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All unicyclic Ramsey (mK_2, P_4)-minimal graphs

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Abstract

For graphs F, G and H, we write $F \to (G, H)$ to mean that if the edges of F are colored with two colors, say red and blue, then the red subgraph contains a copy of G or the blue subgraph contains a copy of H. The graph F is called a Ramsey (G, H) graph if $F \to (G, H)$. Furthermore, the graph F is called a Ramsey (G, H)-minimal graph if $F \to (G, H)$ but $F - e \not\to (G, H)$ for any edge $e \in E(F)$. In this paper, we characterize all unicyclic Ramsey (G, H)-minimal graphs when G is a matching mK_2 for any integer $m \geq 2$ and H is a path on four vertices.

1 Introduction

All the graphs discussed in this paper are finite and simple, without isolated vertices, unless otherwise specified. For any graphs F, G, and H, we write $F \to (G, H)$ to mean that if the edges of F are colored with two colors, say red and blue, then there exists either a red copy of G or a blue copy of H as a subgraph of F. The graph

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F is called a *Ramsey* (G, H) *graph* if $F \rightarrow (G, H)$. The *Ramsey number* $R(G, H)$ is the smallest natural number n such that $K_n \rightarrow (G, H)$. There have been extensive studies on Ramsey numbers $R(G, H)$ for a general graph G versus a graph H; see an interesting survey paper [10] regarding the current progress on the Ramsey numbers for general graphs.

From now on, what we mean by 'coloring' is an edge-coloring of a graph. A (G, H)-*coloring* of F is a red-blue coloring of F such that neither a red copy of G nor a blue copy of H occurs. Furthermore, a Ramsey (G, H) graph F is *minimal* if for any edge $e \in E(F)$, $F - e \nrightarrow (G, H)$. In other words, a Ramsey (G, H) graph F is *minimal* if for every edge $e \in E(F)$, there exists a (G, H)-coloring of $F - e$. The set of all Ramsey (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$. A pair of graphs (G, H) is said to be *Ramsey-infinite* if there are infinitely many minimal graphs F for which $F \rightarrow (G, H)$. If a pair (G, H) is not Ramsey-infinite, then it is said to be *Ramsey-finite*.

The problem of Ramsey-infinite pairs of graphs is studied extensively in the literature; for example, Łuczak [7] showed that for every forest F other than a matching, and every graph H containing a cycle, there exists an infinite number of graphs J such that $J \in \mathcal{R}(F, H)$.

In this paper we focus on a pair of Ramsey-finite graphs. Let us briefly discuss some results concerning Ramsey-finiteness. The problem of characterizing a pair (G, H) that is Ramsey-finite was first addressed by Burr et al. [3] in 1978. It was proved that if G is a matching then (G, H) is Ramsey-finite for any graph H. They stated that in general it is difficult to determine the members of $\mathcal{R}(G, H)$, even if (G, H) is Ramsey-finite. In fact the problem appears to be very difficult for $\mathcal{R}(mK_2, H)$. One trivial case is $\mathcal{R}(K_2, H) = \{H\}$ for an arbitrary graph H. Burr et al. [3] also gave two non-trivial sets $\mathcal{R}(G, H)$, namely, $\mathcal{R}(2K_2, 2K_2)$ and $\mathcal{R}(2K_2, K_3)$. Next, the set $\mathcal{R}(mK_2, 2K_2)$ for $m \in [3, 4]$ is given by Burr et al. [4]. Other results concerning Ramsey-finiteness can be seen in [1, 2, 5, 6, 8, 9, 13]. Most recently, Wijaya et al. [12] showed a relation between Ramsey (mK_2, H)-minimal graphs and $((m-1)K_2, H)$-minimal graphs as follows.

Lemma 1.1. [12] *Let H be a graph and $m \geq 2$. $F \rightarrow (mK_2, H)$ if and only if the following conditions hold:*

(i) *for every $v \in V(F)$, $F - \{v\} \rightarrow ((m-1)K_2, H)$;*

(ii) *for every $K_3 \subseteq F$, $F - E(K_3) \rightarrow ((m-1)K_2, H)$; and*

(iii) *for every $F[S_{2m-1}]$ of F, $F - E(F[S_{2m-1}])$ contains a graph H, where $F[S_{2m-1}]$ is a subgraph of F induced by any $(2m-1)$-set $S_{2m-1} \subseteq V(F)$.*

Theorem 1.2. [12] *Let H be a graph and $m \geq 2$. If $F \in \mathcal{R}(mK_2, H)$, then for every $v \in V(F)$ and $K_3 \subseteq F$, both graphs $F - \{v\}$ and $F - E(K_3)$ contain a Ramsey $((m-1)K_2, H)$-minimal graph.*

In [12], it is also shown that for any connected graph H, the graph $F \cup G \in \mathcal{R}(mK_2, H)$ if and only if $F \in \mathcal{R}(sK_2, H)$ and $G \in \mathcal{R}((m-s)K_2, H)$ for every positive integer $s < m$. Let P_n denote a path on n vertices. Wijaya et al. [11] characterized all unicyclic graphs, namely connected graphs containing exactly one cycle, in $\mathcal{R}(mK_2, P_3)$ for any integer $m \geq 2$. More general results as in the following theorem have been also obtained.

Theorem 1.3. [11]

(a) *There is no tree belonging to $\mathcal{R}(mK_2, P_n)$, for any integers $m, n > 1$.*

(b) *The forest in $\mathcal{R}(mK_2, P_n)$ is only the disjoint union of m paths with n vertices, mP_n.*

(c) *Let $m > 1$ and $n > 2$ be positive integers. A cycle graph C_s belongs to $\mathcal{R}(mK_2, P_n)$ if and only if $mn - n + 1 \leq s \leq mn - 1$.*

In this paper we give the characterization of all unicyclic graphs in $\mathcal{R}(mK_2, P_4)$ for any natural number $m \geq 2$. A *unicyclic graph* is a connected graph containing exactly one cycle. Finding all unicyclic graphs in $\mathcal{R}(mK_2, P_4)$ is not as simple as finding all unicyclic graphs in $\mathcal{R}(mK_2, P_3)$. We prove that the only unicyclic graphs other than cycles in $\mathcal{R}(mK_2, P_4)$ are the graphs formed from a cycle by attaching some pendant paths P_2 and/or P_3 with a certain distribution on them. Note that what we mean by a *pendant path* in a unicyclic graph F is the path with one of the end-vertices in the cycle of F, while the remaining vertices are not in the cycle.

2 Properties of Graphs in $\mathcal{R}(mK_2, P_4)$

In this section we derive some properties of a graph belonging to $\mathcal{R}(mK_2, P_4)$. By considering Theorems 1.2 and 1.3(b), we have the following corollary.

Corollary 2.1. *Let $F \in \mathcal{R}(mK_2, P_n)$, $v \in V(F)$ and $m, n \geq 2$. If $F - \{v\}$ is a forest, then $F - \{v\}$ must contain an $(m-1)P_n$.*

Proof. By Theorem 1.2, for every $v \in V(F)$, $F - \{v\}$ contains a graph G in $\mathcal{R}((m-1)K_2, P_n)$. Since $F - \{v\}$ is acyclic, by Theorem 1.3(b), G must be isomorphic to $(m-1)P_n$. □

Lemma 2.2. *Let $m \geq 2$ and $n \geq 4$ be natural numbers. If $F \in \mathcal{R}(mK_2, P_n)$, then no two vertices of degree 1 have a common neighbor.*

Proof. Let $F \in \mathcal{R}(mK_2, P_n)$. For a contradiction, assume there were two vertices of degree 1 in F, say u_1 and u_2, having a common neighbor v. Now, consider two edges $e_1 = u_1v$ and $e_2 = u_2v$. Since $F \in \mathcal{R}(mK_2, P_n)$, there exists an (mK_2, P_n)-coloring ϕ_1 of $F - e_1$. This means that there are at most $(m-1)$ independent red edges in ϕ_1 of $F - e_1$. Now, if $\phi_1(e_2)$ is red then these $(m-1)$ red edges in $F - e_1$ must include e_2. Therefore, we can define a new red-blue coloring ϕ of F such that

$$\phi(x) = \begin{cases} \phi_1(x) & \text{for } x \in F - e_1, \\ \text{red} & \text{for } x = e_1. \end{cases}$$

Then the new coloring ϕ is an (mK_2, P_n)-coloring of F, which is a contradiction. Therefore $\phi_1(e_2)$ must be blue. Since ϕ_1 is an (mK_2, P_n)-coloring of $F - e_1$, there is neither a red mK_2 nor a blue P_n in $F - e_1$. Now, consider a new red-blue coloring φ of F such that

$$\varphi(x) = \begin{cases} \phi_1(x) & \text{for } x \in F - e_1, \\ \text{blue} & \text{for } x = e_1. \end{cases}$$

However, the new coloring φ is now an (mK_2, P_n)-coloring of F, a contradiction. Therefore there are no two vertices of degree 1 in F having a common neighbor. □

Lemma 2.3. *Let F be a unicyclic graph in $\mathcal{R}(mK_2, P_4)$ with $m \geq 2$. Then there is no P_4 in F consisting of exactly one vertex in the cycle of F.*

Proof. Let F be a unicyclic graph in $\mathcal{R}(mK_2, P_4)$ with $m \geq 2$. On the contrary, assume there were a path P_4 consisting one vertex v in the cycle of F and three vertices a, b, c not in the cycle. By Corollary 2.1, $F - \{v\}$ must contain an $(m-1)P_4$. Clearly the vertices a, b and c are not contained in the forest $(m-1)P_4$. So, together with the vertex v, these three vertices will form a P_4 in F. Therefore F contains mP_4, a contradiction to the minimality of F. □

Theorem 2.4. *Let F be a unicyclic graph in $\mathcal{R}(mK_2, P_4)$ with the cycle C. Then $F - E(C)$ is a linear forest with each component being either P_1, P_2 or P_3.*

Proof. Let F be a unicyclic graph in $\mathcal{R}(mK_2, P_4)$ with the cycle C. Since F is unicyclic, the graph $F - E(C)$ is a linear forest with $|V(C)|$ components. By Lemmas 2.2 and 2.3, each component must be either a singleton vertex or a path with one or two edges. □

We now present a very useful necessary and sufficient condition for any unicyclic graph F satisfying $F \to (mK_2, P_4)$.

Theorem 2.5. *Let F be a unicyclic graph. Then $F \to (mK_2, P_4)$ for any $m \geq 2$ if and only if, for any $v \in V(F)$, the graph $F - \{v\} \supseteq (m-1)P_4$.*

Proof. Let F be a unicyclic graph and say $F \to (mK_2, P_4)$. If F is a cycle, then $F - \{v\}$ is a path for each $v \in V(F)$. By Corollary 2.1, $F - \{v\}$ contains a forest $(m-1)P_4$. Now, if F is not a cycle, then for each $v \in V(F)$, the graph $F - \{v\}$ can be either acyclic or a (connected or disconnected) graph containing exactly one cycle. By Corollary 2.1, if $F - \{v\}$ is an acyclic graph, then F contains an $(m-1)P_4$ and the proof is complete. Now, consider the case $F - \{v\}$ is a (connected or disconnected) graph containing exactly one cycle. Let C be the cycle of $F - \{v\}$. Now, choose the vertex $w \in V(C)$ such that $d(v, w) \leq d(v, u)$ for all $u \in V(C)$. We have that $(F - \{v\}) - \{w\}$ is a forest with two components where the first component is a tree and the second one is a path P_r for some natural number r. By Theorem 2.4, $1 \leq r \leq 2$. By Corollary 2.1, the graph $(F - \{v\}) - \{w\}$ contains a forest $(m-1)P_4$. Clearly the path P_r is not contained in the forest $(m-1)P_4$. Hence the graph $F - \{v\}$

contains the same $(m-1)P_4$ as in $(F-\{v\})-\{w\}$. Therefore, for each $v \in V(F)$, the graph $F-\{v\}$ contains an $(m-1)P_4$.

Conversely, if for each $v \in V(F)$ we have $F-\{v\} \supseteq (m-1)P_4$, then we will show that $F \to (mK_2, P_4)$ provided F is a unicyclic graph. Consider any red-blue coloring of the edges of F containing no red copy of mK_2. Then there are at most $(m-1)$ independent red edges in such a coloring on F. Now, choose any vertex v in F incident to red edge in such a coloring. By the assumption that $F-\{v\} \supseteq (m-1)P_4$ for such a vertex v and since such a coloring has at most $(m-1)$ independent red edges (including one red edge incident with v), then the other $(m-2)$ independent red edges will be distributed in the subgraph $(m-1)P_4$ and leave one path P_4 without red color. It means that there is a blue P_4 in such a coloring. So, $F \to (mK_2, P_4)$. □

The following assertion is a direct consequence of Theorem 2.5.

Corollary 2.6. *Let $m \geq 2$ be a natural number. Let F be a unicyclic graph and $F \to (mK_2, P_4)$. If there is an edge $e \in E(F)$ such that $(F-e)-\{v\} \supseteq (m-1)P_4$ for any vertex $v \in V(F)$, then F is not minimal.*

Proof. Let F be a unicyclic graph and $F \to (mK_2, P_4)$. So, if there is an edge $e \in E(F)$ such that $(F-e)-\{v\} \supseteq (m-1)P_4$ for any vertex $v \in V(F)$, then by Theorem 2.5, we have $(F-e) \to (mK_2, P_4)$. This means that F is not minimal. □

Now we discuss a circumference of a unicyclic graph belonging to $\mathcal{R}(mK_2, P_4)$. The *circumference* of a graph refers to the length of a longest cycle in the graph.

Lemma 2.7. *Let $m \geq 2$ be a natural number. If $F \in \mathcal{R}(mK_2, P_4)$ is a unicyclic graph other than a cycle, then the cycle in F has circumference s with $2m \leq s \leq 4m-4$.*

Proof. Let F be a unicyclic Ramsey (mK_2, P_4)-minimal graph other than a cycle. Then F contains a unique cycle C. By Theorem 1.3(c), the cycle C must have circumference s at most $4m-4$, that is, $s \leq 4m-4$. Otherwise, F contains either a cycle in $\mathcal{R}(mK_2, P_4)$ or a forest mP_4. Now, suppose for a contradiction, that $s \leq 2m-1$. Define a red-blue coloring of the edges of F such that all edges in the cycle C are colored red, and the other edges (namely all edges of pendant paths) are colored blue. By Lemma 2.4, no pendant path in F contains a copy of P_4. So, by such a coloring, there is neither a red mK_2 nor a blue P_4 in F; a contradiction. Therefore $2m \leq s \leq 4m-4$. □

Next, we discuss the lower bound of the number of edges in a unicyclic graph F in $\mathcal{R}(mK_2, P_4)$.

Lemma 2.8. *Let $m \geq 2$ be a natural number. Let $F \in \mathcal{R}(mK_2, P_4)$ be a unicyclic graph other than a cycle. Then $|E(F)| \geq 4m-2$.*

Proof. Let C be the cycle in F and let $v \in V(C)$ be of degree 3. By Theorem 2.5, we have $F-\{v\} \supseteq (m-1)P_4$. Since every two consecutive P_4s in $(m-1)P_4$ must be separated by at least one edge, it follows that we have in total at least $3(m-1)+(m-2)+3 = 4m-2$ edges. □

By Theorem 2.4, we can conclude that each pendant path in a unicyclic Ramsey (mK_2, P_4)-minimal graph must be isomorphic to either P_2 or P_3. Let us define classes of such unicyclic graphs. A unicyclic graph $F \in \mathcal{R}(mK_2, P_4)$ is said to have a *gap sequence* $(a_i)_1^{t-1} = (a_1, a_2, \dots, a_{t-1})$ if all cycle vertices of degree 3 in F can be cyclically ordered as $u_1, u_2, \dots, u_t$ such that a_i is the length of path from u_i to u_{i+1} for each $i \in [1, t-1]$. If we shift the label u_1 to u_2, u_2 to u_3, and so on until u_t to u_1, then a gap sequence of this graph is $(a_2, a_3, \dots, a_{t-1}, a_t)$ where $a_t = s - \sum_{i=1}^{t-1} a_i$. So a gap sequence depends on the labels of vertices of degree 3. For $r = 2$ or 3, denote by $C_s[(t, P_r);\ (a_i)_1^{t-1}]$ the unicyclic graph F with circumference s and having the gap sequence $(a_1, a_2, \dots, a_{t-1})$ such that at every vertex u_i, $i \in [1, t]$, there is a pendant path P_r starting from it. So the order of the graph $C_s[(t, P_r);\ (a_i)_1^{t-1}]$ is $s + (r-1)t$. For example, two graphs in Figure 1 are isomorphic, where the gap sequence depends on the label u_1. To determine all unicyclic Ramsey (mK_2, P_4)-

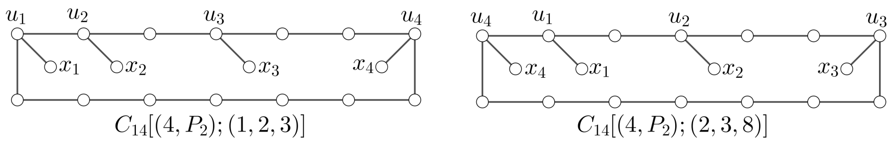


Figure 1: Two isomorphic graphs with distinct gap sequences.

minimal graphs F other than a cycle, we consider whether the graph F contains pendant path P_2 or P_3 only or both.

3 The graph $C_s[(t, P_2); (a_1, a_2, \dots, a_{t-1})]$

In this section we characterize all the graphs $C_s[(t, P_2);\ (a_i)_1^{t-1}]$ with circumference s and gap sequence $(a_1, a_2, \dots, a_{t-1})$ which are Ramsey unicyclic (mK_2, P_4)-minimal graphs.

Lemma 3.1. *Let m, s and t be natural numbers with $m \geq 2$ and $2m \leq s \leq 4m-4$. Let F be a unicyclic graph $C_s[(t, P_2);\ (a_i)_1^{t-1}]$. If there exists some $i \in [1, t-1]$ such that a_i is even and for any $v \in V(F)$, $F - \{v\} \supseteq (m-1)P_4$, then $t \geq 4m - s - 1$.*

Proof. Suppose for each $v \in V(F)$, $F - \{v\} \supseteq (m-1)P_4$. Then $|V(F)| \geq 4(m-1)+1$. For a contradiction, assume $t = 4m - s - 2$. So, F has $t + s$ $(= 4m - 2)$ vertices. Let u_i be the vertex of degree 3 and x_i be the pendant vertex adjacent to u_i for each $i \in [1, t]$. Without loss of generality, we may assume a_1 is even. Then the graph $F - \{u_1\}$ must be isomorphic to a disconnected graph $K_1 \cup T_{4m-4}$ where T_{4m-4} is a tree of order $4m-4$. Since a_1 is even, there is at most one independent P_4 formed by the five vertices (including u_2 and x_2), as depicted in Figure 2. Then the remaining $4m - 9$ vertices are insufficient to form $(m-2)P_4$ in $F - \{u_1\}$, which contradicts the fact that $F - \{v\} \supseteq (m-1)P_4$ for any $v \in F$. Therefore we conclude that $t \geq 4m - s - 1$. □

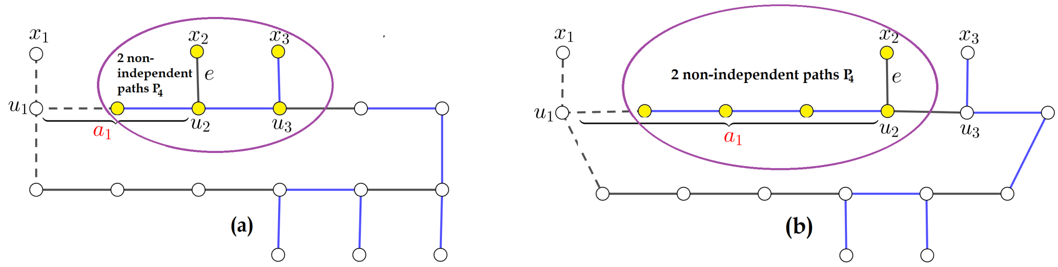


Figure 2: A path P_4 from the five vertices.

Lemma 3.2. *Let m, s and t be natural numbers with $m \geq 2$ and $2m \leq s \leq 4m-4$. Let F be a unicyclic graph $C_s[(t, P_2);\ (a_i)_1^{t-1}]$. If $F \in \mathcal{R}(mK_2, P_4)$, then all the a_i are odd.*

Proof. Let $F \in \mathcal{R}(mK_2, P_4)$ be a unicyclic graph $C_s[(t, P_2);\ (a_i)_1^{t-1}]$ with circumference s and a gap sequence $(a_1, a_2, \ldots, a_{t-1})$. On the contrary, suppose that there exists some $i \in [1, t-1]$ such that a_i is even. Without loss of generality, we can assume a_1 is even. Let u_i be the vertex of degree 3 and x_i be the pendant vertex of F adjacent to u_i for each $i \in [1, t]$. According to Lemma 3.1, $t \geq 4m - s - 1$. Now consider the pendant edge $e = u_2x_2$. Since $F \in \mathcal{R}(mK_2, P_4)$, by Theorem 2.5, for each $v \in V(F)$, $F - \{v\} \supseteq (m-1)P_4$. From the proof of Lemma 3.1, there is a path P_4 not containing the edge e as depicted in Figure 2. This means that for each $v \in V(F)$, $(F-e) - \{v\}$ contains an $(m-1)P_4$, for some pendant edge $e = x_2u_2$. By Corollary 2.6, F is not minimal, which contradicts the fact that $F \in \mathcal{R}(mK_2, P_4)$. Therefore all the a_i are odd. □

For an illustration, consider $F = C_{14}[(5, P_2);\ (2, 1, 3, 1)]$. In this case, $m = 5$, $s = 14$ and $t = 5$. Then, $F \rightarrow (5K_2, P_4)$ as depicted in Figure 3. We can see that for each vertex $v \in V(F)$, $F - \{v\} \supseteq 4P_4$ (in this case, by removing the red vertex of the graph F we have $4P_4$ (in blue)) and the red pendant edge e is not included. Since a gap a_1 is even, for each $v \in V(F)$, $(F-e) - \{v\}$ contains a $4P_4$. So the graph $F = C_{14}[(5, P_2);\ (2, 1, 3, 1)]$ is not minimal.

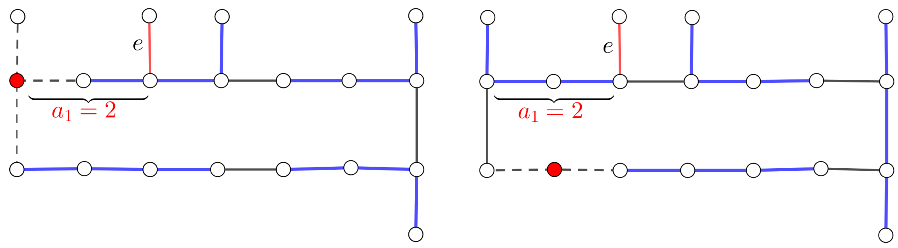


Figure 3: The graph $C_{14}[(5, P_2); (2, 1, 3, 1)]$.

Theorem 3.3. *Let m, s and t be natural numbers with $m \geq 2$ and $2m \leq s \leq 4m-4$. Let F be a unicyclic graph $C_s[(t, P_2); (a_i)_1^{t-1}]$. Then $F \in \mathcal{R}(mK_2, P_4)$ if and only if (i) all the a_i are odd and (ii) $t = 4m - s - 2$.*

Proof. Let F be a unicyclic graph $C_s[(t, P_2); (a_i)_1^{t-1}]$ and $F \in \mathcal{R}(mK_2, P_4)$. First, by Lemma 3.2, all the a_i are odd. Now we will show that $t = 4m-s-2$. By Lemma 2.8, we have $|E(F)| \geq 4m - 2$ and so $|V(F)| \geq 4m - 2$ (since F is a unicyclic graph). Therefore $t \geq 4m - s - 2$. Since every $C_s[(t, P_2); (a_i)_1^{t-1}]$ with $t > 4m - s - 2$ must contain $C_s[(t^*, P_2); (a_i)_1^{t^*-1}]$ with $t^* = 4m - s - 2$ as a subgraph by removing the last consecutive pendant edges, then to get the minimality of F we must have that $F = C_s[(t, P_2); (a_i)_1^{t-1}]$ with $t = 4m - s - 2$.

Conversely, let F be a unicyclic graph $C_s[(t, P_2); (a_i)_1^{t-1}]$ with a gap sequence $(a_i)_1^{t-1}$, where all the a_i are odd and $t = 4m - s - 2$. We can see that for every $v \in V(F)$, $F - \{v\} \supseteq (m-1)P_4$. By Theorem 2.5, we get $F \to (mK_2, P_4)$. Next, to prove the minimality, let e be any edge of F. If e is a pendant edge, then for each vertex w of degree 3, $(F-e) - \{w\} \not\supseteq (m-1)P_4$. If e is an edge in the cycle C of F, then $F - e$ is a tree with $4m - 2$ vertices and $4m - 3$ edges. Now, choose a vertex z in C such that $(F - e) - \{z\}$ is isomorphic to a disconnected graph $P_r \cup G$, where $2 \leq r \leq 3$ and G is a forest having at most two components. So G has q edges, where $4m - 8 \leq q \leq 4m - 6$. In this case, $G \not\supseteq (m-1)P_4$, since G does not have enough edges. Therefore $(F-e) - \{z\} \not\supseteq (m-1)P_4$. So we have shown that for any edge e, $(F - e) \nrightarrow (mK_2, P_4)$. Hence $F \in \mathcal{R}(mK_2, P_4)$. □

In Figure 4, we give an example of graphs $C_{10}[(4, P_2); (a_i)_1^3]$ with all odd a_i that belong to $\mathcal{R}(4K_2, P_4)$.

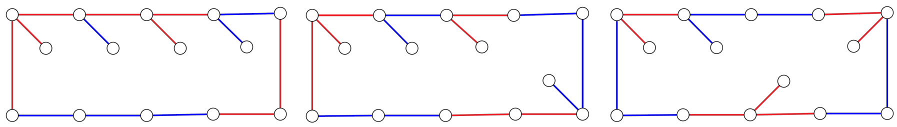


Figure 4: Some examples of the graphs in $\mathcal{R}(4K_2, P_4)$.

4 The graph $C_s[(t, P_3); (b_1, b_2, \ldots, b_{t-1})]$

In this section, we derive necessary and sufficient conditions for unicyclic graphs $C_s[(t, P_3); (b_i)_1^{t-1}]$ with circumference s and a gap sequence $(b_i)_1^{t-1} = (b_1, b_2, \ldots, b_{t-1})$ to be members of $\mathcal{R}(mK_2, P_4)$.

Lemma 4.1. *Let t and m be natural numbers with $m \geq 2$ and $2m \leq s \leq 4m-4$. Let F be a unicyclic graph $C_s[(t, P_3); (b_i)_1^{t-1}]$. If $F \in \mathcal{R}(mK_2, P_4)$, then $b_i \not\equiv 0, 3 \bmod 4$ for each $i \in [1, t-1]$.*

Proof. Let F be a unicyclic graph $C_s[(t,P_3);(b_i)_1^{t-1}]$. For a contradiction, assume there exists $i \in [1,t-1]$ such that $b_i \equiv 0$ or $3 \bmod 4$. Without loss of generality we assume $b_1 \equiv 0$ or $3 \bmod 4$. Let $b_1 \equiv 0 \bmod 4$. Consider now the subgraph B_1 of F obtained by removing all vertices (of degree 1 or 2) in all pendant paths other than two consecutive pendant paths causing a gap b_1. Therefore B_1 is isomorphic to a graph $C_s[(2,P_3);(4k)]$, for some positive integer k. Now, relabeling (if necessary) the vertices of B_1 in such a way we have the graph depicted in Figure 5(a). Consider a path $\mathbb{P}_1 := (x_1, x_2, v_1, v_2, \ldots, v_{1+4k}, y_2, y_1)$ in B_1 of length $4(k+1)$ (depicted with yellow vertices). It is clear that $\mathbb{P}_1 \supseteq (k+1)P_4$ and $\mathbb{P}_1 - \{v_1\} \supseteq kP_4$ where y_1 can be

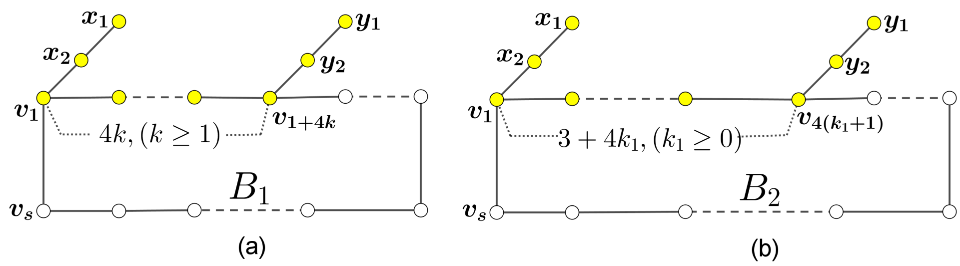


Figure 5: Two unicyclic graphs $B_1 = C_s[(2,P_3);\ (4k, s-4k)]$ for some $k \geq 1$ and $B_2 = C_s[(2,P_3);\ (3+4k_1, s-3-4k_1)]$ for some $k_1 \geq 0$.

included in $V(kP_4)$ but $v_2 \notin V(kP_4)$. This kP_4 is a part of $(m-1)P_4$ in $F - \{v_1\}$. Since the four vertices x_1, x_2, v_1 and v_2 can form a path P_4, it follows that $F \supseteq mP_4$. Hence F is not minimal, a contradiction.

The case $b_1 \equiv 3 \bmod 4$ is treated similarly by considering a path $\mathbb{P}_2 := (x_1, x_2, v_1, v_2, \ldots, v_{4+4k_1}, y_2, y_1)$ in B_2 of length $7+4k_1$ (depicted with yellow vertices) as depicted in Figure 5(b), where B_2 is the subgraph $C_s[(2,P_3);(3+4k_1)]$ of F obtained by deleting all vertices in all pendant paths except two consecutive pendant paths causing a gap b_1. □

Lemma 4.2. *Let F be a unicyclic graph $C_s[(t,P_3);\ (b_i)_1^{t-1}]$. If there are two gaps b_i and b_j with $b_i, b_j \equiv 1 \bmod 4$ for some $i,j \in [1,t-1]$ and for each $v \in V(F)$, $F - \{v\} \supseteq (m-1)P_4$, then $t > 2m - \lceil \frac{s}{2} \rceil$.*

Proof. For a contradiction, assume that $t \leq 2m - \lceil \frac{s}{2} \rceil$. Then $|V(F)| \leq 2t + s = 4m + s - 2\lceil \frac{s}{2} \rceil$. We consider two cases. First, consider the case where b_i and b_j are consecutive. We can assume that $i = 1$ and $j = 2$, namely $b_1 = 1 + 4k_1$ and $b_2 = 1 + 4k_2$ for some positive integers k_1 and k_2. Write $F = C_s[(t,P_3);(1+4k_1, 1+4k_2, b_3, \ldots, b_{t-1})]$. Consider the subgraph $B_{3a} = C_s[(3,P_3);(1+4k_1, 1+4k_2)]$ of F. We relabel the vertices of B_{3a} as depicted in Figure 6(a).

Now consider the subgraph of B_{3a} induced by the set $U = \{v_1, v_2, \ldots, v_{3+4(k_1+k_2)}, x_1, x_2, y_1, y_2, z_1, z_2\}$. Since $F - \{v_1\} \supseteq (m-1)P_4$, it follows that the subgraph induced by the set $U - \{v_1\}$ will contribute $(1+k_1+k_2)P_4$ and $F - U$ must contain $(m-2-k_1-k_2)P_4$. However, there are only at most $4(m-2-k_1-k_2)-1$ vertices

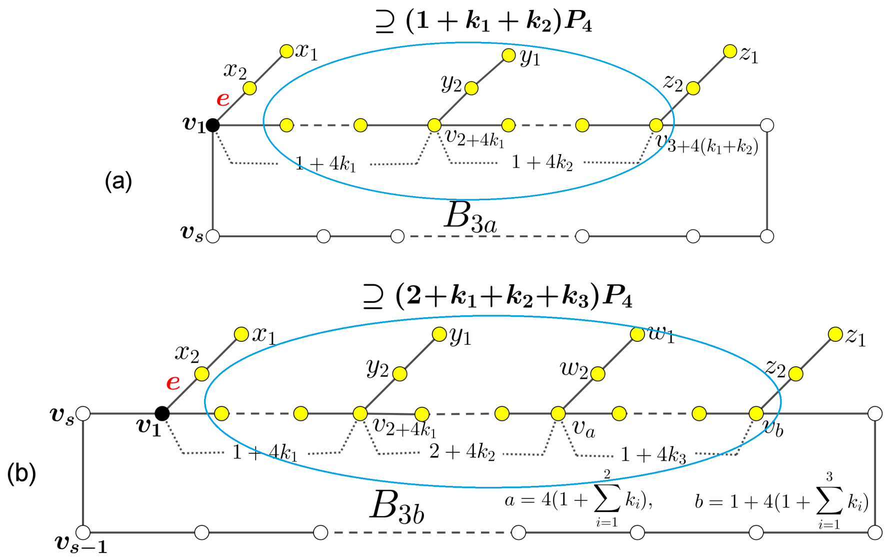


Figure 6: Two graphs $B_{3a} = C_s[(3, P_3); (1 + 4k_1, 1 + 4k_2)]$ and $B_{3b} = C_s[(4, P_3); (1 + 4k_1, 2 + 4k_2, 1 + 4k_3)]$ for some integers $k_1, k_2, k_3 \geq 0$.

in $F - U$ since

$$\begin{aligned} |V(F)| - |U| &\leq (4m + s - 2\lceil \tfrac{s}{2} \rceil) - (4k_1 + 4k_2 + 9) \\ &= 4(m - 2 - k_1 - k_2) - (2\lceil \tfrac{s}{2} \rceil) - s + 1). \end{aligned}$$

Therefore the supposition that $t \leq 2m - \lceil \frac{s}{2} \rceil$ leads to a contradiction. Hence $t > 2m - \lceil \frac{s}{2} \rceil$ if b_i and b_j are consecutive.

Now consider the case where b_i and b_j are not consecutive. Without loss of generality, let $b_1 = 1 + 4k_1$ and $b_2 = 2 + 4k_2$, and $b_3 = 1 + 4k_3$ for some non-negative integers k_1, k_2, and k_3. Write $F = C_s[(t, P_3); (1 + 4k_1, 2 + 4k_2, 1 + 4k_3, b_4, \ldots, b_{t-1})]$. Consider a subgraph $B_{3b} = C_s[(4, P_3); (1 + 4k_1, 2 + 4k_2, 1 + 4k_3)]$ of F. We relabel the vertices of B_{3b} as depicted in Figure 6(b). Consider the subgraph of B_{3b} induced by the set $U = \{v_1, \ldots, v_{5+4(k_1+k_2+k_3)}, w_1, w_2, x_1, x_2, y_1, y_2, z_1, z_2\}$. Since $F - \{v_1\} \supseteq (m - 1)P_4$, it follows that the subgraph induced by the set $U - \{v_1\}$ will contribute at most $(2 + k_1 + k_2 + k_3)P_4$ and the subgraph $F - U$ must contain the remaining $(m - 3 - k_1 - k_2 - k_3)P_4$. However, there are only at most $4(m - 3 - k_1 - k_2 - k_3) - 1$ vertices in $F - U$ since

$$\begin{aligned} |V(F)| - |U| &\leq (4m + s - 2\lceil \tfrac{s}{2} \rceil) - (4k_1 + 4k_2 + 4k_3 + 13) \\ &= 4(m - 3 - k_1 - k_2 - k_3) - (2\lceil \tfrac{s}{2} \rceil) - s + 1). \end{aligned}$$

So this leads to a contradiction. Thus $t > 2m - \lceil \frac{s}{2} \rceil$. □

Lemma 4.3. *Let t and m be natural numbers with $m \geq 2$ and $2m \leq s \leq 4m - 4$. Let F be a unicyclic graph $C_s[(t, P_3); (b_i)_1^{t-1}]$. If $F \in \mathcal{R}(mK_2, P_4)$, then there exists at most one $i_0 \in [1, t - 1]$ such that $b_{i_0} \equiv 1 \bmod 4$, and for the remaining $i \neq i_0$, $b_i \equiv 2 \bmod 4$.*

Proof. Let F be a unicyclic graph $C_s[(t, P_3); (b_i)_1^{t-1}]$ and $F \in \mathcal{R}(mK_2, P_4)$. By Lemma 4.1, we have $b_i \equiv 1$ or $2 \bmod 4$. Now, for a contradiction, suppose that there were two distinct indices i_0 and i_1 such that $b_{i_0} = 1 + 4k_1$ and $b_{i_1} = 1 + 4k_2$ for some positive integers k_1 and k_2. By Lemma 4.2, $t \geq 2m + 1 - \lceil \frac{s}{2} \rceil$. If both b_{i_0} and b_{i_1} are consecutive, then the graph B_{3a} in Figure 6(a) is a subgraph of F (see the proof of Lemma 4.2). If b_{i_0} and b_{i_1} are not consecutive, then F contains the graph B_{3b} as depicted in Figure 6(b). In each of these subgraphs, consider the edge $e = v_1x_2$. We can see that for each $v \in V(F)$, $(F-e)-\{v\}$ contains an $(m-1)P_4$. By Corollary 2.6, $(F - e) \to (mK_2, P_4)$. This means that F is not minimal, a contradiction. Thus we conclude that there is at most one $i_0 \in [1, t-1]$ such that $b_{i_0} \equiv 1 \bmod 4$. □

Theorem 4.4. *Let t, m, s be natural numbers with $m \geq 2$ and $2m \leq s \leq 4m-4$. Let F be a unicyclic graph $C_s[(t, P_3); (b_i)_1^{t-1}]$. Then the graph F satisfies $F \in \mathcal{R}(mK_2, P_4)$ if and only if the following three conditions hold:*

(i) *there exists at most one $i_0 \in [1, t-1]$ such that $b_{i_0} \equiv 1 \bmod 4$ and the remaining b_i satisfy $b_i \equiv 2 \bmod 4$;*

(ii) *s is odd; and*

(iii) $t = 2m - \lceil \frac{s}{2} \rceil$.

Proof. Let F be a unicyclic graph $C_s[(t, P_3); (b_i)_1^{t-1}]$ satisfying the three conditions above. It is easy to check that for each $v \in V(F)$, we have $F - \{v\} \supseteq (m-1)P_4$. By Theorem 2.5, we obtain $F \to (mK_2, P_4)$. To prove the minimality, we consider an edge $e \in E(F)$. If e is an edge of a cycle of F, then choose the vertex w in the cycle of F such that the graph $(F-e)-\{w\}$ is either $P_3 \cup T_a$ or $P_6 \cup T_b$ where T_a or T_b is a tree of order $4m-5$ or $4m-8$, respectively. We obtain $(F-e)-\{w\} \not\supseteq (m-1)P_4$. Next, let e be an edge of a pendant path of F. Choose a vertex w of degree 3 in $F - e$. Then we find that $(F-e)-\{w\} \not\supseteq (m-1)P_4$. Hence, for each $e \in E(F)$, we have $(F - e) \not\to (mK_2, P_4)$. Therefore F is minimal.

Conversely, suppose that $F \in \mathcal{R}(mK_2, P_4)$. First, by Lemma 4.3, there is at most one $i_0 \in [1, t-1]$ such that $b_i \equiv 1 \bmod 4$ and for the remaining $i \neq i_0$, $b_i \equiv 2 \bmod 4$, so (i) holds. We are going to show that s must be odd. Assume, to the contrary, that s were even. Now, if $t \geq 2m - \lceil \frac{s}{2} \rceil$, then $F \supseteq mP_4$. So F is not minimal. If $t < 2m - \lceil \frac{s}{2} \rceil$, then we can choose a vertex u of degree 3 in F to obtain $F - \{u\} \not\supseteq (m-1)P_4$. So $F \not\to (mK_2, P_4)$, a contradiction, and the second condition holds. Next, we prove that the third condition must be satisfied, namely $t = 2m - \lceil \frac{s}{2} \rceil$. For a contradiction, let $t > 2m - \lceil \frac{s}{2} \rceil$. Then F would be not minimal, since F must contain an mP_4. However, if $t < 2m - \lceil \frac{s}{2} \rceil$, then there exists a vertex w of degree 3 in F so that $F - \{w\} \not\supseteq (m-1)P_4$. This means that $F \not\to (mK_2, P_4)$, a contradiction. Therefore the condition $t = 2m - \lceil \frac{s}{2} \rceil$ holds. □

As an illustration, in Figure 7 we provide the graphs $C_{13}[(3, P_3); (2, 2)]$ and $C_{13}[(3, P_3); (1, 2)]$ which are in $\mathcal{R}(5K_2, P_4)$.

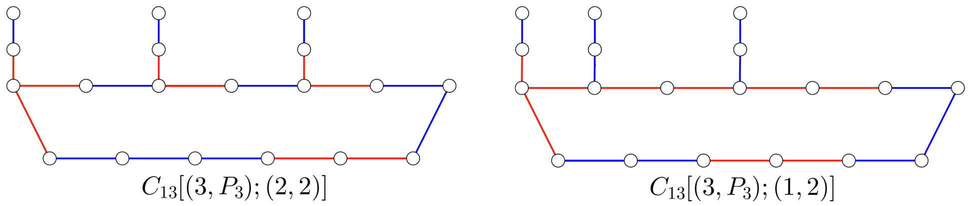


Figure 7: Two examples of the graphs in $\mathcal{R}(5K_2, P_4)$.

5 The graph $C_s[(t, P_2), (t^*, P_3); (a_1, \ldots, a_{t-1}), (b_0, b_1, \ldots, b_{t^*-1})]$

In this section, we characterize all unicyclic graphs G containing both pendant paths P_2 and P_3. First, we discuss the graphs G when all pendant paths P_2 are consecutive. We denote these graphs by $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ where $(a_i)_1^{t-1} = (a_1, \ldots, a_{t-1})$, $(b_j)_0^{t^*-1} = (b_0, b_1, \ldots, b_{t^*-1})$ and b_0 is the distance between the cycle vertex incident with the last pendant path P_2 and the cycle vertex incident with the first pendant path P_3. According to Lemma 3.2, all the a_i are odd for $i \in [1, t-1]$.

Lemma 5.1. *Let m, s, t, t^* be natural numbers and $m \geq 2$. Let F be a unicyclic graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$. If F is a Ramsey (mK_2, P_4)-minimal graph, then $b_0 \equiv 1 \bmod 2$.*

Proof. Let F be a unicyclic Ramsey (mK_2, P_4)-minimal graph of the form $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$. We are going to show that $b_0 \equiv 1 \bmod 2$. Suppose to the contrary that b_0 is even. To do this, we write $F = C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (0 \bmod 2, b_1, \ldots, b_{t^*-1})]$. We consider two cases: $b_0 = 2 + 4k$ or $b_0 = 4(k + 1)$ for some integer $k \geq 0$. We observe the subgraph $C_s[(1, P_2), (1, P_3); (b_0)]$ of F. For $b_0 = 2 + 4k$, consider the graph B_{4a}, while for $b_0 = 4(k + 1)$, consider the graph B_{4b}. Relabel these two graphs as depicted in Figure 8.

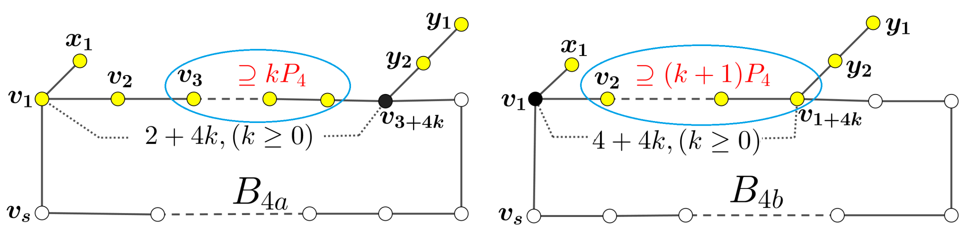


Figure 8: The graphs $B_{4a} = C_s[(1, P_2), (1, P_3); (2+4k)]$ and $B_{4b} = C_s[(1, P_2), (1, P_3); (4 + 4k)]$, for some integer $k \geq 0$.

Consider the subgraph of B_{4a} induced by the set U_a, where $U_a = \{v_1, v_2, \ldots, v_{3+4k}, x_1, y_1, y_2\}$. By Theorem 2.5, the graph $F - \{v_{3+4k}\}$ must contain a forest $(m-1)P_4$, where the path from $v_3, v_4, \ldots, v_{2+4k}$ contains a kP_4. We can see that x_1

and v_2 are the pendant vertivces of $F - e$ (with $e = v_2v_3$). This means that we can exclude the vertex v_2 to form the forest $(m-1)P_4$, and its role is replaced by x_1. However, the path from $v_2, v_3, \ldots, y_1$ contains a $(k+1)P_4$. It forces $F \supseteq mP_4$. Hence F is not minimal; a contradiction. Next we consider the subgraph of B_{4b} induced by the set U_b, where $U_b = \{v_1, v_2, \ldots, v_{1+4k}, x_1, y_1, y_2\}$. By Theorem 2.5, the graph $F - \{v_1\} \supseteq (m-1)P_4$. The subgraph induced by the set $U_b - \{v_1\}$ must contain a $(k+1)P_4$, and exclude the vertices y_1 and y_2. Since the induced subgraph $F[U_b]$ contains a $(k+2)P_4$, it forces $F \supseteq mP_4$. So F is not minimal; a contradiction. □

In the next corollary we show that there is no unicyclic graph $C_s[(1, P_2), (1, P_3); (b_0)]$ in $\mathcal{R}(mK_2, P_4)$ for any integers $m \geq 2$ and $s \geq 1$.

Corollary 5.2. *The graph $C_s[(1, P_2), (1, P_3); (b_0)]$ is not in $\mathcal{R}(mK_2, P_4)$ for any positive integers s and $m \geq 2$.*

Proof. Let F be a unicyclic graph $C_s[(1, P_2), (1, P_3); (b_0)]$ with any $s \geq 1$. By contradiction, assume that $F \in \mathcal{R}(mK_2, P_4)$. It follows from Theorem 3.3 that $C_{4m-4}[(2, P_2); (1 \bmod 2)]$ is in $\mathcal{R}(mK_2, P_4)$. Let F be a unicyclic graph $C_s[(1, P_2), (1, P_3); (b_0)]$. By Lemma 5.1, b_0 must be odd. For $s = 4m-4$, $F \supseteq C_s[(2, P_2); (1 \bmod 2)]$. So $F \notin \mathcal{R}(mK_2, P_4)$. For $s \leq 4m-5$, for each vertex u of degree 3 incident with the pendant path P_3, we have $F - \{w\} \not\supseteq (m-1)P_4$. This means that $F \nrightarrow (mK_2, P_4)$. This leads to a contradiction. □

Now we discuss the gap sequence $(b_j)_0^{t^*-1}$ for pendant paths P_3. It follows from Lemma 4.1 that $b_j \not\equiv 0, 3 \bmod 4$. By Lemma 4.3, there exists at most one $i_0 \in [1, t]$ such that $b_{j_0} \equiv 1 \bmod 4$ and for the remaining $i \neq i_0$, $b_j \equiv 2 \bmod 4$.

Lemma 5.3. *Let m, s, t and t^* be natural numbers with $m \geq 3$ and $2m \leq s \leq 4m-5$. Let F be a unicyclic graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ with all the a_i and b_0 odd. If $F \in \mathcal{R}(mK_2, P_4)$, then $b_j \equiv 2 \bmod 4$ for all $j \in [1, t^*-1]$.*

Proof. For a contradiction, assume that $b_j \not\equiv 2 \bmod 4$ for some $j \in [1, t^*-1]$. According to Lemmas 4.1 and 4.3, there is exactly one $j_0 \in [1, t^*-1]$ such that $b_{j_0} \equiv 1 \bmod 4$ and for the remaining j, $b_j \equiv 2 \bmod 4$. Therefore F contains B_5 as a subgraph, where $B_5 = C_s[(1, P_2), (2, P_3); (1+2k_1, 1+4k_2)]$ for some natural numbers $k_1, k_2 \geq 0$. Relabeling all vertices of B_5 in such a way, we have the graph as depicted in Figure 9(a). By Theorem 3.3, we have $C_{4m-5}[(3, P_2); (1 \bmod 2)] \in \mathcal{R}(mK_2, P_4)$. Consequently, for $s = 4m-5$, F is not minimal since F contains $C_{4m-5}[(3, P_2); (1 \bmod 2)]$.

Now, consider s even and $2m \leq s \leq 4m-6$. Since b_0 is odd, clearly $t^* \geq 2$. By relabeling the graph B_5 with opposite direction (with v_1 fixed, v_s becomes v_2, $v_{3+2k_1+4k_2}$ becomes $v_{s-1-2k_1-4k_2}$, and so on; see Figure 9(b)), we obtain that the length of the path from the vertex v_1 to $v_{s-1-2k_1-4k_2}$ is b_0, where b_0 is even, which contradicts the fact that b_0 is odd.

Now consider the case s odd and $2m+1 \leq s \leq 4m-7$. If we take $s = 4m-7$, $t^* = 2$ and $t = 1$, then $F - \{v_1\} \not\supseteq (m-1)P_4$. So $F \nrightarrow (mK_2, P_4)$, a contradiction. If $t^* > 2$, F is not minimal since F contains a graph $C_{4m-7}[(3, P_3); (1 \bmod 4, 2 \bmod 4)] \in$

$\mathcal{R}(mK_2, P_4)$ (by Theorem 4.4). If $t > 1$ and t is even, then by relabeling the graph F with opposite direction we find that the length of the path from the vertex incident with the last pendant path P_3 to the vertex incident with the first pendant path P_2 is even, which produces a contradiction. Hence, for $s = 4m - 7$, it should be $b_j \equiv 2 \bmod 4$ for all $j \in [1, t^*-1]$. Any other odd values of s with $2m+1 \leq s \leq 4m-9$ can be proved in a similar fashion. □

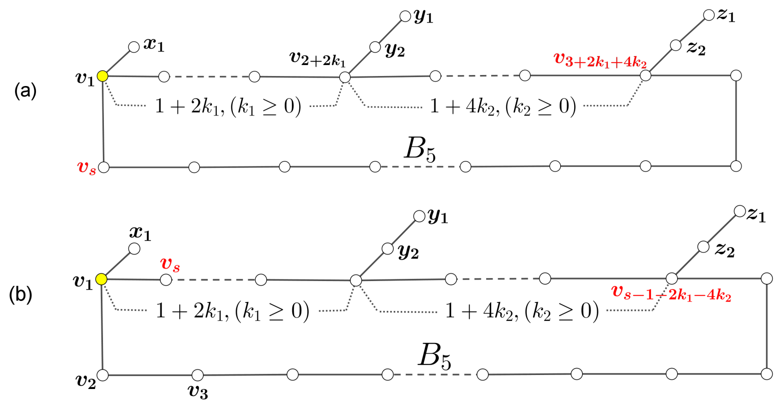


Figure 9: The graph $B_5 = C_s[(1, P_2), (2, P_3); (1+2k_1), (1+4k_2)]$ for some non-negative integers k_1 and k_2 with two different labelings.

According to Lemmas 2.8, 3.2, 4.1, 5.1, and 5.3 we have the following consequence.

Corollary 5.4. *If a unicyclic graph $C_s[(t, P_2)(t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ is Ramsey (mK_2, P_4)-minimal, then the following three conditions hold:*

(i) *both b_0 and all the a_i are odd;*

(ii) *$b_j \equiv 2 \bmod 4$ for each $j \in [1, t^* - 1]$;*

(iii) *$t + 2t^* \geq 4m - s - 2$.*

Proof. Let $F \in \mathcal{R}(mK_2, P_4)$ be a unicyclic graph $C_s[(t, P_2)(t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$. By Lemmas 3.2, 4.1, 5.1 and 5.3, the conditions of (i) and (ii) hold. By Lemma 2.8, we obtain $|E(F)| = s + t + 2t^* \geq 4m - 2$. So $t + 2t^* \geq 4m - s - 2$, that is, the condition (iii) holds. □

Lemma 5.5. *Let m, s, t and t^* be natural numbers with $m \geq 2$ and $2m + 1 \leq s \leq 4m - 6$. Let F be a unicyclic graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ with all the a_i and b_0 odd, and $b_j \equiv 2 \bmod 4$ for $i \in [1, t-1], j \in [1, t^* - 1]$. If s and t are the same parity, then $F \notin \mathcal{R}(mK_2, P_4)$.*

Proof. Let F be a unicyclic graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ with all the a_i and b_0 odd, and $b_j \equiv 2 \bmod 4$ for $i \in [1, t - 1], j \in [1, t^* - 1]$. By Corollary 5.4(c), we have $t + 2t^* \geq 4m - s - 2$. Let s and t both be odd. For $t + 2t^* = 4m - s - 2$, by choosing the vertex u of degree 3 incident with a pendant path P_3, we obtain $F - \{u\} \not\supseteq (m-1)P_4$. So $F \not\rightarrow (mK_2, P_4)$. Now, for $t + 2t^* > 4m - s - 1$, we have $F \supseteq mP_4$. This implies that F is not minimal. Therefore, in each case, we obtain $F \notin \mathcal{R}(mK_2, P_4)$. Similarly we can show the result in the case that s and t are both even. □

Theorem 5.6. *Let m, s, t and t^* be natural numbers with $m \geq 2$ and $2m + 1 \leq s \leq 4m - 5$. Let F be a unicyclic graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ for odd s. The graph $F \in \mathcal{R}(mK_2, P_4)$ if and only if the following conditions are satisfied:*

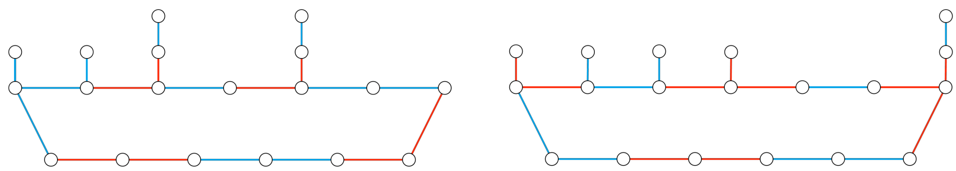
(i) *t is even and $t + 2t^* = 4m - s - 1$;*

(ii) *all the a_i and b_0 are odd, and $b_j \equiv 2 \bmod 4$ for $i \in [1, t-1]$, $j \in [1, t^* - 1]$.*

Proof. Let F be a unicyclic graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$ satisfying the two conditions above. It is easy to verify that for each $v \in V(F)$, the graph $F - \{v\} \supseteq (m-1)P_4$. So $F \rightarrow (mK_2, P_4)$. Next, we prove the minimality property of F. Let e be an edge of F. First we consider that e is an edge of a pendant path. Then, by choosing a cycle vertex u incident with a pendant path P_3, we obtain $(F - e) - \{u\} \not\supseteq (m-1)P_4$. Meanwhile, if e is an edge of the cycle of F, then $F - e$ is a tree. If possible, choose a vertex w of degree 2 such that $(F - e) - \{w\} = P_3 \cup T$, where T is a tree; otherwise, choose a cycle vertex z incident with a pendant path P_3. Then we obtain $(F - e) - \{z\} \not\supseteq (m-1)P_4$. Therefore the graph F is minimal.

Conversely, for a contradiction, assume t is odd. Since s is odd, by Lemma 5.5, we obtain $F \notin \mathcal{R}(mK_2, P_4)$ which leads to a contradiction. Hence t must be even. Next, by Corollary 5.4, $t + 2t^* \geq 4m - s - 2$. If $t + 2t^* = 4m - s - 2$, then we take a cycle vertex u incident with a pendant path P_3, such that $F - \{u\} \not\supseteq (m-1)P_4$. So $F \not\rightarrow (mK_2, P_4)$. However, if $t + 2t^* > 4m - s - 1$ then F is not minimal, since $F \supseteq mP_4$. Hence $t + 2t^* = 4m - s - 1$. Next, by Corollary 5.4, condition (ii) holds. □

The graphs in Figure 10 are examples of unicyclic graphs with circumference 13 belonging to $\mathcal{R}(5K_2, P_4)$.



$C_{13}[(2, P_2), (2, P_3); (1), (1, 2)]$ $C_{13}[(4, P_2), (1, P_3); (1, 1, 1), (3)]$

Figure 10: Two non-isomorphic unicyclic graphs with circumference 13 both belong to $\mathcal{R}(5K_2, P_4)$.

Theorem 5.7. *Let m, s, t and t^* be natural numbers and $m \geq 3$ and $2m \leq s \leq 4m-6$. Let F be a unicyclic graph $C_s[(t,P_2),(t^*,P_3);(a_i)_1^{t-1},(b_j)_0^{t^*-1}]$ for even s. The graph $F \in \mathcal{R}(mK_2,P_4)$ if and only if the following conditions are satisfied.*

(i) *t is odd and $t+2t^* = 4m-s-1$;*

(ii) *for all $i \in [1,t-1]$, $j \in [1,t^*-1]$, $a_i \equiv 1 \bmod 2$, $b_0 \equiv 1 \bmod 2$ and $b_j \equiv 2 \bmod 4$.*

Proof. Let F be a unicyclic graph $C_s[(t,P_2),(t^*,P_3),(a_i)_1^{t-1},(b_j)_0^{t^*-1}]$ for even s satisfying the above conditions (i) and (ii). Since for every $v \in V(F)$, the graph $F-\{v\} \supseteq (m-1)P_4$, we have $F \to (mK_2,P_4)$. Now we prove the minimality. Consider an edge $e \in E(F)$. If e is an edge of a pendant path, then choose any cycle vertex u of degree 3 of $F-e$; we obtain $(F-e)-\{u\} \not\supseteq (m-1)P_4$. Furthermore, if e is an edge of the cycle of F, then, if possible, choose a vertex w of degree 2 of the cycle such that $(F-e)-\{w\} = P_3 \cup T$, where T is a tree; otherwise choose a vertex z of degree 3 incident with a pendant path P_3. We again obtain $(F-e)-\{z\} \not\supseteq (m-1)P_4$. Hence F is minimal.

Conversely, assume, to the contrary, that t is even. Since s is even, by Lemma 5.5, $F \notin \mathcal{R}(mK_2,P_4)$. Next, by Corollary 5.4, $t+2t^* \geq 4m-s-2$. If $t+2t^* = 4m-s-2$, then we choose any vertex u of degree 3 incident with a pendant path P_3, and we get $F-\{u\} \not\supseteq (m-1)P_4$. So $F \nrightarrow (mK_2,P_4)$. However, if $t+2t^* > 4m-s-1$ then F is not minimal, since F contains an mP_4. Therefore the supposition that t is even or $t+2t^* \neq 4m-s-1$ leads to a contradiction. Therefore t must be odd and $t+2t^* = 4m-s-1$. The second condition holds by applying Corollary 5.4. □

For example, to illustrate Theorem 5.7, we give two non-isomorphic graphs with circumference 14 belonging to $\mathcal{R}(5K_2,P_4)$ in Figure 11.

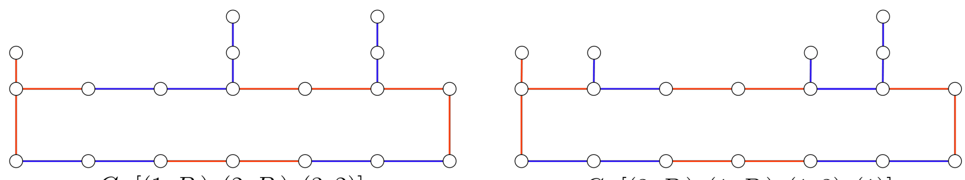


Figure 11: Two non-isomorphic graphs with circumference 14 that are in $\mathcal{R}(5K_2,P_4)$.

Now we are investigating a unicyclic graph F with pendant paths P_2 and P_3 alternating in a cycle C_s. We denote this graph by $C_s[(t,P_2,P_3);\ (a_i)_1^{t-1}]$, i.e., a unicyclic graph with circumference s and a gap sequence $(a_i)_1^{t-1} = (a_1,a_2,\ldots,a_{t-1})$ with pendant paths P_2 and P_3 alternating.

Let $V(C_s) = \{v_1,v_2,\ldots,v_s\}$ be the vertex set of the cycle of F. Hence there are t vertices of C_s having degree 3. Next, let $u_1,u_2,\ldots,u_t$ be the vertices of degree 3. A vertex u_i is said to be *close* to u_j if there is no other vertex of degree 3 between u_i and u_j in the cycle. In this case, we also say that a pendant path incident with

u_i is close to a pendant path incident with u_j. According to Lemmas 3.2, 4.3, 5.1, and 5.3, we have the remark below.

Remark 5.8. *Let m, s and t be natural numbers with $m \geq 2$. Let F be a unicyclic graph $C_s[(t, P_2, P_3); (a_i)_1^{t-1}]$, where pendant paths P_2 and P_3 are alternating in the cycle C_s. Let $u_1, u_2, \ldots, u_t$ be the vertices of degree 3 in the cycle C_s. If $F \in \mathcal{R}(mK_2, P_4)$, then the following conditions must be satisfied.*

(i) If a pendant path P_2 incident with u_i is close to either a pendant path P_2 or P_3 incident with u_j, then $d(u_i, u_j)$ is odd.

(ii) If a pendant path P_3 incident with u_i is close to a pendant path P_3 incident with u_j, then $d(u_i, u_j) \equiv 2 \bmod 4$.

A sequence of pendant paths appearing in distances $(a_i)_1^{t-1}$ of the graph $C_s[(t, P_2, P_3); (a_i)_1^{t-1}]$ is called a *pendant path sequence*. For example, the graph in Figure 12 has a pendant path sequence (P_2, P_3, P_2, P_3).

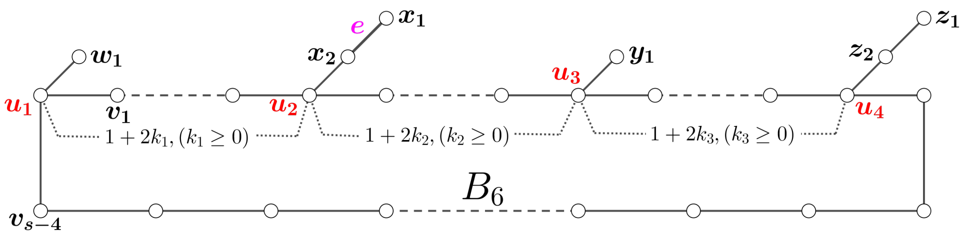


Figure 12: The graph $B_6 = C_s[(4, P_2, P_3); (1 + 2k_1, 1 + 2k_2, 1 + 2k_3)]$.

Theorem 5.9. *Let m, s and t be natural numbers with $m \geq 2$. There is no a unicyclic graph $C_s[(t, P_2, P_3); (a_i)_1^{t-1}]$ in $\mathcal{R}(mK_2, P_4)$.*

Proof. Let F be a unicyclic graph $C_s[(t, P_2, P_3); (a_i)_1^{t-1}]$. For a contradiction, assume that $F \in \mathcal{R}(mK_2, P_4)$. Without loss of generality, we could consider a subgraph of F by removing all pendant paths except any four pendant paths with the sequence (P_2, P_3, P_2, P_3). By Remark 5.8, we consider the unicyclic graph $B_6 = C_s[(t, P_2, P_3); (a_i)_1^{t-1}]$ having a gap sequence $a_i = 1 \bmod 2$ for each $i \in [1, 3]$. Now, relabeling (if necessary) the vertices of B_6 in such a way we have the graph depicted in Figure 12. Consider now the pendant edge $e = x_1x_2$ of a pendant path P_3 (see Figure 12). For each $v \in V(F)$, we get $(F - e) - \{v\} \supseteq (m-1)P_4$. By Corollary 2.6, F is not minimal, which is a contradiction. □

6 Conclusion

To conclude this paper, we present the characterization of all unicyclic Ramsey (mK_2, P_4)-minimal graphs in the following theorem (as a summary from Theorems 1.3, 3.3, 4.4, 5.6, 5.7 and 5.9).

Theorem 6.1. *Let F be a unicyclic Ramsey (mK_2, P_4)-minimal graph. Then graph F is one of the following forms:*

(i) *a cycle C_s, where $s \in \{4m-3, 4m-2, 4m-1\}$;*

(ii) *a graph $C_s[(t, P_2); (a_i)_1^{t-1}]$, where $2m \le s \le 4m-4$, $t = 4m-s-2$ and all the a_i are odd;*

(iii) *a graph $C_s[(t, P_3); (b_i)_1^{t-1}]$, where $2m+1 \le s \le 4m-5$ and s is odd, $t = 2m - \lceil \frac{s}{2} \rceil$ and there is at most one $i_0 \in [1, t-1]$ such that $b_{i_0} \equiv 1 \bmod 4$ and the remaining b_i satisfy $b_i \equiv 2 \bmod 4$; or*

(iv) *a graph $C_s[(t, P_2), (t^*, P_3); (a_i)_1^{t-1}, (b_j)_0^{t^*-1}]$, where $2m \le s \le 4m-5$, $t + 2t^* = 4m - s - 1$, all the a_i and b_0 are odd, and $b_j \equiv 2 \bmod 4$ for $j \in [1, t^*-1]$.* □

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