k+5} n$) and thus $|B| \ge 1$. Now, if $|B| = 1$, then G is a graph obtained from a disjoint union of cycles by identifying one vertex from each cycle into one vertex, and clearly in this case $\gamma(G) \le n/3 < \frac{k+2}{3k+5} n$, a contradiction. Hence $|B| \ge 2$. If there are two adjacent vertices $x, y \in B$, then, by the choice of G, we must have $\gamma(G - xy) \le \frac{k+2}{3k+5} n$, and since the domination number does not increase by the removal of any edge, we get a contradiction. Hence B is an independent set of G. On the other hand, suppose that there is a (maximal) 2-path $P = v_1 \ldots v_t$ between two vertices $x, y \in B$ but not including them, where $xv_1, yv_t \in E(G)$ and $t \equiv 0 \pmod 3$. By our choice of G, $\gamma(G - V(P)) \le \frac{k+2}{3k+5}(n-t)$, and, clearly, any $\gamma(G - V(P))$-set can be extended to a dominating set of G by adding the vertices v_{3i+2} for $0 \le i \le t/3 - 1$. It follows that $\gamma(G) \le \gamma(G - V(P)) + t/3$ and thus

$$\begin{aligned}
\gamma(G) &\le \frac{k+2}{3k+5}(n-t) + t/3 \\
&< \frac{k+2}{3k+5}(n-t) + \frac{k+2}{3k+5} t \\
&= \frac{k+2}{3k+5} n,
\end{aligned}$$

a contradiction. Therefore we can assume that such a path does not exist. For the remainder, we need to state some claims.

Claim 1. G has no special cycle.
Proof of the claim. Suppose, to the contrary, that $C = q_1 P_1 q_2 P_2 \ldots q_t P_t q_1$ $(t \ge 2)$ is a special cycle in G. Note that C cannot be a tailed-cycle because $t \ge 2$. For each $j \in \{1, \ldots, t\}$, let $P_j = x_1^j x_2^j \ldots x_{s_j}^j$. Let G' be the graph obtained from G, by first deleting, for each j, either the vertices $x_2^j, \ldots, x_{s_j-1}^j$ if $s_j \ge 3$ or the edge $x_1^j x_2^j$ if $s_j = 2$, and then adding the edge $x_{s_j}^j x_1^{j+1}$ for each j. Observe that $d^*(G') \le d^*(G)$.

If $d^*(G') = d^*(G)$, then G' would have more tailed-triangles and triangular vertices than G, which contradicts the choice of G. Hence we must have $d^*(G') < d^*(G)$. It follows that $\gamma(G') \leq \frac{k+2}{3k+5}n(G')$ and clearly G' has a $\gamma(G')$-set S containing the vertices $q_1, \ldots, q_t$. Moreover, S can be extended to a dominating set of G of size $|S| + \sum_{j=1}^{t} \frac{s_j-2}{3}$, implying that

$$\gamma(G) \;\leq\; \frac{k+2}{3k+5}(n - \sum_{j=1}^{t}(s_j - 2)) \;<\; \frac{k+2}{3k+5}n,$$

a contradiction. ♦

Claim 2. G has no special path.

Proof of the claim. Suppose, to the contrary, that G has a special path, and let $P = q_1P_1q_2P_2 \ldots q_tP_tq_{t+1}$ ($t \geq 1$) be a longest one in G. For each $j \in \{1, \ldots, t\}$, let $P_j = x_1^j x_2^j \ldots x_{s_j}^j$. We claim that not both q_1 and q_{t+1} can be contained in tailed-cycles. Assume, to the contrary, that both q_1 and q_{t+1} are contained in tailed-cycles. Let $C_1 = (q_1u_1 \ldots u_pq_1)$ be a tailed cycle containing q_1 and $C_2 = (q_{t+1}w_1 \ldots w_mq_{t+1})$ be a tailed cycle containing q_{t+1}. If $p \equiv 0 \pmod 3$ and $\deg_G(q_1) \geq 4$, then let $G' = G - \{u_1, \ldots, u_p\}$. By the choice of G, we have $\gamma(G') \leq \frac{k+2}{3k+5}n(G')$ and clearly any $\gamma(G')$-set can be extended to a dominating set of G by adding the vertices u_{3i-1} for $1 \leq i \leq p/3$, yielding $\gamma(G) < \frac{k+2}{3k+5}\,n$, a contradiction. Hence $p \not\equiv 0 \pmod 3$ or $\deg_G(q_1) = 3$. Likewise we have $m \not\equiv 0 \pmod 3$ or $\deg_G(q_{t+1}) = 3$. Assume first that $t = 1$, and let G' be the graph obtained from G by deleting the vertices $x_1^1, x_2^1, \ldots, x_{s_1}^1$. By the choice of G, we have $\gamma(G') \leq \frac{k+2}{3k+5}n(G')$. We shall show that G' has a $\gamma(G')$-set that contains q_1 and q_2. Let S be a $\gamma(G')$-set such that $|S \cap \{q_1, q_2\}|$ is as large as possible. If $q_1, q_2 \in S$, then we are done. Hence assume, without loss of generality, that $q_1 \notin S$. This implies that $\deg_G(q_1) \geq 4$. Our earlier assumption implies that $p \not\equiv 0 \pmod 3$. Since $q_1 \notin S$, we have $|S \cap \{u_1, \ldots, u_p\}| \geq \lfloor p/3 \rfloor + 1$ and then $(S - \{u_1, \ldots, u_p\}) \cup \{q_1, u_{3i} \mid 1 \leq i \leq \lfloor p/3 \rfloor\}$ is a $\gamma(G')$-set containing q_1 which contradicts the choice of S. Thus G' has a $\gamma(G')$-set S that contains q_1 and q_2. Obviously, S can be extended to a dominating set of G of size $|S| + \frac{s_1-2}{3}$, yielding $\gamma(G) < \frac{k+2}{3k+5}n$, a contradiction. Assume now $t \geq 2$, and let G' be the graph obtained from G by first deleting the vertices $x_1^1, x_{s_t}^t$, as well as, for every j, either the vertices $x_2^j, \ldots, x_{s_j-1}^j$ if $s_j \geq 3$ or the edge $x_1^j x_2^j$ if $s_j = 2$ and then adding the edge $x_{s_j}^j x_1^{j+1}$ for each j. The graph G' satisfies $d^*(G') < d^*(G)$ and thus, by the choice of G, we have $\gamma(G') \leq \frac{k+2}{3k+5}n(G')$. Also since G' has a $\gamma(G')$-set S containing $q_1, \ldots, q_t$, and such a set S can be extended to a dominating set of G of size $|S| + \sum_{j=1}^{t} \frac{s_j-2}{3}$, we get $\gamma(G) < \frac{k+2}{3k+5}n$, which is a contradiction. Therefore q_1 and q_{t+1} cannot both be contained in tailed-cycles. Now, to achieve the proof of Claim 2, let us assume, without loss of generality, that q_{t+1} is not contained in a tailed-cycle. Since $\deg(q_{t+1}) \geq 3$, let a_1, b_1 be two neighbors of q_{t+1} different from $x_{s_t}^t$. By the choice of G and our earlier assumption, a_1 and b_1 belong to 2-paths $Q_1 = a_1 \ldots a_p$, $Q_2 = b_1 \ldots b_q$, respectively. Let $a_pa, b_qb \in E(G)$, where $a, b \in B$. Note that $a_1 \neq a$ and $b_1 \neq b$. Since G does not contain (maximal) 2-paths of order $\equiv 0 \pmod 3$, we have $|V(Q_i)| \not\equiv 0 \pmod 3$. Also, by the choice of P as being longest we have

$|V(Q_i)| \not\equiv 2 \pmod 3$. Hence $|V(Q_i)| \equiv 1 \pmod 3$ for each $i \in \{1, 2\}$.

Consider the following two cases.

Case 1. $a \neq b$.
Let G' be the graph obtained from G by first deleting either the vertices $a_2, \ldots, a_p$ if $p \geq 2$ or the edge $a_1 a$; either the vertices $b_2, \ldots, b_q$ if $q \geq 2$ or the edge $b_1 b$; and then adding the edge $a_1 b_1$. The graph G' satisfies $d^*(G') < d^*(G)$ and thus we have $\gamma(G') \leq \frac{k+2}{3k+5} n(G')$. Moreover, G' has a $\gamma(G')$-set S containing the vertex q_{t+1}, and obviously S can be extended to a dominating set of G of size $|S| + \frac{p-1}{3} + \frac{q-1}{3}$, yielding $\gamma(G) < \frac{k+2}{3k+5} n$, which is a contradiction.

Case 2. $a = b$.
If $\deg(a) \geq 4$, then using an argument similar to that described in Case 1 leads to the same contradiction. Hence we assume that $\deg(a) = 3$. Let z_1 be the neighbor of a different from a_p, b_q. We distinguish two situations.

Subcase 2.1. $z_1 = q_1$.i
Let G' be the graph obtained from G by first deleting for every j, either the vertices $x_2^j, \ldots, x_{s_j-1}^j$ if $s_j \geq 3$ or the edge $x_1^j x_2^j$ when $s_j = 2$; either the vertices $a_2, \ldots, a_p$ if $p \geq 2$ or the edge $a_1 b$ when $p = 1$, and the vertices $b_1, \ldots, b_q$, and then adding the edges $x_1^1 b, a_1 x_{s_t}^t$ and $x_{s_j}^j x_1^{j+1}$ for each j (for an example, see Figure 1). Clearly, G' satisfies $\gamma(G') \leq \frac{k+2}{3k+5} n(G')$, and has a $\gamma(G')$-set S containing vertices $q_1, \ldots, q_t$. Now, since S can be extended to a dominating set of G of size $|S| + \frac{p-1}{3} + \frac{q-1}{3} + \sum_{j=1}^{t} \frac{s_j - 2}{3}$, we obtain $\gamma(G) < \frac{k+2}{3k+5} n$, a contradiction.

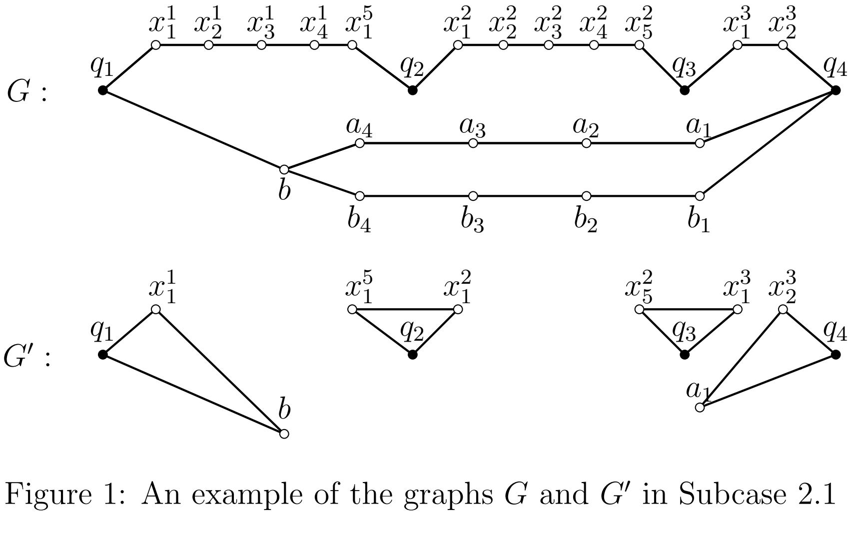

Figure 1: An example of the graphs G and G' in Subcase 2.1

Subcase 2.2. $z_1 \neq q_1$.
Suppose first that q_1 is not contained in a tailed-cycle. By symmetry with vertex q_{t+1} and noting the above situations, we may assume that there are two 2-paths $Q_1' = a_1' a_2' \ldots a_{p_1}'$ and $Q_2' = b_1' b_2' \ldots b_{q_1}'$ and a vertex a' such that $q_1 a_1', q_1 b_1', a' a_{p_1}', a' b_{q_1}' \in E(G)$ and $\deg(a') = 3$. Let G' be the graph obtained from G by first deleting either the vertices $a_2, \ldots, a_p$ if $p \geq 2$ or the edge $a_1 a$; either the vertices $a_2', \ldots, a_{p_1}'$ if $p_1 \geq 2$

or the edge $a_1'a'$, and for every j either the vertices $x_2^j, \ldots, x_{s_j-1}^j$ if $s_j \geq 3$ or the edge $x_1^j x_2^j$ when $s_j = 2$; and then adding the edges $a_1 x_{s_t}^t, x_1^1 a_1', x_{s_j-1}^j x_1^{j+1}$ for each j. By the choice of G, we have $\gamma(G') \leq \frac{k+2}{3k+5} n(G')$ and clearly G' has a $\gamma(G')$-set S containing $q_1, \ldots, q_t$. Moreover, since S can be extended to a dominating set of G of size $|S| + \frac{p-1}{3} + \frac{p_1-1}{3} + \sum_{j=1}^{t} \frac{s_j-2}{3}$, we deduce that $\gamma(G) < \frac{k+2}{3k+5} n$, a contradiction. Assume now that q_1 is contained in a tailed-cycle C. Let G' be the graph obtained from G by first deleting either the vertices $a_2, \ldots, a_p$ if $p \geq 2$ or the edge $a_1 a$; for every j, either the vertices $x_2^j, \ldots, x_{s_j-1}^j$ if $s_j \geq 3$ or the edge $x_1^j x_2^j$ when $s_j = 2$, and then adding the edges $a_1 x_{s_t}^t$, $x_{s_j-1}^j x_1^{j+1}$ for each j. Clearly G' is the union of the graph $C + q_1 x_1^1$ and a graph G'' with $\delta(G'') \geq 2$. By the induction hypothesis, G'' satisfies $\gamma(G'') \leq \frac{k+2}{3k+5} n(G'')$ and has a $\gamma(G'')$-set S_1 containing $q_2, \ldots, q_{t+1}$. On the other hand, clearly $\gamma(C + q_1 x_1^1) \leq \frac{k+2}{3k+5} n(C + q_1 x_1^1)$ and $C + q_1 x_1^1$ has a $\gamma(C + q_1 x_1^1)$-set S_2 containing q_1. Now $S_1 \cup S_2$ can be extended to a dominating set of G of size $|S_1 \cup S_2| + \frac{p-1}{3} + \sum_{j=1}^{t} \frac{s_j-2}{3}$, which leads to $\gamma(G) < \frac{k+2}{3k+5} n$, a contradiction. ♦

Claim 3. G has no cycle C such that $|V(C) \cap B| \geq 2$.

Proof of the claim. Suppose, to the contrary, that G has a cycle C such that $|V(C) \cap B| \geq 2$. Thus there are vertices $q_1, \ldots, q_k \in B$ and 2-paths $P_1, \ldots, P_k$ such that $C = q_1 P_1 q_2 P_2 \ldots, q_k P_k q_1$. Let $P_j = x_1^j x_2^j \ldots x_{s_j}^j$ for each j. Suppose first that $k \geq 3$. By the argument before Claim 1 and according to Claim 2, G has no 2-path of length $\equiv 0, 2 \pmod 3$ between two vertices of B and thus we have $|V(P_i)| \equiv 1 \pmod 3$. Let G' be the graph obtained from G by first deleting either the vertices $x_1^1, \ldots, x_{s_1-1}^1$ if $s_1 \geq 2$ or the edge $q_1 x_1^1$ when $s_1 = 1$, and either the vertices $x_2^2, \ldots, x_{s_2}^2$ if $s_2 \geq 2$ or the edge $q_3 x_{s_2}^2$ when $s_2 = 1$ and then adding the edges $x_{s_1}^1 x_1^2$. The graph G' satisfies $d^*(G') < d^*(G)$ and thus we have $\gamma(G') \leq \frac{k+2}{3k+5} n(G')$. Furthermore, G' has a $\gamma(G')$-set S containing q_2 and since S can be extended to a dominating set of G of size $|S| + \frac{s_1-1}{3} + \frac{s_2-1}{3}$, we obtain $\gamma(G) < \frac{k+2}{3k+5} n$, a contradiction.

Assume now that $k = 2$. If $\deg(q_2) \geq 4$, then by considering the graph G' obtained from G by first deleting either the vertices $x_2^1, \ldots, x_{s_1}^1$ if $s_1 \geq 2$ or the edge $q_2 x_1^1$ if $s_1 = 1$, either the vertices $x_1^2, \ldots, x_{s_2-1}^2$ if $s_2 \geq 2$ or the edge $q_2 x_{s_2}^2$ if $s_2 = 1$ and then adding the edges $x_{s_2}^2 x_1^1$, we get a contradiction as above. Hence let $\deg(q_2) = 3$, and, by symmetry, we have $\deg(q_1) = 3$. Since $C = q_1 P_1 q_2 P_2 q_1$ is a cycle of length $3r + 1$ for some r, we deduce from the assumption $B_{2k+2} = \emptyset$ that $r \geq 2k + 3$ and so $n \geq 3r + 1 \geq 6k + 10$. If there exists another 2-path $x_1^3 x_2^3 \ldots x_{s_3}^3$ between q_1 and q_2, then we have $B = \{q_1, q_2\}$ and the set $S = \{q_1, q_2\} \cup \{x_{3i}^1, x_{3j}^2, x_{3t}^3 \mid 1 \leq i \leq (s_1 - 1)/3, 1 \leq j \leq (s_2 - 1)/3, 1 \leq t \leq (s_3 - 1)/3\}$ is a dominating set of G of size $\frac{n+1}{3} < \frac{k+2}{3k+5} n$, which leads to a contradiction. Hence we can assume that P_1 and P_2 are the only 2-paths between q_1 and q_2. Let a_1 be the neighbor of q_1 different from $x_1^1, x_{s_2}^2$ and b_1 be the neighbor of q_2 different from $x_{s_1}^1, x_1^2$. Then a_1 belongs to a 2-path $Q_1 = a_1 \ldots a_p$ and b_1 belongs to a 2-path $Q_2 = b_1 \ldots b_q$. Let $a_p z, b_q z' \in E(G)$ where $z, z' \in B$. By our earlier assumption $p \equiv 1 \pmod 3$ and $q \equiv 1 \pmod 3$. Now, if $z = z'$, then G contains a cycle $q_1 P_1 q_2 Q_2 z Q_1^{-1} q_1$ which contains three vertices of B, and as above we get a contradiction. Hence, we may assume that $z \neq z'$. In this case, let G' be the graph obtained from G by first deleting either the vertices $x_2^1, \ldots, x_{s_1}^1$ if

$s_1 \geq 2$ or the edge $x_1^1 q_2$ if $s_1 = 1$, either the vertices $x_2^2, \ldots, x_{s_2}^2$ if $s_2 \geq 2$ or the edge $q_1 x_1^2$ otherwise, either the vertices $a_2, \ldots, a_p$ if $p \geq 2$ or the edge $a_1 z$ otherwise, either the vertices $b_2, \ldots, b_q$ if $q \geq 2$ or the edge $b_1 z'$ otherwise, and then adding the edges $a_1 x_1^1, b_1 x_1^2$. The graph G' has minimum degree at least 2, having in particular two components which are triangles: one contains the vertex q_1 and the other contains the vertex q_2. Moreover, G' satisfies $d^*(G') < d^*(G)$, and thus $\gamma(G') \leq \frac{k+2}{3k+5} n(G')$. Since G' has a $\gamma(G')$-set S containing q_1, q_2, and such a set S can be extended to a dominating set of G of size $|S| + \frac{s_1 - 1}{3} + \frac{s_2 - 1}{3} + \frac{p-1}{3} + \frac{q-1}{3}$, we obtain $\gamma(G) < \frac{k+2}{3k+5} n$, which leads to a contradiction again. ♦

We deduce from Claim 3 that all cycles of G are tailed-cycles. Let G_1 be the graph obtained from G by deleting the vertices of all cycles apart from the vertices with degree at least 3. Clearly, G_1 is a tree.

Claim 4. G_1 is a path.
Proof of the claim. We only need to prove that $\Delta(G_1) \leq 2$. Suppose, to the contrary, that G_1 has a vertex x with degree at least 3. Assume that $x_1^1, \ldots, x_1^t$ are the neighbors of x in G_1. Then for each i, vertex x_1^i belongs to a maximal 2-path $x_1^i x_2^i \ldots x_{s_i}^i$ in G. By our earlier assumption, we have $s_i \equiv 1 \pmod 3$ for each i. Assume first that x belongs to no cycle in G. In this case, let G' be the graph obtained from G by deleting the vertex x and the vertices $x_1^i, \ldots, x_{s_i}^i$ for each i. Clearly $\delta(G') \geq 2$ and by the choice of G we have $\gamma(G') < \frac{k+2}{3k+5} n(G')$. Obviously, any $\gamma(G')$-set can be extended to a dominating set of G of size $\gamma(G') + 1 + \sum_{i=1}^{t} \frac{s_i - 1}{3}$ and it follows that $\gamma(G) < \frac{k+2}{3k+5} n$, which is a contradiction. Assume now that x belongs to a cycle C in G. In this case, let G' be the graph obtained from G by deleting the vertices $x_1^i, \ldots, x_{s_i}^i$ for each i. Clearly $\delta(G') \geq 2$ and G satisfies $\gamma(G') < \frac{k+2}{3k+5} n(G')$. Moreover, there is a $\gamma(G')$-set that contains x, and such a set can be extended to a dominating set of G of size $\gamma(G') + \sum_{i=1}^{t} \frac{s_i - 1}{3}$, which leads as before to a contradiction. ♦

Thus G_1 is a path. It follows that all vertices in B contained in a tailed cycle. Let x be a vertex of degree 1 in G_1, and let $x a_1 \ldots a_p y$ be a subpath of G_1 such that $y \in B$ and $\deg(a_i) = 2$ for each $i \in \{1, \ldots, p\}$. By our earlier assumption $p \equiv 1 \pmod 3$. Let G' be the graph obtained from G by deleting the vertices $a_1, \ldots, a_p$. By the choice of G, we have $\gamma(G') \leq \frac{k+2}{3k+5} n(G')$ and clearly G' has a $\gamma(G')$-set S containing x. Now, since S can be extended to a dominating set of G of size $|S| + \frac{p-1}{3}$, we obtain $\gamma(G) < \frac{k+2}{3k+5} n$, which leads to a contradiction. This proves the bound.

To see the sharpness of the bound, let G be the connected graph obtained from $m \geq 2$ disjoint cycles $C_{3k+5}^j = (x_1^j x_2^j \ldots x_{3k+5}^j x_1^j), 1 \leq j \leq m$, by adding the edges $x_1^1 x_1^2, \ldots, x_1^1 x_1^m$. Clearly, $\delta(G) \geq 2$, $n(G) = m(3k+5)$, $\mathcal{A}_k \cup \mathcal{B}_{2k+2} = \emptyset$ and $\gamma(G) = m(k+2) = \frac{k+2}{3k+5} n(G)$. This completes the proof. □

It is worth mentioning that, for $k = 1$, Theorem 1 immediately follows from Theorem 3. Moreover, for $k \geq 2$, Theorem 3 provides an improvement on the $\frac{3}{8}n$-upper bound for graphs without special cycles.

We conclude this paper with an open problem and a conjecture.

Problem. Characterize all graphs G on n vertices with $\delta(G) \geq 2$ and with $\mathcal{A}_k \cup \mathcal{B}_{2k+2} = \emptyset$ such that $\gamma(G) = \dfrac{k+2}{3k+5}n$.

Conjecture. Let k be a nonnegative integer and let G be a multigraph of order $n \geq 6k+10$ with $\delta(G) \geq 2$. If G has no induced cycle of lengths $3r+1, 3r+2$ for $0 \leq r \leq k$, then $\gamma(G) < \dfrac{k+2}{3k+5}n$.

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