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Directed embeddings of 2-regular diplanar digraphs on surfaces of low Euler genus

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Abstract

A *directed embedding* of an Eulerian digraph on a surface is a 2-cell embedding of its underlying graph on the surface with a property that each face is bounded by a directed cycle. A digraph is 2-*regular* if each vertex has both indegree and outdegree two, and is *diplanar* if it is directedly embeddable on the sphere (or the plane). As an expansion of Whitney's theorem to directed embeddings, Archdeacon et al. [*Australas. J. Combin.* 67 (2017), 159–165] proved that every strongly 2-edge-connected 2-regular diplanar digraph is uniquely directedly embeddable on the sphere.

We indicate the close relationship between an embedding of a 3-connected graph on a surface and a directed embedding of a strongly 2-edge-connected digraph on the surface, and give a simpler proof of the above theorem. Moreover, we give the complete structures of directed embeddings of strongly 2-edge-connected 2-regular diplanar digraphs on the projective-plane, the torus and the Klein bottle. This enables us to evaluate the total number of such directed embeddings.

1 Introduction

Let S be a surface, which is a compact 2-dimensional manifold without boundary. An *embedding* $f : G \to S$ of an undirected graph G on S is an injective continuous map from G to S. In other words, an embedding is a drawing of a graph on a surface without edge-crossings. We often consider a graph G is already mapped on a surface and denote its image by G itself. The *faces* are the component of the open set $S - f(G)$. An embedding of a graph is 2-*cell* if each face is homeomorphic to an open 2-cell, which contains neither handles nor crosscaps. In this paper, we consider only 2-cell embeddings. For terminologies of topological graph theory, we refer to [10].

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Two embeddings $f_1, f_2 : G \to S$ of a graph G on a surface S are *equivalent* if there is a homeomorphism $h : S \to S$ such that $hf_1 = f_2$, and they are *inequivalent* otherwise. A graph G is *uniquely embeddable* (up to equivalent) on a surface S if any two embeddings of G on S are equivalent. Whitney's theorem [14, 15] is one of the most important results on plane embeddings, and states that every 3-connected planar graph is uniquely embeddable on the sphere. This theorem was obtained as a corollary of a stronger result that one of any two embeddings of a 2-connected planar graph can be obtained from the other by a sequence of local re-embeddings, called *Whitney flips* (see [10, Sections 2 and 5] for details).

A digraph is *Eulerian* if each vertex has the same indegree and outdegree. A *directed embedding* of an Eulerian digraph D on a surface S is defined as a 2-cell embedding of its underlying graph on S with a property that each face is bounded by a directed closed walk. Hence, in- and out-edges alternate in the rotation around each vertex of a directedly embedded digraph. An Eulerian digraph D is *diplanar* if D has a directed embedding on the sphere (or the plane). This type of embeddings first appeared in Tutte's work [13], and has been studied extensively (see for example [1–6, 8, 9]).

For an integer $k \geq 1$, a digraph D is *k-regular* if each vertex of D has both indegree and outdegree k, and is *strongly k-edge-connected* if for any $X \subseteq E(D)$ with $|X| < k$, $D - X$ is strongly connected. In this paper, we mainly focus on strongly 2-edge-connected 2-regular diplanar digraph. Thus, our digraphs have no loops and no *multiple edges*, that is, several copies of an edge uv, while they may have *opposite edges*, that is, two anti-directed edges uv and vu. Archdeacon et al. [3] proved an analogue of Whitney's theorem:

Theorem 1.1 (Archdeacon et al. [3]). *Every strongly 2-edge-connected 2-regular diplanar digraph is uniquely directedly embeddable on the sphere.*

As with Whitney's theorem, Theorem 1.1 was obtained as a corollary of the stronger result that one of any two directed embeddings of a connected 2-regular diplanar digraph can be obtained from the other by a sequence of "directed" Whitney flips. Moreover, in [3], they proved an analogue of Tutte's peripheral cycles theorem, which is also a stronger result than Theorem 1.1. Unfortunately, Theorem 1.1 does not hold for arbitrary Eulerian digraphs. That is, there are infinitely many strongly 2-edge-connected Eulerian diplanar digraphs having at least two directed embeddings on the sphere.

In Section 2, we indicate the close relationship between an embedding of a 3-connected graph on a surface and a directed embedding of a strongly 2-edge-connected digraph on the surface, which enables us to give a simple proof of Theorem 1.1. Moreover, we focus on directed embeddings of diplanar digraphs on non-spherical surfaces. As for embeddings of planar undirected graphs, Mohar, Robertson and Vitray [12], and Mohar and Robertson [11] studied embeddings of 2-connected planar graphs on non-spherical surfaces and showed that such embeddings have special structures, called "patch structures". Recently, the author [7] completely characterized structures of embeddings of 3-connected 3-regular planar graphs on the projective-plane, the torus and the Klein bottle, which are useful for enumerating such embeddings and counting their total number. In Section 3, we extend the above result to directed embeddings of digraphs, that is, we characterize structures of directed embeddings of strongly 2-edge-connected 2-regular diplanar digraph on the projective-plane, the torus and the Klein bottle. In addition to this, we evaluate the number of such directed embeddings.

2 A simple proof of Theorem 1.1

For the sake of arguments in later sections, we introduce the combinatorial way of describing embeddings of a (undirected) simple graph, called "embedding schemes". A general description of an embedding scheme can be found in [10].

For an embedding $f : G \to S$ of a graph G on a surface S, there are two possible cyclic orderings of edges incident with each vertex v of G. Choose one of them, called the *rotation* around v, and denote it by ρ_v. A *signature* of $E(G)$ is a map assigning 1 or -1 to each edge of G, denoted by λ, such that for an edge $e = uv$ with its endvertices u and v, $\lambda(e) = 1$ if a subwalk induced by the three edges $\rho_u(e)$, e and $\rho_v^{-1}(e)$ is included in a facial walk, otherwise $\lambda(e) = -1$. It can be shown that this definition of the signature λ is consistent, that is, $\lambda(uv) = \lambda(vu)$ for every edge uv. The pair (ρ, λ), where $\rho = \{\rho_v : v \in V(G)\}$, is obtained by the above procedure, is called an *embedding scheme* for $f(G)$. An embedding scheme determines exactly one embedding of G up to equivalence.

For an embedded graph G associated with a given embedding scheme, we call an operation of replacing the signatures of some edges with their inverses *twisting* these edges, and the embedding associated with the resulting embedding scheme the *re-embedding* of G obtained by twisting these edges.

Now we introduce a transforming operation of a 4-regular graph. Let G be a 4-regular graph embedded on a surface, and v be a vertex of G adjacent to u_1, u_2, u_3 and u_4 such that the rotation around v corresponds to this order. A *truncation* of a vertex v is an operation of replacing a small neighborhood around v with a facial cycle of order 4, called a *truncated cycle*, shown in Fig. 1; delete v and add four vertices v_1, v_2, v_3 and v_4 together with edges v_iu_i and v_iv_{i+1} with indices taken modulo 4.

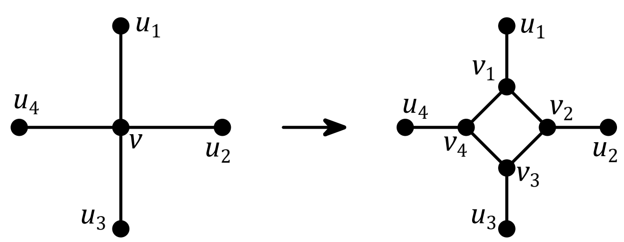

Fig. 1: Truncation of a vertex

The *truncated graph* of a 4-regular graph G embedded on a surface is the embedded graph obtained from G by truncating all vertices, denoted by $tr(G)$. We call the edges of $tr(G)$ contained in a truncated cycle *truncated edges* and the others *original edges*. It is clear that $tr(G)$ is 3-regular and each vertex is incident with two truncated edges and one original edge. In this paper, we mainly focus on 2-regular digraphs D. Since our digraphs have no loops, for the underlying graph G of D, the truncated graph $tr(G)$ is simple.

Note that the truncating operation depends on the rotation around a vertex. That is, if two embeddings $f_1(G)$ and $f_2(G)$ of G on a surface are inequivalent then the truncated graphs $tr(f_1(G))$ and $tr(f_2(G))$ may not be isomorphic to each other. However, we do not have to consider such a situation when G is the underlying graph of a connected 2-regular digraph D.

Lemma 2.1. *For any two directed embeddings $f_1(D)$ and $f_2(D)$ of a connected 2-regular digraph D on a surface with its underlying graph G, there is an isomorphism from $tr(f_1(G))$ to $tr(f_2(G))$ that preserves truncated cycles.*

Proof. Since G is the underlying graph of D, which is 2-regular, there are only two possible rotations around each vertex of G, and one of them is the inverse of the other. This implies that the sets of truncated cycles of $tr(f_1(G))$ and $tr(f_2(G))$ are the same, and hence $tr(f_1(G))$ and $tr(f_2(G))$ are isomorphic to each other. □

Lemma 2.2. *If a connected 2-regular digraph D directedly embedded on a surface is strongly 2-edge-connected then the truncated graph of the underlying graph of D is 3-connected.*

Proof. It is easy to see that if D is strongly 2-edge-connected then the underlying graph G is 2-connected and 4-edge-connected. Suppose that the truncated graph $tr(G)$ of G is not 3-connected. As $tr(G)$ is 3-regular, it is not 3-edge-connected.

We first assume that $tr(G)$ has a bridge e. As a truncated edge is contained in a truncated cycle, e is not a truncated edge. Thus, e is an original edge and hence it is also a bridge in G, a contradiction.

Next, we assume that $tr(G)$ has two edges e_1 and e_2 forming an edge-cut, both of which are not bridges. If one of them is truncated, then the other must be contained in the same truncated cycle, denoted by C. In this situation, the vertex of G corresponding to C is a cut vertex, a contradiction. Thus, both of e_1 and e_2 are not truncated. However, this implies that they are original edges and hence they form an edge-cut of G, which contradicts the 4-edge-connectivity of G. □

Using Lemmas 2.1 and 2.2, we can prove Theorem 1.1 easily.

Proof of Theorem 1.1. Let D be a strongly 2-edge-connected 2-regular diplanar digraph with its underlying graph G, and $f_1(D)$ and $f_2(D)$ be two directed embeddings of D on the sphere. By Lemma 2.1, the two truncated graphs $tr(f_1(G))$ and $tr(f_2(G))$ of $f_1(G)$ and $f_2(G)$, respectively, are isomorphic. Moreover, by Lemma 2.2, they are

3-connected and hence equivalent to each other (by Whitney's theorem). This implies that $f_1(D)$ and $f_2(D)$ are equivalent. □

3 Directed embeddings on non-spherical surfaces

3.1 Directed embedding structures

The author [7] proved the following theorems:

Theorem 3.1 (Enami [7]). *A* 3*-connected* 3*-regular graph embedded on the projective-plane is planar if and only if it has one of the two structures* ($P1$) *and* ($P2$) *shown in Fig.* 2.

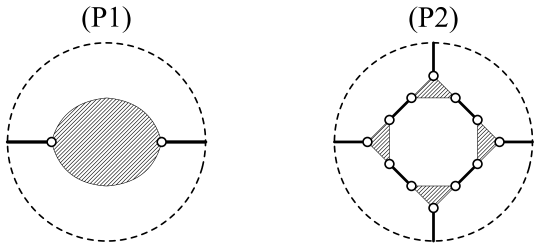

Fig. 2: Embedding structures on the projective-plane

Theorem 3.2 (Enami [7]). *A* 3*-connected* 3*-regular graph embedded on the torus is planar if and only if it has one of the two structures* ($T1$), ($T2$) *and* ($T3$) *shown in Fig.* 3.

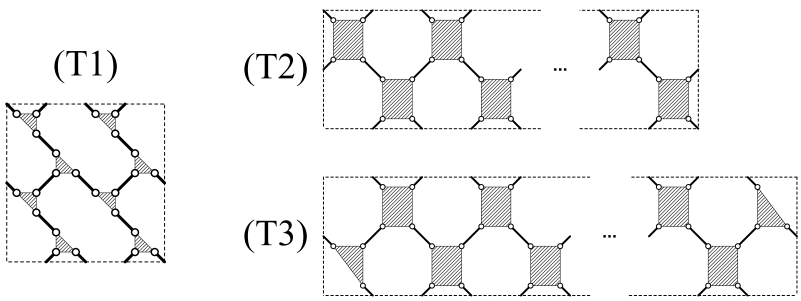

Fig. 3: Embedding structures on the torus

Theorem 3.3 (Enami [7]). *A* 3*-connected* 3*-regular graph embedded on the Klein bottle is planar if and only if it has one of the eight structures* ($K1$) *to* ($K8$) *shown in Fig.* 4.

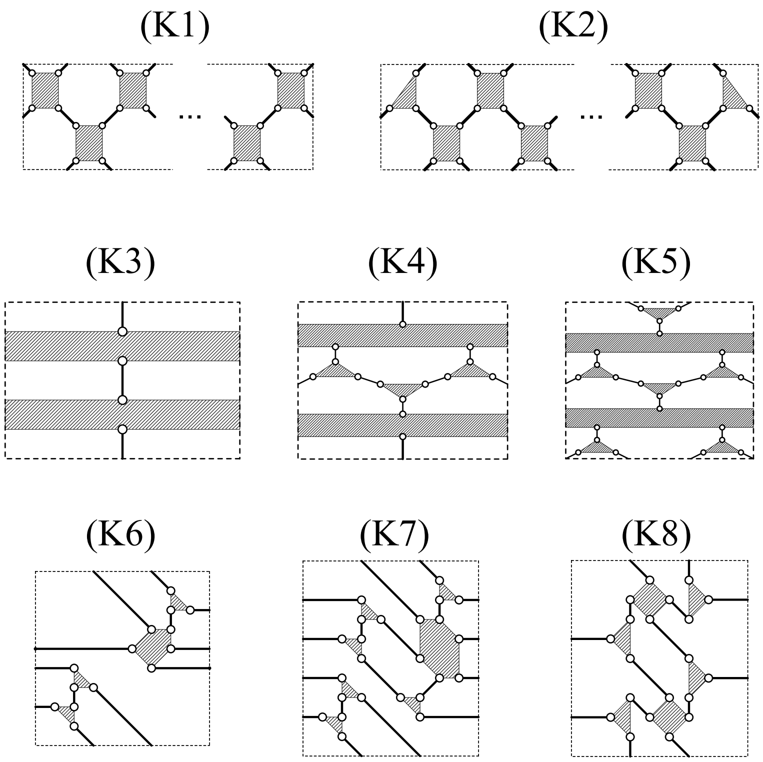

Fig. 4: Embedding structures on the Klein bottle

In Fig. 2, each pair of antipodal points on the dashed circle should be identified to recover the projective-plane. Similarly, in Fig. 3, to recover the torus, both pairs of opposite sides of dashed rectangle should be identified in the same direction, and in Fig. 4, to recover the Klein bottle, the top and bottom sides of the dashed rectangle should be identified in the same direction while the left and right sides should be identified in the opposite direction. In these figures, each of the shaded areas corresponds to a component of the graph obtained from the original graph by deleting all edges not bounded by shaded areas. Some vertices on the boundary of such an area may not be different from each other, that is, the edges not bounded by shaded areas may not be a matching. We omit a series of shaded rectangles from $(T2)$, $(T3)$, $(K1)$ and $(K2)$. Both $(T2)$ and $(T3)$ have an even number of shaded rectangles ($(T3)$ may have no shaded rectangle), while both $(K1)$ and $(K2)$ have an odd number of shaded rectangles. In [12], the structures of embeddings of a 2-connected planar graph on the projective-plane was analysed, and the structures in Theorem 3.1 is special cases in [12].

Remark 3.4. Let G be a 3-connected 3-regular planar graph. Suppose that G is already embedded on the projective-plane, the torus or the Klein bottle, and hence it has one of the structures shown in Figs. 2, 3 and 4. We can obtain the re-embedding $f(G)$ of G on the sphere by twisting all edges not bounding a shaded area. Because of that, every facial walk of G containing an edge not bounding a shaded area, that is, a closed walk bounding an empty area in the structure, is not facial in $f(G)$.

Now we show expansions of Theorems 3.5, 3.6 and 3.7 to directed embeddings of strongly 2-edge-connected 2-regular diplanar digraphs:

Theorem 3.5. *A strongly* 2*-edge-connected* 2*-regular digraph directedly embedded on the projective-plane is diplanar if and only if it has the structure shown in Fig.* 5*.*

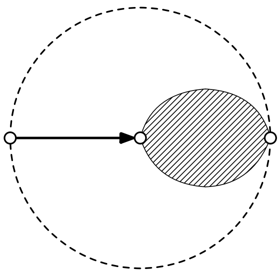

Fig. 5: Directed embedding structures on the projective-plane

Theorem 3.6. *A strongly* 2*-edge-connected* 2*-regular digraph directedly embedded on the torus is diplanar if and only if it has the structure shown in Fig.* 6*.*

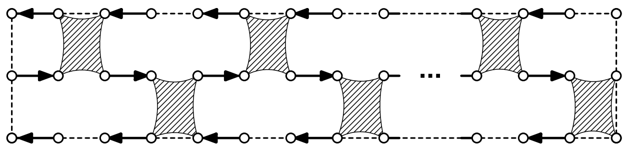

Fig. 6: Directed embedding structures on the torus

Theorem 3.7. *A strongly* 2*-edge-connected* 2*-regular digraph directedly embedded on the Klein bottle is diplanar if and only if it has one of the two structures shown in Fig.* 7*.*

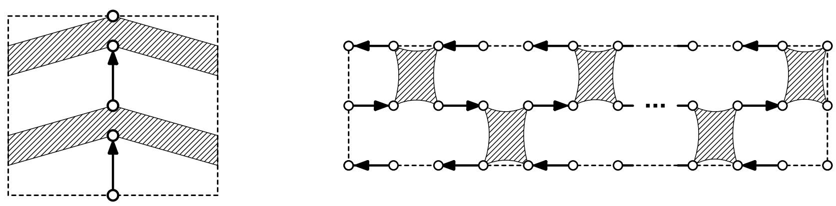

Fig. 7: Directed embedding structures on the Klein bottle

As with the structures of embeddings shown in Figs. 2, 3 and 4, in the structures of directed embeddings shown in Figs. 5, 6 and 7, each of the shaded areas corresponds to a component of the digraph obtained form the original digraph by deleting all directed edges not bounded by shaded areas.

Proof of Theorems 3.5, 3.6 and 3.7. If a directed embedding of a strongly 2-edge-connected 2-regular digraph on the projective-plane, the torus or the Klein bottle has one of the structures shown in Figs. 5, 6 and 7, then it has one of the directed embeddings on the sphere shown in Fig. 8. Hence, we only have to show that any directed embedding of a strongly 2-edge-connected 2-regular diplanar digraph D on the projective-plane, the torus or the Klein bottle must have one of the structures shown in Figs. 5, 6 and 7.

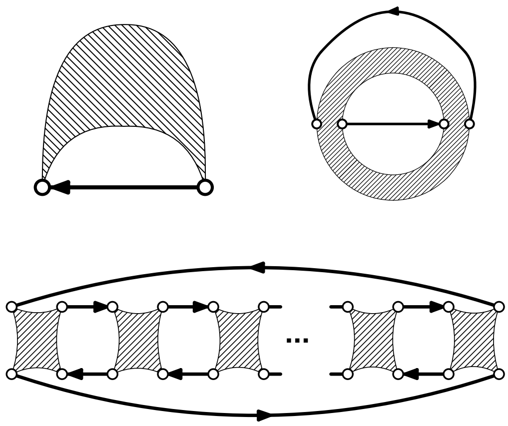

Fig. 8: Three structures of a directed embedding on the sphere

Suppose that D is already directedly embedded on the projective-plane, the torus or the Klein bottle with a directed embedding $f_1(D)$, giving an embedding $f_1(G)$ of the underlying graph G. By Lemma 2.2, the truncated graph $T_1 = tr(f_1(G))$ of G is 3-connected. Moreover, we now show that every edge-cut of order 3 in T_1 is *trivial*, that is, its edges have a common end-vertex. Suppose that T_1 has a non-trivial edge-cut of order 3 (for a contradiction). If an edge in the edge-cut is truncated then one of the others is contained in the same truncated cycle. Since the edge-cut is non-trivial, in this situation, we can find a vertex-cut of order at most two in T_1, which contradicts the 3-connectivity of T_1. Thus, each of them is an original edge. However, this implies that G has an edge-cut of order 3, which contradicts the strong 2-edge-connectivity of D. Therefore, every edge-cut of order 3 in T_1 is trivial.

The directed embedding $f_1(D)$ gives a derived embedding $g_1(T_1)$ in the same surface. Since T_1 is 3-connected, 3-regular and planar, by Theorems 3.1, 3.2 and 3.3, the embedding $g_1(T_1)$ on the projective-plane, the torus or the Klein bottle has one of the structures shown in Figs. 2, 3 and 4. In the structure, if $g_1(T_1)$ has a shaded triangle then the three edges incident with this triangle form an edge-cut. Thus, this triangle corresponds to only one vertex, denoted by v, and if C is the truncated cycle of T_1 incident with v, then $g_1(C)$ bounds an empty area in the structure. This implies that for any re-embedding $g_1'(T_1)$ of T_1 on the sphere, $g_1'(T_1)$ is not facial (See Remark 3.4).

Now D has another directed embedding $f_2(D)$ on the sphere, from which we derive an embedding of $T_2 = tr(f_2(G))$ on the sphere, dnoted by $g_2(T_2)$. (By Theorem 1.1, D has an unique directed embedding on the sphere.) For every truncated cycle C of T_2, $g_2(C)$ is a facial cycle in $g_2(T_2)$. By Lemma 2.1, there is an isomorphism $\phi : T_1 \to T_2$ which preserves truncated cycles, and hence $g_2(\phi(T_1)) = g'_1(T_1)$, where $g'_1 = g_2 \circ \phi$, is an embedding if T_1 on the sphere. Since C is a truncated cycle in T_1, $\phi(C)$ is a truncated cycle in T_2, and hence $g_2(\phi(C))$ is a facial cycle in $g_2(T_2)$. In other words, $g'_1(C)$ is a facial cycle in $g'_1(T_1)$, which is a contradiction.

Therefore, $g_1(T_1)$ has no shaded triangle, that is, the structure is one of the four $(P1)$, $(T2)$, $(K1)$ and $(K3)$. From the above argument, an edge contained in a truncated cycle appear in a shaded area. Thus, $f_1(G)$ has the same structure, and hence $f_1(D)$ has one of the structures shown in Figs. 5, 6 and 7.

□

3.2 The number of directed embeddings

For a directed embedding of a 2-regular digraph D on a surface, the operation of twisting some edges of D can be defined as with the case of embeddings of (undirected) graphs. This operation holds the property that in- and out-edges alternate in the rotation at each vertex of D. That is, the resulting mapping of D on a surface is a re-embedding of D. Actually, for a digraph D directedly embedded on the sphere with one of the structure shown in Fig. 8, we obtain the re-embedding of D with one of the structure shown in Figs. 5, 6 and 7 by twisting all edges which are not contained in shaded area. It can be shown that for two distinct edge-sets X_1 and X_2, the two directed embeddings of D obtained from the original directed embedding of D by twisting the edges in X_1 and X_2 are inequivalent or mapped on distinct surfaces.

Proposition 3.8. *Every connected 2-regular diplanar digraph D with n vertices has at least $2n$ inequivalent directed embeddings on the projective-plane. If D is strongly 2-edge-connected, then D has at least n inequivalent directed embeddings on the torus, and at least $n(2n-1)$ inequivalent directed embeddings on the Klein bottle.*

Proof. Let D be a 2-regular diplanar digraph with n vertices, and suppose that D is now directedly embedded on the sphere.

Twisting an edge of D, we obtain a re-embedding of D on the projective-plane, which has the structure shown in Fig. 5. Then D has at least $|E(D)| = 2n$ inequivalent directed embeddings on the projective-plane.

Hereafter, suppose that D is strongly 2-edge-connected. First, choose one vertex v of D. Then D has the structure shown in the bottom of Fig. 8, when there are exactly two shaded rectangles and one of them represents only one vertex. Since D is strongly 2-edge-connected, the other shaded rectangle represent a connected subgraph of D. Hence, twisting the four edges incident with a vertex of D, we obtain a re-embedding of D on the torus, which has the structure shown in Fig. 6. (If D is not strongly 2-edge-connected and v is a cut vertex, then the re-embedding of D is on the sphere.) Then D has at least $|V(D)| = n$ inequivalent directed embeddings on the torus.

Second, choose two edges of D and twist them. Since D is strongly 2-edge-connected, the resulting directed embedding of D is on a non-spherical surface, that is, this operation cannot be a directed Whitney flip. In this situation, this directed embedding is on the Klein bottle, which has one of the two structures shown in the left of Fig. 7 or the right of Fig. 7 when there is exactly one shaded rectangle. If we choose two edges contained in the same facial directed walk, then the re-embedding has the structure shown in the left of Fig. 7. Otherwise, it has the structure shown in the right of Fig. 7. Thus, we can give at least $\binom{|E(G)|}{2} = n(2n-1)$ inequivalent directed embeddings of D on the Klein bottle. □

By Theorem 3.5, there are no directed embeddings of a strongly 2-edge-connected 2-regular diplanar digraph on the projective-plane other than those in Proposition 3.8.

Corollary 3.9. *Every strongly 2-edge-connected 2-regular diplanar digraph with n vertices has exactly $2n$ inequivalent directed embeddings on the projective-plane.* □

Next, we show a family of digraphs attaining the lower bounds in Proposition 3.8 on the torus and the Klein bottle. An undirected graph is *cyclically k-edge-connected* if there is no set of at most $k-1$ edges such that the graph obtained by deleting these edges has at least two components having a cycle.

Corollary 3.10. *If the underlying graph of a strongly 2-edge-connected 2-regular diplanar digraph with n vertices is cyclically 5-edge-connected, then it has exactly n inequivalent directed embeddings on the torus and $n(2n-1)$ inequivalent directed embeddings on the Klein bottle. Moreover, in the case on the torus, the converse holds.*

Proof. Let D be a a strongly 2-edge-connected 2-regular digraph D with n vertices. Suppose that the underlying graph G of D is cyclically 5-edge-connected, and D has more than n inequivalent directed embeddings on the torus or $n(2n-1)$ inequivalent directed embeddings on the Klein bottle. Thus, there is a directed embedding of D on the torus or the Klein bottle which is not counted in Proposition 3.8. This directed embedding has the structure shown in Fig. 6 or the right of Fig. 7 having at least two shaded rectangle, each of which does not represent just one vertex, that is, has a cycle. In this situation, we can find four edges of G such that the graph obtained by deleting these edges has exactly two components having cycles, a contradiction.

If G is not cyclically 5-edge-connected then there are four edges such that the graph obtained by deleting these edges has two components having a cycle. Twisting these edges, we obtain a re-embedding of D on the torus, which has the structure shown in Fig. 6 when there are exactly two shaded rectangles. This directed embedding is not counted in Proposition 3.8. □

The underlying graph of a digraph attaining the lower bounds in Proposition 3.8 on the Klein bottle is not necessarily cyclically 5-edge-connected. For example, Fig. 9 presents the underlying graph of such a digraph D.

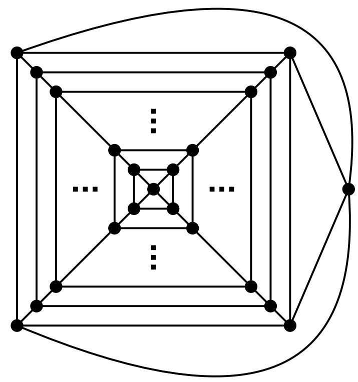

Fig. 9: The underlying graph of D

In addition to Corollary 3.10, we can give a polynomial-time algorithm for counting the number of inequivalent directed embeddings of a given strongly 2-edge-connected 2-regular diplanar digraph on the torus or the Klein bottle and a polynomial-delay algorithm for enumerating them. The author [7] already showed such algorithms in the case of embeddings of 3-connected 3-regular planar graphs. To imitate this algorithm, we can easily obtain the desired algorithm, and hence we omit details here.

4 Concluding remarks

In this paper, we studied directed embeddings of strongly 2-edge-connected 2-regular diplanar digraphs on the sphere, the projective-plane, the torus and the Klein bottle. In order to extend our results to surfaces with higher Euler genera, we should show the complete lists of structures of 3-connected 3-regular planar graphs embedded on these surfaces like Figs. 2, 3 and 4. However, we think that there are a large number of re-embedding types even on surfaces with Euler genus at least 4. Then, it seems to be difficult to give such complete lists without additional assumptions.

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