AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **81(1)** (2021), Pages 187–207

Extended strict order polynomial of a poset and fixed elements of linear extensions

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Abstract

In this paper, we extend the concept of the strict order polynomial $\Omega_P^\circ(n)$, which enumerates strictly order-preserving maps $\phi : P \to \boldsymbol{n}$ for a poset P, to the extended strict order polynomial $\mathrm{E}_P^\circ(n,z) = \sum_{Q \subseteq P} \Omega_Q^\circ(n) z^{|Q|}$, which enumerates analogous maps for all induced subposets of P. Richard Stanley showed that the strict order polynomial $\Omega_P^\circ(n)$ can be expressed as the sum $\Omega_P^\circ(n) = \sum_{w \in \mathcal{L}(P)} \binom{n+\mathrm{des}(w)}{p}$, where $\mathcal{L}(P)$ is the set of linear extensions of P, $\mathrm{des}(w)$ is the number of descents of w, and p is the number of elements of P. This reduces the computation of $\mathrm{E}_P^\circ(n,z)$ to the enumeration of linear extensions of subposets of P by descents. We show that every linear extension v of every induced subposet of P can be associated with a linear extension w of P. The number of linear extensions of subposets of size k associated with a given linear extension w of P is $\binom{p-\mathrm{fix}_P(w)}{k-\mathrm{fix}_P(w)}$, where $\mathrm{fix}_P(w)$ is the number of fixed elements of w defined in the text. Consequently, the extended strict order polynomial $\mathrm{E}_P^\circ(n,z)$ can be represented as

$$\mathrm{E}_P^\circ(n,z) = \sum_{w \in \mathcal{L}(P)} \sum_{k=0}^{p} \binom{p-\mathrm{fix}_P(w)}{k-\mathrm{fix}_P(w)} \binom{n+\mathrm{des}(w)}{k} z^k.$$

1 Notation and Definitions

1.1 Standard terminology

The current communication closely follows the poset terminology introduced in Stanley's book [19]. The reader familiar with the terminology can jump directly

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ISSN: 2202-3518

to Subsection 1.2. A *partially ordered set* P, or *poset* for short, is a set together with a binary relation $<_P$. In this manuscript, we are concerned with finite posets P consisting of p elements and with strict partial orders, meaning that the relation $<_P$ is irreflexive, transitive and asymmetric. An *induced subposet* $Q \subset P$ is a subset of P together with the order $<_Q$ inherited from P which is defined by $s <_P t \iff s <_Q t$. The symbol $<$ shall denote the usual relation "larger than" in $\mathbb{N}$. The symbol $[\,n\,]$ stands for the set $\{1, 2, \ldots, n\}$, and (n, m) stands for the set $\{n+1, n+2, \ldots, m-1\}$. The symbol $\boldsymbol{n}$ represents the chain $1 < 2 < 3 < \ldots < n$. A map $\phi : P \to \mathbb{N}$ is a *strictly order-preserving map* if it satisfies $s <_P t \Rightarrow \phi(s) < \phi(t)$. A *natural labeling* of a poset P is an order-preserving bijection $\omega : P \to [\,p\,]$. A *linear extension* of P is an order-preserving bijection $\sigma : P \to \boldsymbol{p}$. A linear extension σ can be represented as a permutation $\omega \circ \sigma^{-1}$ expressed as a sequence $w = w_1 w_2 \ldots w_p$ of labels $w_i = \omega(\sigma^{-1}(i))$; the sequence w shall also be referred to as a linear extension in the following. The set of all such sequences w is denoted by $\mathcal{L}\,(P)$ and is referred to as the *Jordan-Hölder set* of P. If two subsequent labels w_i and w_{i+1} in w stand in the relation $w_i > w_{i+1}$, then the index i is called a *descent* of w. The total number of descents of w is denoted by $\operatorname{des}(w)$. The *strict order polynomial* $\Omega^\circ_P(n)$ of a poset P [19, 20, 21] enumerates the strictly order-preserving maps $\phi : P \to [\,n\,]$. Stanley showed (in [20, 21], see also [19, Sections 3.15.8 and 3.15.12]) that the strict order polynomial can be expressed as a sum over the set of linear extensions of P:

$$\Omega^\circ_P(n) = \sum_{w \in \mathcal{L}(P)} \binom{n + \operatorname{des}(w)}{p} \tag{1}$$

The idea behind the proof of Eq. (1) is to associate every strictly order-preserving map $\phi : P \to [\,n\,]$ with a compatible linear extension $w \in \mathcal{L}(P)$. Here, ϕ and w are compatible if $\phi(\omega^{-1}(w_i)) \le \phi(\omega^{-1}(w_{i+1}))$ whenever i is a descent in w, and $\phi(\omega^{-1}(w_i)) < \phi(\omega^{-1}(w_{i+1}))$ otherwise. Thus, for every linear extension $w \in \mathcal{L}(P)$, there are $\binom{n+\operatorname{des}(w)}{p}$ strictly order-preserving maps $\phi : P \to [\,n\,]$, and $\Omega^\circ_P(n)$ is given by Eq. (1).

1.2 Non-standard terminology

We will often construct—by a slight abuse of notation—a subposet of P by specifying a set of labels $S \subset [\,p\,]$: The expression $P \setminus S$ stands for the induced subposet with the elements $\{p \in P \,|\, \omega(p) \notin S\}$; and $P \cap S$ stands for the induced subposet with the elements $\{p \in P \,|\, \omega(p) \in S\}$. Clearly the subposet $P \cap S$ constructed in this way has $|S|$ elements; and the set $\mathcal{P}(P)$ of subposets of P stands in a direct correspondence to the power set of $[\,p\,]$: $\mathcal{P}(P) = \{P \cap S \,|\, S \in \mathcal{P}([\,p\,])\}$. Similarly, if w is a sequence in $\mathcal{L}\,(P)$ and $S \subset [\,p\,]$ is a set of labels, let us denote by $w \setminus S$ the subsequence obtained by deleting all the labels of S from w, and by $w \cap S$ the subsequence obtained by deleting all the labels that are not in S from w. For example, $13245 \setminus \{1, 4\} = 13245 \cap \{2, 3, 5\} = 325$. Clearly, deleting some arbitrary set S from two different sequences may produce the same subsequence: for example, $13245 \setminus \{1, 4\} = 325 = 32154 \setminus \{1, 4\}$. We will later (in Def. 6) classify the labels in each linear extension into *fixed* and *deletable* labels, and show (in Lemma 10) that deleting deletable labels from two distinct sequences always results in two distinct subsequences.

Further, let us slightly modify the standard representation of linear extensions of subposets: Normally, one would assign to each subposet $Q = P \cap S$ a new natural labeling $\omega^Q : Q \to [\,q\,]$, and then express the linear extensions of Q as sequences of the elements of $[\,q\,]$. Instead, we avoid re-labeling each subposet, and use instead the labeling $\omega : Q \to S$ inherited from P. Then, a linear extension σ of Q is represented by a sequence $w = w_1 \dots w_q$ defined in the usual way: $w_i = \omega(\sigma^{-1}(i))$. The set of such sequences shall still be denoted by $\mathcal{L}(Q)$. Using this notation, it is now easy to see (with a proper demonstration coming later in Lemma 12) that if w is a linear extension of P, then $w \cap S$ is a linear extension of $P \cap S$.

2 Main results

In this paper we extend the concept of the strict order polynomial $\Omega^\circ_P(n)$ to the extended strict order polynomial $\mathrm{E}^\circ_P(n, z)$ given by Eq. (2), which enumerates and classifies the totality of strictly order-preserving maps $\phi : Q \to \boldsymbol{n}$ with $Q \subset P$. We show below in Theorem 2 that there exists a compact combinatorial expression characterizing $E^\circ_P(n, z)$. In the following, we shall always assume that P is a poset with p elements, a strict order $<_P$, and a natural labeling ω. Subposets of P are always assumed to be induced.

Definition 1. The *extended strict order polynomial* $\mathrm{E}^\circ_P(n, z)$ of a poset P is defined as

$$\mathrm{E}^\circ_P(n, z) = \sum_{Q \subset P} \Omega^\circ_Q(n) z^{|Q|}, \tag{2}$$

where the sum runs over all the induced subposets Q of P.

A compact expression for $\mathrm{E}^\circ_P(n, z)$ can be obtained directly by applying the following theorem.

Theorem 2. *The extended strict order polynomial is given by*

$$\mathrm{E}^\circ_P(n, z) = \sum_{w \in \mathcal{L}(P)} \sum_{k=0}^{p} \binom{p - \mathrm{fix}_P(w)}{k - \mathrm{fix}_P(w)} \binom{n + \mathrm{des}(w)}{k} z^k, \tag{3}$$

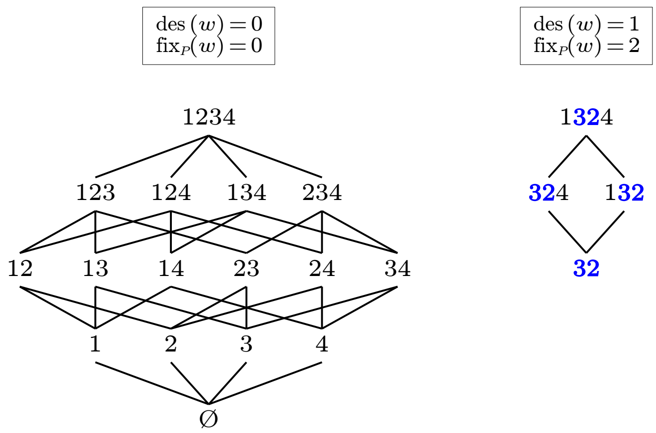
where $\mathrm{fix}_P(w)$ *denotes the number of fixed labels in* w*.*

Theorem 2 is based on the fact that every element v of $\bigcup_{Q \subset P} \mathcal{L}(Q)$ can be uniquely associated with some element w of $\mathcal{L}(P)$. We construct a partition of the set $\bigcup_{Q \subset P} \mathcal{L}_Q$ into blocks B_w indexed by the elements of $\mathcal{L}(P)$ such that each block B_w contains w but no other element of $\mathcal{L}(P)$. The elements within B_w have $\mathrm{des}(w)$ descents, and can be obtained from w by deleting some of its deletable labels, while retaining all fixed labels in the same order. We will see that each B_w contains $\binom{p-\mathrm{fix}_P(w)}{k-\mathrm{fix}_P(w)}$ elements of length k, which leads directly to Eq. (3). This concept is illustrated below in Examples 3 and 4. One may notice that the specific linear extension $w \in \mathcal{L}(P)$ associated with each $v \in \bigcup_{Q\subset P} \mathcal{L}(Q)$, and thus also the partitioning into blocks B_w, depends on the choice of the natural labeling ω. However, the number of blocks of each size, and thus the final result given in Theorem 2, is unaffected by the choice of ω. The proof of Theorem 2 will be given at the end of this paper after formalizing the concept of fixed and deletable labels and proving some technical lemmata.

Example 3. Let us consider the lattice $P = \mathbf{2}\times\mathbf{2}$ together with the labeling specified in Fig. 2. We find $\mathcal{L}(P) = \{1234, 1324\}$ and

$$\bigcup_{Q\subset P} \mathcal{L}(Q) = \{\varnothing, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 32, 123, 124, 134, 234, 324, 132, 1234, 1324\}\,.$$

Our results allow us to partition the set $\bigcup_{Q\subset P} \mathcal{L}(Q)$ of linear extensions into two blocks B_{1234} and B_{1324}:



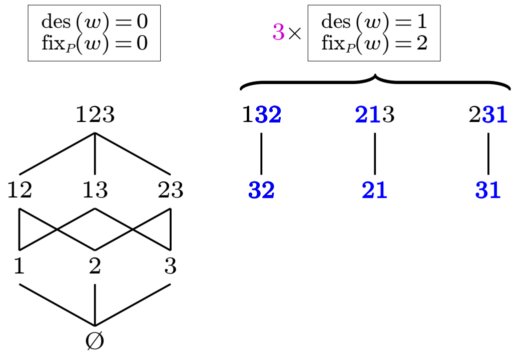
The first block originates from the linear extension 1234, which has zero descents ($\mathrm{des}(w) = 0$) and an empty set of fixed labels $\mathrm{Fix}_P(w) = \varnothing$, meaning that $\mathrm{fix}_P(w) = 0$. Therefore, the first block B_{1234} contains $\binom{4-0}{k-0}$ sequences of each length k. The second block originates from the linear extension 1324, which has one descent ($\mathrm{des}(w) = 1$) and the set of fixed labels $\mathrm{Fix}_P(w) = \{\mathbf{2}, \mathbf{3}\}$, meaning that $\mathrm{fix}_P(w) = 2$. Therefore, the second block B_{1234} contains $\binom{4-2}{k-2}$ sequences of each length k. Consequently, the extended strict order polynomial is given by

$$\mathrm{E}^{\circ}_P(n, z) = \sum_{k=0}^{4} \left(\binom{4-0}{k-0}\binom{n}{k} + \binom{4-2}{k-2}\binom{n+1}{k} \right) z^k.$$

Example 4. Let us consider the poset $P = \{a, b, c\}$ of three non-comparable elements. We have $\mathcal{L}(P) = \{123, 132, 213, 231, 312, 321\}$ and

$$\bigcup_{Q\subset P} \mathcal{L}(Q) = \{\varnothing, 1, 2, 3, 12, 21, 13, 31, 23, 32, 123, 132, 213, 231, 312, 321\}\,.$$

Our results allow us to classify the linear extensions in $\bigcup_{Q\subset P}\mathcal{L}(Q)$ into six blocks



each of which is associated with a pair of numbers $(\mathrm{des}(w), \mathrm{fix}_P(w))$ specified above. The extended strict order polynomial is given by

$$\mathrm{E}^\circ_P(n,z)=\sum_{k=0}^{3}\left(\binom{3}{k}\binom{n}{k}+3\binom{3-2}{k-2}\binom{n+1}{k}+\binom{3-3}{k-3}\binom{n+1}{k}+\binom{3-3}{k-3}\binom{n+2}{k}\right)z^k.$$

The introduced concept of the extended strict order polynomial $\mathrm{E}^\circ_P(n,z)$ can be used for solving the following combinatorial problem:

Example 5. Consider a family of three shepherds: Fiadh, Fiadh's father Aidan, and Aidan's father Lorcan. Every day, some of the shepherds go out and each herds a flock of at least one and at most n sheep. Aidan always herds more sheep than Fiadh, and Lorcan always herds more sheep than both Fiadh and Aidan. How many possible ways are there of assigning flock sizes to the shepherds?

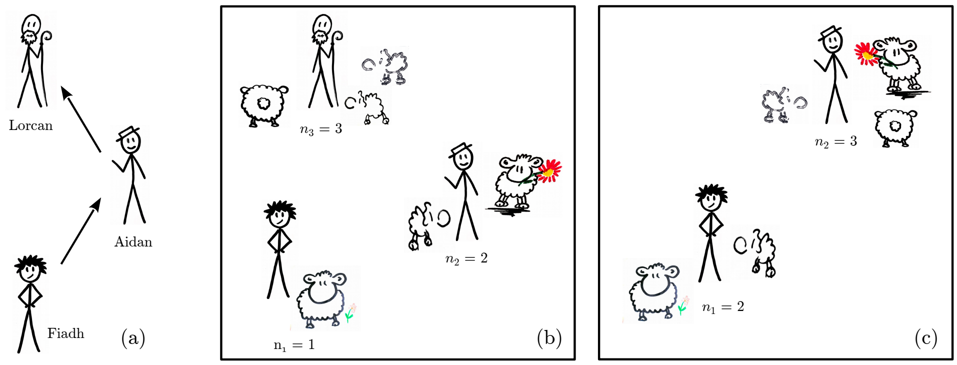


Figure 1: The poset formed by the three shepherds, shown in (a), is isomorphic to the chain **3**. Situations such as the one depicted in (b), where all three shepherds herd a flock of sheep, are counted by the strict order polynomial $\Omega^\circ_P(n)$. The extended strict order polynomial $\mathrm{E}^\circ_P(n,z)$ also counts situations such as the one shown in (c), where only a subset of the shepherds are present.

The three shepherds together with the seniority relation form a poset P isomorphic to the chain **3**: Fiadh $<_P$ Aidan $<_P$ Lorcan, see Fig. 1 (a). Let us denote the number of sheep in Fiadh's flock by n_1, the size of Aidan's flock by n_2 and the size of Lorcan's flock by n_3; then the above conditions tell us that $1 \leq n_1 < n_2 < n_3 \leq n$. On a day when all three shepherds go to work, such as depicted in Fig. 1 (b), the numbers n_1, n_2 and n_3 can be chosen in $\Omega_P^\circ(n) = \binom{n}{3}$ ways. Assume now that k of the three shepherds go to work, such as depicted in Fig. 1 (c) for the subposet $Q = \{\text{Fiadh}, \text{Aidan}\}$. We may choose the working shepherds in $\binom{3}{k}$ ways, and their respective flock sizes in $\binom{n}{k}$ ways. Therefore, the extended strict order polynomial $\mathrm{E}_P^\circ(n, z)$ has the form

$$\mathrm{E}_P^\circ(n, z) \quad = \quad \sum_{k=0}^{3} \binom{3}{k}\binom{n}{k} z^k.$$

3 Classification of linear extensions

Within every sequence in $\mathcal{L}(Q)$, we distinguish between fixed and deletable labels:

Definition 6. Consider a sequence $w = w_1 w_2 \dots w_p$ which represents a linear extension σ of P. A label w_i is *fixed* in w if

(1) $i - 1$ or i is a descent, or

(2) the set of positions of preceding larger labels i $L(w_i) := \{l \mid l < i, w_l > w_i\}$ and the set of positions of necessarily preceding labels $J(w_i) := \{j \mid \omega^{-1}(w_j) <_P \omega^{-1}(w_i)\}$ satisfy $L(w_i) \neq \varnothing$ and $\max(L(w_i)) > \max(J(w_i))$.

A label w_i is *deletable* from w if it is not fixed. The set of fixed labels of w is denoted by $\mathrm{Fix}_P(w)$, and its cardinality by $\mathrm{fix}_P(w) = |\mathrm{Fix}_P(w)|$.

Removing a deletable label w_i from w and reinserting it at any earlier position cannot result in a linear extension with the same labels involved in the descent pairs; in this sense deletable labels appear in w "as early as possible". This concept constitutes the main idea behind associating a linear extension $v \in \bigcup_{Q \subset P} \mathcal{L}(Q)$ with a unique linear extension $w \in \mathcal{L}(P)$ developed in detail later in this communication.

Let us now demonstrate how Definition 6 can be used in a direct manner to distinguish between deletable and fixed labels in linear extensions.

Example 7. Consider again the poset $P = \mathbf{2} \times \mathbf{2}$ shown in Fig. 2 and its two linear extensions $w = 1234$ and $w' = 1324$. Let us first determine which of the labels are deletable from 1234. We have no descents in 1234, so condition (1) of Definition 6 is not satisfied for any of the labels. None of the labels are preceded by any larger labels, meaning that $L(w_i) = \varnothing$ and condition (2) is not satisfied for any of the labels. Therefore, all four labels in the linear extension 1234 are deletable: $\mathrm{Fix}_P(1234) = \varnothing$ and $\mathrm{fix}_P(1234) = 0$. This is not the case for the linear extension 1324. The labels 3 and 2 are fixed by condition (1) since the position $i = 2$ is a descent. The labels 1 and 4 are deletable due to $L(1) = L(4) = \varnothing$. Consequently, the set of fixed labels of 1324 is $\mathrm{Fix}_P(1324) = \{2, 3\}$ and thus $\mathrm{fix}_P(1324) = 2$.

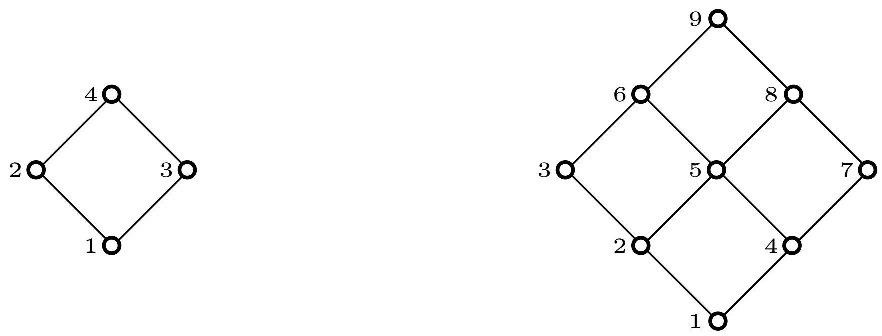


Figure 2: Hasse diagram of two posets $P = \mathbf{2} \times \mathbf{2}$ and $P = \mathbf{3} \times \mathbf{3}$ together with (natural) labelings ω.

Example 8. Consider the linear extension $w = 124753689$ of the poset $P = \mathbf{3} \times \mathbf{3}$ shown in Fig. 2. By condition (1) of Definition 6, the labels $3, 5$ and 7 are fixed; the remaining labels might be fixed if they satisfy condition (2) of Definition 6. For the labels $w_i = 1, 2, 4, 8, 9$ we find $L(w_i) = \varnothing$, thus they are deletable. The only remaining label for which the deletability (or fixedness) is not immediately obvious—and hence the machinery of Definition 6 must be fully put to work—is $w_7 = 6$. The only larger, preceding label is $w_4 = 7$, thus $L(w_7) = \{4\}$. Inspection of Fig. 2 shows that the labels that necessarily precede $w_7 = 6$ are $w_1 = 1$, $w_2 = 2$, $w_3 = 4$, $w_5 = 5$ and $w_6 = 3$, and therefore $J(w_7) = \{1, 2, 3, 5, 6\}$. Since $\max(J(w_7)) = 6 > 4 = \max(L(w_7))$, condition (2) is not satisfied and $w_7 = 6$ is deletable. Thus, $\mathrm{Fix}_P(w) = \{3, 5, 7\}$ and $\mathrm{fix}_P(w) = 3$.

We are now ready to investigate formally the relation between the linear extensions of P and the linear extensions of its subposets.

3.1 Correspondence between linear extensions of a poset and of its subposets

Deleting deletable labels does not affect the number of descents:

Lemma 9. *Consider a linear extension $w \in \mathcal{L}(P)$ and a set $F \subset [p]$ such that $\mathrm{Fix}_P(w) \subset F$. Then the subsequence $v = w \cap F$ satisfies $\mathrm{des}(w) = \mathrm{des}(v)$.*

Proof. Let us augment the linear extension w with two auxiliary fixed labels $w_0 = 0$ and $w_{p+1} = p + 1$. Then any deletable label of w is located between two fixed labels w_i and w_j, which can be selected in such a way that all the labels $w_{i+1}, \ldots, w_{j-1}$ in between are deletable. If there is any $k \in (i, j)$ such that $w_k > w_{k+1}$, k would be a descent in w and w_k and w_{k+1} would be fixed according to condition 1) of Def. 6, contradicting the choice of w_i and w_j. Therefore, we have $w_i < w_{i+1} < \ldots < w_{j-1} < w_j$. Every deletable label belongs to such an interval containing monotonically increasing deletable labels flanked by two fixed labels. Therefore, constructing $v = w \cap F$ by deleting only deletable labels from w does not remove or introduce any descents. $\square$

The following two lemmata establish the correspondence between the elements of $\mathcal{L}(P \cap F)$ and the elements of $\mathcal{L}(P)$.

Lemma 10. *Let $F \subset [\,p\,]$, and let $Q = P \cap F$ be a subposet of P. For every linear extension v of Q, there exists exactly one linear extension w of P such that $v = w \cap F$ and $\mathrm{Fix}_P(w) \subset F$.*

Proof. Since the following considerations are more transparent in terms of deletable rather than fixed labels, let us denote by $\mathrm{Del}_P(w)$ the set of deletable labels $[\,p\,] \setminus \mathrm{Fix}_P(w)$, and by D the complement $[\,p\,] \setminus F$ of F. Clearly, $\mathrm{Fix}_P(w) \subset F$ if and only if $D \subset \mathrm{Del}_P(w)$ and $w \cap F = w \setminus D$ for all $w \in \mathcal{L}(P)$. In the following, let $q = |Q| = |F| = p - |D|$, the length of v. For the sake of brevity of the following exposition, assume during this proof that $v_0 = 0$ and $v_{q+1} = p + 1$. Let us attempt to construct a sequence w of the labels in $[\,p\,]$ such that

(a) $v = w \setminus D$,

(b) $w \in \mathcal{L}(P)$,

(c) $D \subset \mathrm{Del}_P(w)$.

A sequence w can only satisfy condition (a) if it contains the labels $v_1, \ldots, v_q$ appearing in the same order as in v, preceded, interleaved, and/or succeeded by the elements of D. Let us denote by D^0 the set of elements of D appearing in w before v_1, by D^1 the set of elements appearing between v_1 and v_2, and so on. Clearly, D is a disjoint union of the subsets $D^0, D^1, \ldots, D^q$. We can thus construct a sequence w in two steps:

Step 1: Partition D into $q+1$ (possibly empty) subsets $D^0, D^1, \ldots, D^q$ containing $m_0, m_1, \ldots, m_q$ elements, respectively.

Step 2: Arrange the elements of each subset D^i into a subsequence $d^i_1 d^i_2 \ldots d^i_{m_i}$ and form a sequence w by concatenating the labels in v and the consecutive subsequences $d^0_1 d^0_2 \ldots d^0_{m_0}, \ldots, d^q_1 d^q_2 \ldots d^q_{m_q}$ in the following way:

$$w = d^0_1 d^0_2 \ldots d^0_{m_0} v_1 d^1_1 d^1_2 \ldots d^1_{m_1} v_2 \ldots v_q d^q_1 d^q_2 \ldots d^q_{m_q}.$$

Obviously, many different sequences w can be constructed in this way by choosing different partitionings of D and by selecting distinct orders of the elements in each D^i; we show during the following construction process that the conditions (b) and (c) restrict this abundance to a single, unique sequence w.

Every $d \in D$ must be inserted in such a way that that $w_{i-1} < w_i \equiv d < w_{i+1}$, otherwise, d would (by condition (1) of Definition 6) be fixed in w, thus violating condition (c). Therefore, the subsequence $d^i_1 d^i_2 \ldots d^i_{m_i}$ inserted between v_i and v_{i+1} must satisfy $v_i < d^i_1 < d^i_2 < \ldots < d^i_{m_i} < v_{i+1}$. This shows that, in Step 2, when we augment v with the elements of a subset D^i, the only choice is to arrange these elements into a monotonically increasing sequence before doing so. Moreover, this requirement seriously reduces the number of allowed partitions of D into subsets D^i, as each $d \in D^i$ needs to satisfy the condition $v_i < d < v_{i+1}$.

Consider a label $d \in D$. Let us now narrow down the family of subsets D^i into which d may be placed. Let $j_d = \max\{j \in [\,q\,] \,|\, \omega^{-1}(v_j) <_P \omega^{-1}(d)\}$, or $j_d = 0$ if this set is empty. In order to not violate condition (b), d must be in some D^i with $j_d \leq i$. Denote by $I_d = \{i \,|\, i \geq j_d,\, v_i < d < v_{i+1}\}$ the set of possible choices for i limited by the so far derived conditions $i \geq j_d$ and $v_i < d < v_{i+1}$. The set I_d is nonempty: It follows from the order-preserving nature of ω that $v_{j_d} < d < v_{q+1}$, so there must be at least one value of i with $j_d \leq i \leq q$ such that $v_i < d < v_{i+1}$. Let $i_d = \min\, I_d$.

We will now show by *reductio ad absurdum* that placing d into a subset other than D^{i_d} leads to a violation of condition (c). (Recollect that every deletable label appears in w "as early as possible".) Assume that $d \in D^i$ with $i \in I_d$ and $i > i_d$. Then, by definition of I_d, we know that $d < v_{i_d+1}$. Consider now the sets $L(d)$ and $J(d)$ from Definition 6. Since $d < v_{i_d+1}$ and v_{i_d+1} precedes d in w, we have $\sigma(\omega^{-1}(v_{i_d+1})) \in L(d)$, where σ denotes the map $\sigma : P \to \boldsymbol{p}$, $w_i \mapsto i$ implied by w. Since $L(d) \neq \varnothing$, by condition (2) of Definition 6, the only way for d to be deletable is if $\max(L(d)) \geq \sigma(\omega^{-1}(v_{i_d+1})) \not> \max(J(d))$, that is, if there is a label $e \in [\,p\,]$ such that $\omega^{-1}(e) <_P \omega^{-1}(d)$ which appears in w between v_{i_d+1} and d. Denote by E the set of such labels: $E = \{e \in [\,p\,] \,|\, \omega^{-1}(e) <_P \omega^{-1}(d), \sigma(\omega^{-1}(v_{i_d+1})) < \sigma(\omega^{-1}(e)) < \sigma(\omega^{-1}(d))\}$. If there is an $e \in E$ with $e \notin D$, then e must appear in v at some position k, $e \equiv v_k$. Since $\omega^{-1}(e) <_P \omega^{-1}(d)$, according to the definitions of j_d and i_d we find that $k \leq j_d \leq i_d$, in contradiction with the requirement that v_{i_d+1} precede $e = v_k$ in v. Therefore, $e \in D$ and thus $E \subset D$. Consider now the label $c = \min\, E$. It follows from $\omega^{-1}(c) <_P \omega^{-1}(d)$ that $c < d < v_{i_d+1}$. In order for c to be deletable from the finished sequence w, there must be a label $e' \in [\,p\,]$ such that $J(c) \ni \sigma(\omega^{-1}(e')) > \sigma(\omega^{-1}(v_{i_d+1})) \in L(c)$, that is, a label e' with $\omega^{-1}(e') <_P \omega^{-1}(c)$ which appears in w between v_{i_d+1} and c. Because $\omega^{-1}(c) <_p \omega^{-1}(d)$, and since c must precede d in w, the aforementioned label e' must be in E, and therefore $\min\, E = c < e'$. At the same time, since ω is a natural labeling, $\omega^{-1}(e') <_P \omega^{-1}(c)$ implies $e' < c$. This contradiction shows that c cannot be deletable from w, $c \notin \mathrm{Del}_P(w)$. However we have found before that $c \in D$, which means that the assumption $i > i_d$ leads to a violation of condition (c). Therefore, we must have $i \leq i_d$. Since i_d is defined as the minimum allowed value of i, we have $i = i_d$.

To summarize, we have shown until now that the only way to construct a sequence w in a way that does not contradict conditions (a)–(c) is to follow the construction introduced above, which can be described in the following way:

Step 1: For every $d \in D$, let

$$j_d = \max\left(\{j \in [\,q\,] \,|\, \omega^{-1}(v_j) <_P \omega^{-1}(d)\} \cup \{0\}\right) \tag{4}$$
$$\text{and } i_d = \min\left(\{i \,|\, j_d \leq i \leq q,\, v_i < d < v_{i+1}\}\right), \tag{5}$$

and assign d to the set D^{i_d}.

Step 2: For every $0 \leq i \leq q$, insert between v_i and v_{i+1} the elements of D^i in increasing order:

$$w = d_1^0 d_2^0 \dots d_{m_0}^0 v_1 d_1^1 d_2^1 \dots d_{m_1}^1 v_2 \dots v_q d_1^q d_2^q \dots d_{m_q}^q,$$

where $d_q^i \in D^i$ and $v_i < d_1^i < d_2^i < \dots < d_{m_i}^i < v_{i+1}$.

It remains to be demonstrated that the sequence w uniquely defined in this way indeed satisfies all conditions (a)–(c). Condition (a) is satisfied by construction.

Next, let us verify that w satisfies condition (b). Consider two arbitrary elements $s, t \in P$ such that $s <_P t$. Each of their labels $\omega(s)$ and $\omega(t)$ can be in D or in $F = [\,p\,] \setminus D$. For each case, we have to show that $\omega(s)$ precedes $\omega(t)$ in w.

- If $\omega(s), \omega(t) \in F$, then $\omega(s) \equiv v_i$ and $\omega(t) \equiv v_j$ for some $i, j \in [\,q\,]$. Since Q is an induced subposet of P, we have $s <_Q t$, and therefore we know that $i < j$, i.e., $v_i \equiv \omega(s)$ precedes $v_j \equiv \omega(t)$ in v. Then, by construction, $\omega(s)$ precedes $\omega(t)$ also in w.

- If $\omega(s) \in F$ and $\omega(t) \in D$, then $\omega(s) \equiv v_k$ for some $k \in [\,q\,]$. Step 1 defines two numbers $j_{\omega(t)}$ and $i_{\omega(t)}$. Since $s <_P t$, k is in $\{j \in [\,q\,] \,|\, \omega^{-1}(v_j) <_P t\}$, and thus by Eq. (4) $k \leq j_{\omega(t)}$. From Eq. (5) it is clear that $j_{\omega(t)} \leq i_{\omega(t)}$. Consequently, the label $\omega(t)$ is assigned to $D^{i_{\omega(t)}}$ with $k \leq i_{\omega(t)}$, which means that $\omega(t)$ appears in w after $\omega(s) \equiv v_k$.

- If $\omega(s) \in D$ and $\omega(t) \in F$, then $\omega(t) \equiv v_k$ for some $k \in [\,q\,]$. Any $v_l \in F$ with $\omega^{-1}(v_l) <_P s$ also satisfies $\omega^{-1}(v_l) <_P s <_P t = \omega^{-1}(v_k)$, and therefore $l < k$. Therefore, application of Step 1 to $\omega(s)$ results in $j_{\omega(s)} < k$ and, due to the fact that $\omega(s) < \omega(t)$, we have $i_{\omega(s)} < k$. Thus, $\omega(s)$ is assigned to a $D^{i_{\omega(s)}}$ with $i_{\omega(s)} < k$, and therefore it appears in w before $\omega(t)$.

- If $\omega(s), \omega(t) \in D$, then for any v_k with $\omega^{-1}(v_k) <_P s$, it follows directly that $\omega^{-1}(v_k) <_P t$. Therefore, in Step 1, we find $j_{\omega(s)} \leq j_{\omega(t)}$, and as a result, in addition to $j_{\omega(s)} \leq i_{\omega(s)}$ (due to Eq. (5)) we also know that $j_{\omega(s)} \leq i_{\omega(t)}$. By construction of $j_{\omega(s)}$ and $i_{\omega(t)}$ as well as the order-preserving nature of ω, we find $v_{j_{\omega(s)}} < \omega(s) < \omega(t) < v_{i_{\omega(t)}+1}$. Therefore, there must be at least one value of i in the interval $j_{\omega(s)}, \dots, i_{\omega(t)}$ such that $v_i < \omega(s) < v_{i+1}$. It follows that $i_{\omega(s)} \leq i_{\omega(t)}$. If $i_{\omega(s)} < i_{\omega(t)}$, then obviously $\omega(s)$ appears in w before $v_{i_{\omega(t)}}$, which in turn appears before $\omega(t)$. Finally, even if $i_{\omega(s)} = i_{\omega(t)}$, in Step 2 the elements of each D^i are inserted into the sequence w in increasing order, so in any case $\omega(s)$ will be inserted before $\omega(t)$.

We have shown that the constructed sequence w satisfies the condition (b), $w \in \mathcal{L}(P)$.

Finally let us verify that w satisfies condition (c). Consider a label $d \in D$ which is inserted into w at some position k, thus $d \equiv w_k$. We have to show that d is not fixed, i.e. that it does not satisfy either of the conditions (1) and (2) of Def. 6. The construction process ensures that $w_{k-1} < d \equiv w_k < w_{k+1}$, therefore condition

(1) of Def. 6 is not satisfied. If the set $L(w_k) = \{l \,|\, l < k \text{ and } w_l > w_k\}$ is empty, then condition (2) of Def. 6 is trivially not satisfied. If $L(w_k)$ is nonempty, let $l = \max L$. If we can find a value of $j \in (l, k)$ such that $\omega^{-1}(w_j) <_P \omega^{-1}(w_k)$, then $\max(L(w_k)) = l < j \leq \max(J(w_k))$, thus ensuring that condition (2) of Def. 6 is not satisfied. We show below that this is indeed the case.

It follows directly from the definition of l that $w_l > w_k$. Since $w_{k-1} < w_k$, it is clear that $l \neq k-1$, and thus $l+1 < k$. Thus, we also have $w_k > w_{l+1}$ (since $l+1 \notin L(w_k)$); meaning that $w_l > w_{l+1}$, and l is a descent. We already have seen that condition (1) of Def. 6 is not satisfied for any of the labels in D; therefore w_l and w_{l+1} are not in D, but appear somewhere in the original sequence v, in the form $w_l \equiv v_{\tilde{l}}$ and $w_{l+1} \equiv v_{\tilde{l}+1}$ for some $\tilde{l} \in [\,q\,]$. Since $l+1 < k$, during Step 1, d must have been assigned to some D^{i_d} with $\tilde{l} < i_d$.

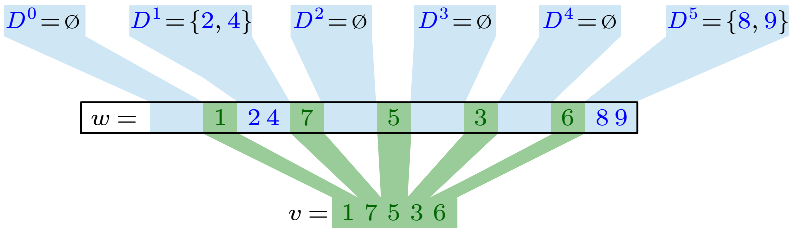
Let us assume that $j_d < \tilde{l}$; we will see that this assumption leads to a contradiction. From $\omega^{-1}(v_{j_d}) <_P \omega^{-1}(d)$ it follows that $v_{j_d} < d$. By construction of $\tilde{l}$, we have $d < v_{\tilde{l}}$. Therefore, if $j_d < \tilde{l}$, then there must be a $i \in (j_d - 1, \tilde{l})$ such that $v_i < d < v_{i+1}$. Then, by definition of i_d, we would have $i_d \leq i < \tilde{l}$, in contradiction with $\tilde{l} < i_d$. Therefore, the assumption $j_d < \tilde{l}$ made at the beginning of this paragraph must be wrong and we have $\tilde{l} \leq j_d$. Since $v_{\tilde{l}} > d$, we find $\omega^{-1}(v_{\tilde{l}}) \not<_P \omega^{-1}(d)$, and therefore (by Eq. (4)) $j_d \neq \tilde{l}$. This reasoning shows that $\tilde{l} < j_d$.

The entry v_{j_d} of v appears in w in the form $v_{j_d} \equiv w_j$ at some position $j \in [\,p\,]$. Since $\tilde{l} < j_d \leq i_d$ and labels in v appear in the same order in w, we find $l < j < k$. By construction of l, we have $w_m < w_k$ for all $m \in (l, k)$ (and thus especially for all $m \in (j, k)$); and by construction of j, we have $\omega^{-1}(w_j) <_P \omega^{-1}(w_k)$. Therefore, as discussed above, condition (2) of Def. 6 is not satisfied. It follows that every $d \in D$ is deletable in w, meaning that $D \subset \mathrm{Del}_P(w)$.

We have demonstrated that there is exactly one sequence w of labels of $[\,p\,]$ that satisfies conditions (a)–(c) given at the beginning of this proof. In other words, there is exactly one linear extension w such that $v = w \setminus D = w \cap F$ and $D \subset \mathrm{Del}_P(w)$ if and only if $\mathrm{Fix}_P(w) \subset F$. $\square$

Example 11. Let us demonstrate the insertion process described in Steps 1 and 2 during the proof above. Consider the sequence $v = 17536 \in \mathcal{L}(P \setminus D)$ with the poset $P = \mathbf{3} \times \mathbf{3}$ shown in Fig. 2 and the set of deleted labels $D = \{2, 4, 8, 9\}$. For the labels $d = 2$ and 4, we find for the set in Eq. (4) $\{j \in [\,5\,] \,|\, \omega^{-1}(v_j) <_P \omega^{-1}(d)\} = \{j \in [\,5\,] \,|\, v_j \in \{1\}\} = \{1\}$, and thus $j_2 = j_4 = 1$. Since $v_1 = 1 < 2, 4 < v_2 = 7$, in Eq. (5) we find $i_2 = i_4 = 1$. Therefore, the labels 2 and 4 are assigned to the subset D^1, and will be inserted into the sequence v between $v_1 = 1$ and $v_2 = 7$. For the label 8, we find that in Eq. (4), $\{j \in [\,5\,] \,|\, \omega^{-1}(v_j) <_P \omega^{-1}(8)\} = \{j \in [\,5\,] \,|\, v_j \in \{5, 7\}\} = \{2, 3\}$, and thus $j_8 = 3$. Since however the label 8 is larger than any of the following labels in v, $v_3 = 5$, $v_4 = 3$ and $v_5 = 6$, we find in Eq. (5) that $\{i \,|\, j_8 = 3 \leq i \leq 5,\ v_i < 8 < v_{i+1}\} = \{5\}$ and therefore $i_8 = 5$. Finally, for the label 9, we find that in Eq. (4), $\{j \in [\,5\,] \,|\, \omega^{-1}(v_j) <_P \omega^{-1}(9)\} = \{j \in [\,5\,] \,|\, v_j \in \{6, 8\}\} = \{5\}$, and thus $j_9 = 5$ and $i_9 = 5$. To summarize, in Step 1, we split the set $D = \{2, 4, 8, 9\}$ into the subsets $D^0 = \varnothing$, $D^1 = \{2, 4\}$, $D^2 = \varnothing$, $D^3 = \varnothing$, $D^4 = \varnothing$ and $D^5 = \{8, 9\}$. In

Step 2, the elements of each subset are arranged into a growing sequence, specifically $d_1^1 d_2^1 = 24$ and $d_1^5 d_2^5 = 89$, and inserted into v:



Lemma 12. *Let w be a linear extension of P with the set of fixed labels $\mathrm{Fix}_P(w)$. For every set of labels $F \supset \mathrm{Fix}_P(w)$, the sequence $w \cap F$ is a linear extension of $P \cap F$.*

Proof. Consider two elements $s, t \in P \cap F$ with $s <_{P\cap F} t$. In order to show that $w \cap F$ is a linear extension of $P \cap F$, we have to demonstrate that $\omega(s)$ appears in v before $\omega(t)$. Since $P \cap F$ is an induced subposet of P, it follows from $s <_{P\cap F} t$ that $s <_P t$, and since w is a linear extension, this implies that $\omega(s)$ precedes $\omega(t)$ in w. Clearly then, by construction of v, $\omega(s)$ also precedes $\omega(t)$ in v. □

By combining the previous two lemmata, we find that

Lemma 13. *The (disjoint) union of the Jordan-Hölder sets of all subposets of P is given by the disjoint union*

$$\bigsqcup_{Q \subset P} \mathcal{L}(Q) = \bigsqcup_{w \in \mathcal{L}(P)} \ \bigsqcup_{\mathrm{Fix}_P(w) \subset F \subset [p]} \{w \cap F\}.$$

Proof. Consider first a set $F \subset [p]$ and the corresponding subposet of P given by $P \cap F$. It follows directly from Lemmata 10 and 12 that the collection of linear extensions of $P \cap F$ can be written as

$$\mathcal{L}(P \cap F) = \bigsqcup_{\substack{w \in \mathcal{L}(P) \\ \text{for which} \\ F \supset \mathrm{Fix}_P(w)}} \{w \cap F\}.$$

Therefore, the set of linear extensions of subposets of P is given by

$$\bigsqcup_{Q \subset P} \mathcal{L}(Q) = \bigsqcup_{F \subset [p]} \mathcal{L}(P \cap F) = \bigsqcup_{F \subset [p]} \ \bigsqcup_{\substack{w \in \mathcal{L}(P) \\ \text{for which} \\ F \supset \mathrm{Fix}_P(w)}} \{w \cap F\} = \bigsqcup_{w \in \mathcal{L}(P)} \ \bigsqcup_{\mathrm{Fix}_P(w) \subset F \subset [p]} \{w \cap F\}.$$

□

4 Proof of Theorem 2

We are now ready to combine the lemmata derived so far into the derivation of the closed from of the extended strict order polynomial given in the first section.

Proof. (of Theorem 2) By application of Eqs. (1) and (2) as well as (in line 3) Lemmata 9 and 13, we find

$$\begin{aligned}
\mathrm{E}^{\circ}_{P}(n,z) &= \sum_{Q\subset P} \Omega^{\circ}_{Q}(n) z^{|Q|} \\
&= \sum_{Q\subset P} \sum_{v\in\mathcal{L}(Q)} \binom{n+\mathrm{des}(v)}{|Q|} z^{|Q|} \\
&= \sum_{w\in\mathcal{L}(P)} \sum_{\mathrm{Fix}_P(w)\subset F\subset[p]} \binom{n+\mathrm{des}(w)}{|F|} z^{|F|} \\
&= \sum_{w\in\mathcal{L}(P)} \sum_{k=0}^{p} \left|\left\{\mathrm{Fix}_P(w)\subset F\subset[p] \,\middle|\, |F|=k\right\}\right| \cdot \binom{n+\mathrm{des}(w)}{k} z^{k} \\
&= \sum_{w\in\mathcal{L}(P)} \sum_{k=0}^{p} \binom{p-\mathrm{fix}_P(w)}{k-\mathrm{fix}_P(w)} \cdot \binom{n+\mathrm{des}(w)}{k} z^{k}.
\end{aligned}$$

□

5 Perspectives

Our main motivation to develop the extended strict order polynomial $\mathrm{E}^{\circ}_{P}(n,z)$ introduced in the current communication is its close relation to the Zhang-Zhang polynomial [26, 27, 28] (also known as Clar covering polynomial or ZZ polynomial) enumerating Clar covers of benzenoid hydrocarbons [6], a topic to which we have devoted feverish activity in our laboratory for almost a decade now [2, 5, 4, 25, 24, 11, 12]. A *benzenoid* is a finite 2-connected subgraph of the hexagonal grid [7]; see for example Fig. 3 (a). A *Clar cover* of a benzenoid $\boldsymbol{B}$ is a spanning subgraph of $\boldsymbol{B}$ whose connected components are either edges (K_2) or hexagons (C_6) [6]. Usually a Clar cover is displayed together with its underlying benzenoid, with its hexagon components represented as circles in the corresponding hexagon of $\boldsymbol{B}$, see e.g. Fig. 3 (b). The Clar covers of a benzenoid can be enumerated using the *Zhang-Zhang polynomial*

$$\mathrm{ZZ}(\boldsymbol{B}) = \sum_{k=0}^{Cl} c_k x^k,$$

where the coefficient c_k denotes the number of Clar covers with k hexagons, and Cl is the maximum number of hexagons that can appear in the Clar covers of $\boldsymbol{B}$.

Our recent contribution, introducing the interface theory of benzenoids [11, 12], demonstrated that the enumeration of Clar covers of a benzenoid $\boldsymbol{B}$ can be efficiently performed by studying distributions of only the horizontal edges of its Clar covers (assuming $\boldsymbol{B}$ is oriented such that some of its edges are horizontal). The Clar covers of a benzenoid follow certain rules governing the number and the relative positions of their horizontal edges. We may thus associate each such horizontal edge with an element of a (Clar cover independent) poset $\mathcal{S}(\boldsymbol{B})$, whose partial order $<_{\mathcal{S}(\boldsymbol{B})}$ encodes the relative positions between these edges. The enumeration of Clar covers becomes especially simple for a certain subclass of benzenoids, the so-called regular benzenoid strips. These structures consist of fused columns of hexagons, each of which is, at either end, only half a hexagon longer or shorter than its neighbouring columns; and furthermore the first and last columns need to consist of the same number n of hexagons. All examples of benzenoids shown in this subsection fall into this category. For a regular strip $\boldsymbol{B}$ of length n, every strictly order-preserving map σ from a subposet $Q \subset \mathcal{S}(\boldsymbol{B})$ to the chain $\boldsymbol{n}$ corresponds to a well-defined set of $(1+x)^{|Q|}$ Clar covers, which have horizontal edges in the positions specified by σ. As a result, for any regular strip $\boldsymbol{B}$ that has a non-zero number of Clar covers, there exists a corresponding poset $\mathcal{S}(\boldsymbol{B})$ such that the extended strict order polynomial $\mathrm{E}^{\circ}_{\mathcal{S}(\boldsymbol{B})}(n,z)$ coincides with the Zhang-Zhang polynomial $\mathrm{ZZ}\,(\boldsymbol{B},x)$ of $\boldsymbol{B}$ (with $z = x+1$). A detailed proof of this fact is given in [13], and it has been used in [14, 15] to automatically and efficiently compute the Zhang-Zhang polynomials for a large amount of benzenoids (and in the same breath, the extended strict order polynomials for a large amount of posets, listed in [15]), a feat which was previously impossible. The equivalence between the extended strict order polynomials $\mathrm{E}^{\circ}_{P}(n,z)$ developed in the current study and the Zhang-Zhang polynomials $\mathrm{ZZ}\,(\boldsymbol{B},x)$ of regular benzenoid strips $\boldsymbol{B}$ allows us to recognize a large collection of facts about $\mathrm{E}^{\circ}_{P}(n,z)$ due to the previously discovered facts about the ZZ polynomials. Among others, the following facts are easy to deduce:

(a) (b)

Figure 3: Example of (a) a benzenoid $\boldsymbol{B}$, (b) a Clar cover of $\boldsymbol{B}$.

1. The chain $P = \boldsymbol{p}$ corresponds to a parallelogram $M\,(p,n)$ shown in Fig. 4.

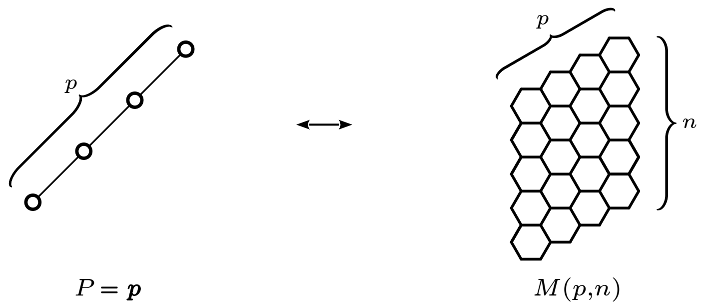


Figure 4: The extended strict order polynomial $\mathrm{E}^{\circ}_{P}(n,z)$ of a chain $P = \boldsymbol{p}$ is equivalent to the ZZ polynomial $\mathrm{ZZ}\left(M\left(m,n\right),x\right)$ of a parallelogram $M\left(p,n\right)$.

for which the ZZ polynomial is given by $\mathrm{ZZ}\left(M\left(m,n\right),x\right) = \sum_{k=0}^{p}\binom{m}{k}\binom{n}{k}(x+1)^k$ [8, 2, 3]; consequently, we have

$$\mathrm{E}^{\circ}_{\boldsymbol{p}}(n,z) = \sum_{k=0}^{p}\binom{p}{k}\binom{n}{k}z^k. \tag{6}$$

This result is also directly obvious from Theorem 2: The Jordan-Hölder set of $\boldsymbol{p}$ consists of only one element, $\mathcal{L}(\boldsymbol{p}) = \{123\ldots p\}$, for which $\mathrm{fix}_{\boldsymbol{p}}(123\ldots p) = 0$ and $\mathrm{des}_{\boldsymbol{p}}(123\ldots p) = 0$. Thus, Eq. (3) immediately assumes the form of Eq. (6).

2. The poset P containing p non-comparable elements corresponds, according to the interface theory of benzenoids, to a prolate rectangle $Pr\left(p,n\right)$ shown in Fig. 5 for which the ZZ polynomial is given by

$$\mathrm{ZZ}\left(Pr\left(p,n\right),x\right) = \left(1+n\left(x+1\right)\right)^p$$

[28, 1]; consequently, we have

$$\mathrm{E}^{\circ}_{[p]}(n,z) = \left(1+nz\right)^p.$$

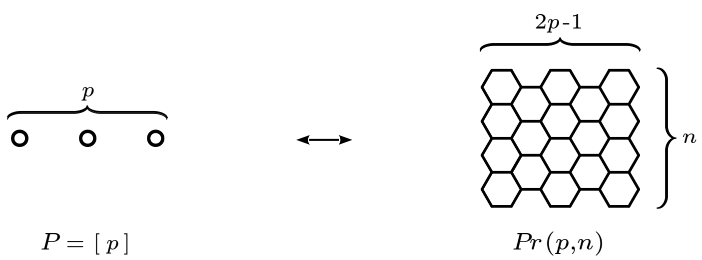


Figure 5: The extended strict order polynomial $\mathrm{E}^{\circ}_{P}(n,z)$ of a poset P of p non-comparable elements is equivalent to the ZZ polynomial $\mathrm{ZZ}\left(Pr\left(p,n\right),x\right)$ of a prolate rectangle $Pr\left(p,n\right)$.

This result can also be derived directly from Eq. (2): Every subposet $Q \subset [\,p\,]$ is an antichain of $|Q|$ incomparable elements, and thus has the strict order polynomial $\Omega^\circ_Q(n) = n^{|Q|}$. It follows that $\mathrm{E}^\circ_P(n,z) = \sum_{Q\subset P}\Omega^\circ_Q(n)z^{|Q|} = \sum_{Q\subset P}(nz)^{|Q|} = \sum_k \binom{p}{k}(nz)^k = (1+nz)^p$.

3. The poset $P = \mathbf{2}\times\boldsymbol{m}$ corresponds to a hexagonal graphene flake $O\,(2,m,n)$ shown in Fig. 6.

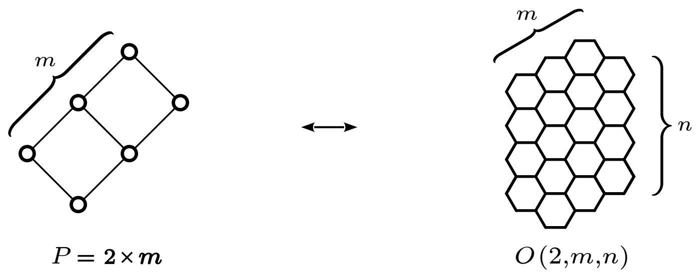


Figure 6: The extended strict order polynomial $\mathrm{E}^\circ_P(n,z)$ of a lattice $P = \mathbf{2}\times\boldsymbol{m}$ is equivalent to the ZZ polynomial $\mathrm{ZZ}\,(O\,(2,m,n)\,,x)$ of a hexagonal flake $O\,(2,m,n)$.

It follows from the ZZ polynomial $\mathrm{ZZ}\,(O\,(2,m,n)\,,x)$ [9, 10, 22] that the extended strict order polynomial has the form of a 2×2 determinant

$$\mathrm{E}^\circ_{\mathbf{2}\times\boldsymbol{m}}(n,z) = \begin{vmatrix} \sum\limits_{k=0} \binom{m}{k}\binom{n}{k} z^k & \sum\limits_{k=1} \binom{m+1}{k+1}\binom{n-1}{k-1} z^k \\ \sum\limits_{k=1} \binom{m-1}{k-1}\binom{n+1}{k+1} z^k & \sum\limits_{k=0} \binom{m}{k}\binom{n}{k} z^k \end{vmatrix}. \tag{7}$$

This determinantal formula for $\mathrm{ZZ}\,(O\,(2,m,n)\,,x)$ is a conjecture which has been discovered and verified via extensive numerical tests inspired by the John-Sachs Theorem [18]. The missing proof for this formula can likely be provided using the relation to the extended strict order polynomial and Eq. (3); note especially the similarity between Eq. (3) and the entries of the determinant in Eq. (7).

4. The extended strict order polynomial for the lattice $P = \boldsymbol{l}\times\boldsymbol{m}$ is unknown, following the fact that this poset corresponds to the hexagonal flake $O\,(l,m,n)$ shown in Fig. 7. The ZZ polynomial $\mathrm{ZZ}\,(O\,(l,m,n)\,,x)$ of this structure constitutes the hardest unsolved problem in the theory of ZZ polynomials [1, 4, 9, 23].

5. The fence $P = Q(1,m)$ with m elements corresponds to a multiple zigzag chain $Z(m,n)$ shown in Fig. 8.

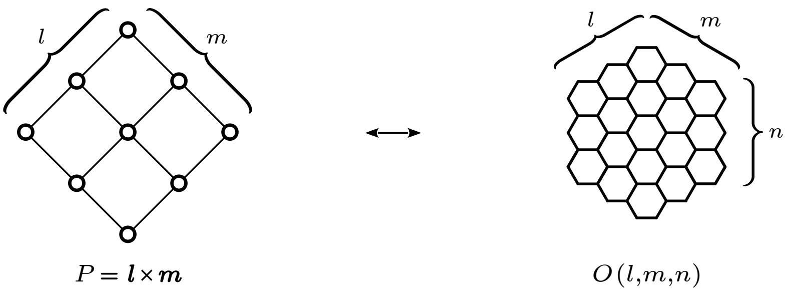


Figure 7: The extended strict order polynomial $\mathrm{E}^{\circ}_{P}(n,z)$ of a lattice $P = \boldsymbol{l} \times \boldsymbol{m}$ is equivalent to the ZZ polynomial $\mathrm{ZZ}\left(O\left(l,m,n\right),x\right)$ of a hexagonal flake $O\left(l,m,n\right)$.

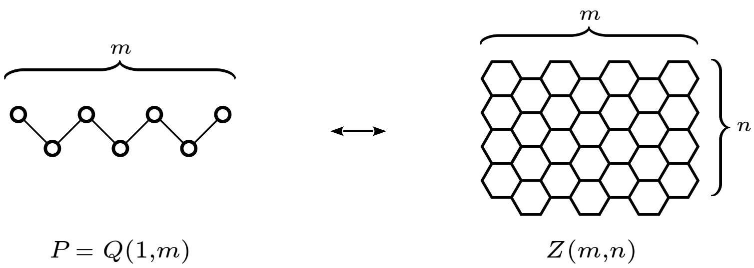


Figure 8: The extended strict order polynomial $\mathrm{E}^{\circ}_{P}(n,z)$ of a fence $P = Q(1,m)$ is equivalent to the ZZ polynomial $\mathrm{ZZ}\left(Z\left(m,n\right),x\right)$ of a multiple zigzag chain $Z(m,n)$.

The expression for $\mathrm{ZZ}\left(Z\left(m,n\right),x\right)$, and consequently $\mathrm{E}^{\circ}_{P}(n,z)$, is given by a lengthy formula [1, 4, 16], but the associated generating function has the form of a continued fraction [16].

$$\sum_{m=0}^{\infty} \mathrm{E}^{\circ}_{Q(1,m)}\left(n,z\right)t^{m} = \cfrac{-1}{t + \left.\cfrac{-1}{zt + \cfrac{-1}{\ddots + \cfrac{\ddots}{zt + \cfrac{-1}{zt + \cfrac{-1}{zt - \left(-1\right)^{n}}}}}}\right\} n} \tag{8}$$

An analogous generating function with respect to n is unknown. The generating function for $\mathrm{ZZ}\left(Z\left(m,n\right),x\right)$ was derived by utilizing certain recurrence relations between Zhang-Zhang polynomials of generalized multiple zigzag chains $Z_k\left(m,n\right)$, i.e., multiple zigzag chains with one incomplete row of length $k < n$. Finding the rather surprising result given in Eq. (8) through the preexisting methods of poset theory would likely be difficult; however, further interesting connections might be waiting to be discovered here.

The extended strict order polynomial $\mathrm{E}^{\circ}_P(n,z)$ can be also computed in an efficient fashion directly from Eq. (3) through an algorithm based on a graph of "compatible" antichains of P. Propagating weights through this graph in a certain way yields the extended strict order polynomial without ever having to construct the entire set $\mathcal{L}(P)$. This algorithm has been implemented in Maple 16 [17] and will be reported later. For the example of the poset $P = \mathbf{3} \times \mathbf{3}$ depicted in Fig. 2, we obtain in this way

$$\begin{aligned}\mathrm{E}^{\circ}_{\mathbf{3}\times\mathbf{3}}(n,z) = \sum_{k=0}^{9} \Bigg(& \left(\binom{9}{k}\binom{n}{k} + \left(9\binom{9-2}{k-2} + \binom{9-3}{k-3}\right)\right)\binom{n+1}{k} \\ & + \left(\binom{9-3}{k-3} + 17\binom{9-4}{k-4} + 2\binom{9-5}{k-5}\right)\binom{n+2}{k} \\ & + \left(2\binom{9-5}{k-5} + 7\binom{9-6}{k-6} + \binom{9-7}{k-7}\right)\binom{n+3}{k} + \binom{9-7}{k-7}\binom{n+4}{k}\Bigg) z^k. \end{aligned} \tag{9}$$

We suspect that the coefficients $e_{l,j}\,(P)$ appearing in $\mathrm{E}^{\circ}_P(n,z)$ in front of the terms $\binom{p-2l-j}{k-2l-j}\binom{n+l}{k}$ are #P-complete to compute, in close analogy to the coefficients $e\,(P)$ corresponding to the number of linear extensions of P. These coefficients are growing very fast with the size of the poset P. The largest of the coefficients $e_{l,j}\,(\mathbf{3}\times\mathbf{3})$ is only 17 (as can be easily seen from Eq. (9)), but larger P are characterized by much greater coefficients, e.g., $\max\left(e_{l,j}\,(\mathbf{4}\times\mathbf{4})\right)$=3765, $\max\left(e_{l,j}\,(\mathbf{4}\times\mathbf{5})\right)$=200440, $\max\left(e_{l,j}\,(\mathbf{5}\times\mathbf{5})\right)$=61885401, and $\max\left(e_{l,j}\,(\mathbf{5}\times\mathbf{6})\right)$=27950114975.

Stanley's strict order polynomial $\Omega^{\circ}_P(n)$ can be used to enumerate the linear extensions of a poset P by their number of descents. We have

$$\sum_{n=0}^{\infty} \Omega^{\circ}_P(n) t^n = \frac{\sum_{w\in\mathcal{L}(P)} t^{p-\mathrm{des}(w)}}{(1-t)^{p+1}}. \tag{10}$$

By computing the left-hand side of Eq. (10) and multiplying it by $(1-t)^{p+1}$, we obtain a polynomial in t, whose coefficients give the number of linear extensions of each type. This approach can be extended to the extended strict order polynomial $\mathrm{E}^{\circ}_P(n,z)$ in order to enumerate the linear extensions of a poset P simultaneously by their number of descents and by their number of fixed labels. We have

$$\sum_{n=0}^{\infty} \mathrm{E}^{\circ}_P(n,z) t^n \;=\; \frac{(1-t+zt)^p}{t^p\,(1-t)^{p+1}} \sum_{w\in\mathcal{L}(P)} \left(\frac{zt}{1-t+zt}\right)^{\mathrm{fix}_P(w)} t^{p-\mathrm{des}(w)}. \tag{11}$$

Now, by introducing a new variable $y = \frac{zt}{1-t+zt}$, we have

$$(1-t)\,t^p\,(1-y)^p \sum_{n=0}^{\infty} \mathrm{E}^{\circ}_P\left(n, \frac{y\,(1-t)}{t\,(1-y)}\right) t^n = \sum_{w\in\mathcal{L}(P)} y^{\mathrm{fix}_P(w)}\, t^{p-\mathrm{des}(w)}. \tag{12}$$

Computing the left-hand side of Eq. (12) produces a bivariate polynomial in y and t, whose coefficients give the number of linear extensions of each type. We anticipate that the actual computation of the left-hand side of Eq. (12) can be a rather difficult task but for the simplest posets.

Acknowledgements

We thank Professors Richard Stanley, Tomasz Łuczak, and Chih-wen Weng for helpful explanations and literature suggestions in early stages of work on this paper and two anonymous referees for improving the mathematical rigor of our exposition. This work was financially supported by Ministry of Science and Technology of Taiwan (MOST108-2113-M-009-010-MY3 and MOST110-2923-M-009-004-MY3) and the Center for Emergent Functional Matter Science of National Chiao Tung University from the Featured Areas Research Center Program within the framework of the Higher Education Sprout Project by the Ministry of Education (MOE), Taiwan.

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(Received 17 Oct 2020; revised 22 Mar 2021, 16 June 2021)