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4-regular 4-connected Hamiltonian graphs with a bounded number of Hamiltonian cycles

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Abstract

We prove that there exists an infinite family of 4-regular 4-connected Hamiltonian graphs with a bounded number of Hamiltonian cycles. We do not know whether there exists such a family of 5-regular 5-connected Hamiltonian graphs.

1 Introduction

There is a variety of 3-regular 3-connected graphs with no Hamiltonian cycles. Much less is known about 4-regular 4-connected graphs. Thus the Petersen graph (on 10 vertices) is the smallest 3-regular 3-connected non-Hamiltonian graph whereas it was an open problem of Nash-Williams as to whether there exists a 4-regular 4-connected non-Hamiltonian graph, until Meredith [9] gave an infinite family, the smallest of which has 70 vertices; see [1, p. 239]. Tait's conjecture that every 3-regular 3-polyhedral graph has a Hamiltonian cycle was open from 1880 till Tutte [15] in 1946 found a counterexample; see [1, p. 161]. By Steinitz' theorem, the Tutte graph (and subsequently many others) also show that there are infinitely many 3-regular 3-polyhedral graphs which are non-Hamiltonian, whereas it is a longstanding conjecture of Barnette that every 4-regular 4-polyhedral graph has a Hamiltonian

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cycle [5, p. 1145] (see also [6, p. 389a] and [4, p. 375]). There are infinitely many 3-regular hypohamiltonian graphs whereas it is a longstanding open problem as to whether there exists a hypohamiltonian graph of minimum degree at least 4; see [12]. Starting with the complete graph on 4 vertices and successively replacing vertices by triangles, we obtain an infinite family of graphs with precisely three Hamiltonian cycles. Cantoni's conjecture says that these are precisely the planar 3-regular graphs with exactly three Hamiltonian cycles; see [16].

Recently, Haythorpe [8] conjectured that 4-regular graphs behave differently from the 3-regular graphs, also in this respect, in that the number of Hamiltonian cycles increases as a function of the number of vertices. The purpose of this note is to answer this in the negative. So in this respect, the 4-regular 4-connected graphs behave in a similar way as the 3-regular 3-connected graphs. We do not know if this also holds for the 5-regular 5-connected graphs.

Andrew Thomason [11] proved that every Hamiltonian graph whose vertices are all of odd degree has a second Hamiltonian cycle. Thomason's theorem was inspired by (and extends) Smith's result stating that every cubic graph has an even number of Hamiltonian cycles through each edge; see [15]. Sheehan [10] conjectured that the same holds for 4-regular graphs: if they are Hamiltonian, then they contain at least two Hamiltonian cycles. In [3] this was shown to hold up to order 21. If true in general, this would imply that, for every natural number $k \geq 3$, every k-regular Hamiltonian graph has a second Hamiltonian cycle. The latter statement was verified in [14] for all $k > 72$ and subsequently in [7] for all $k > 20$.

For restricted classes of 4-regular 4-connected graphs, it is still possible that the number of Hamiltonian cycles in a Hamiltonian graph must increase (perhaps even exponentially) as a function of the number of vertices. This may be true for planar graphs where it is known that the number of Hamiltonian cycles increases at least as a linear function; [2]. And it may hold for bipartite graphs where it is known that the number of Hamiltonian cycles in a Hamiltonian k-regular graph increases more than exponentially as a function of k; [13].

2 4-regular 4-connected graphs with a bounded number of Hamiltonian cycles

As mentioned earlier, the Meredith graph is a 4-regular 4-connected non-Hamiltonian graph.

Figure 1 indicates an infinite family of 4-regular Hamiltonian graphs with a bounded number of Hamiltonian cycles.

All graphs in this infinite family have connectivity 2. We shall now describe 4-regular 4-connected graphs with a bounded (positive) number of Hamiltonian cycles.

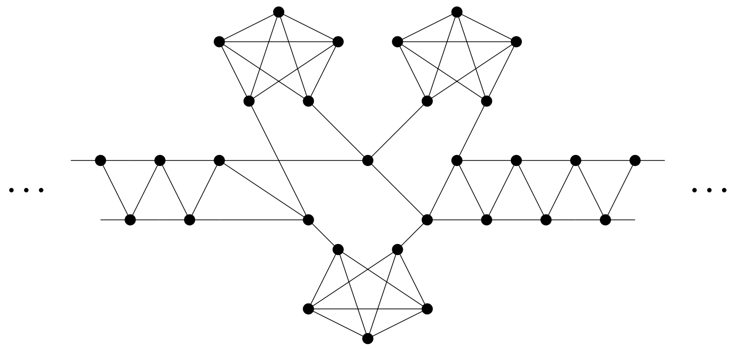

Fig. 1: 4-regular graphs, each having exactly 216 Hamiltonian cycles. (The left-most and right-most part of the graph are to be connected in the obvious way.)

Theorem. *There exists a constant* $c > 0$ *such that there are infinitely many* 4*-regular* 4*-connected graphs, each containing exactly* c *Hamiltonian cycles.*

Proof. Assume G is a 4-regular 4-edge-connected graph containing a path $abcd$ such that

(i) G has no Hamiltonian cycle;

(ii) G has a 2-factor consisting of two cycles C and C' such that C contains ab and C' contains cd; and

(iii) $G - bc$ has no Hamiltonian path joining two of a, b, c, d. If v is any vertex in $\{a, b, c, d\}$, then $G - v - bc$ has no Hamiltonian path joining two of a, b, c, d.

We construct the graph H_G as follows. Let G' be a copy of G and $\ell \geq 1$ a natural number. Take the disjoint union of G, G'; for a vertex v in G denote its copy in G' by v'; delete the edges $ab, bc, cd, a'b', b'c', c'd'$; and add four pairwise disjoint paths P_a, P_b, P_c, P_d joining a, c' and b, b' and c, a' and d, d', respectively, where P_a and P_d have length $\ell + 1$, while P_b and P_c have length ℓ. Thereafter, add a zig-zag path between P_a and P_b, as well as a zig-zag path between P_c and P_d. The construction of H_G is illustrated in Figure 2. Every Hamiltonian cycle in H_G will contain the paths P_a, P_b, P_c, P_d. Note that the number of Hamiltonian cycles of H_G is independent of ℓ.

The required graph G will be the modification of the Petersen graph shown in Figure 3 where the bold edges form the 2-factor satisfying (ii). It is straightforward to infer from the non-Hamiltonicity of Petersen's graph that G satisfies (i), and it is left to the reader to verify that it satisfies (iii).

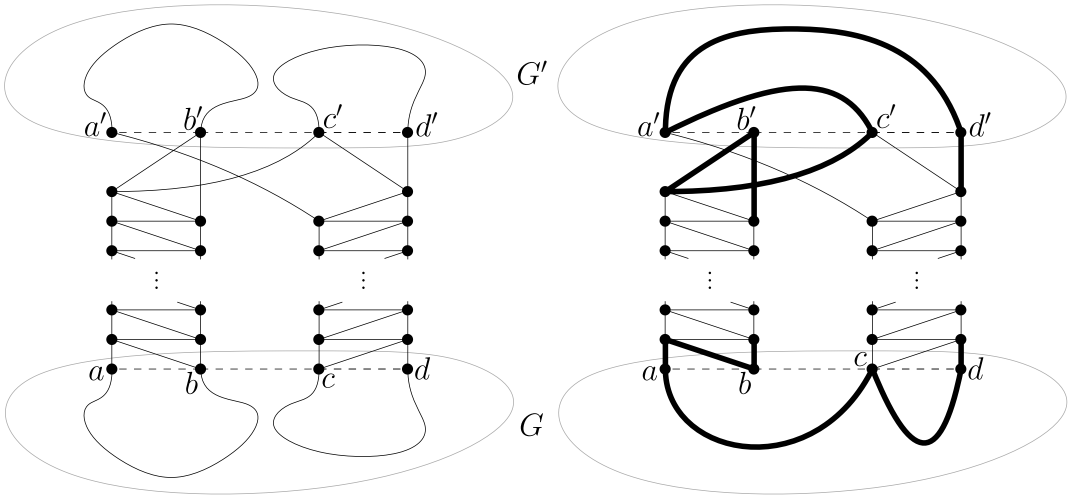

Fig. 2: On the left, the graph H_G is shown, with relevant 2-factors in G and G'. On the right, we illustrate why conditions (i), (ii), and the first statement in (iii) would not suffice: if G and G' are traversed as shown, then the number of Hamiltonian cycles in H_G would increase with ℓ.

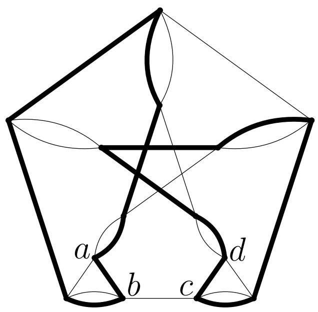

Fig. 3: A useful modification of Petersen's graph.

Using two copies of G, we construct H_G as described above. One final issue remains: there are double edges occurring in H_G. We shall make use of the idea behind Meredith's classical construction in which a vertex is replaced by a complete bipartite graph $K_{3,4}$; see [1, p. 161]. In the Meredith graph the operation is performed on each vertex of the Petersen graph (in which a 1-factor is replaced by double edges). In the present note the operation is performed only on both ends of each double edge in H_G. □

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