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On s-fully cycle extendable line graphs

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Abstract

A graph G is said to be fully cycle extendable if every vertex of G lies in a triangle and for every non-Hamiltonian cycle C there is a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. We investigate the trail extendability in a graph and the cycle extendability in its line graph. For simple connected graphs that are neither paths nor cycles, we define $l(G) = \max\{m : G$ has a divalent path of length m that is not both of length 2 and in a $K_3\}$, where a divalent path is a path whose internal vertices have degree two in G. If the removal of any s or fewer vertices in G results in a fully cycle extendable graph, we say G is an s-fully cycle extendable graph. The s-fully cycle extendable index, $\text{fce}_s(G)$, of a simple connected graph G, is the least nonnegative integer m such that $L^m(G)$ is s-fully cycle extendable. Let $s \geq 0$ be an integer and G be a simple connected graph that is not a path, a cycle or a $K_{1,3}$. We show that

$$\text{fce}_s(G) \leq \begin{cases} l(G) + s + 1 & \text{if } 0 \leq s \leq 1, \\ l(G) + \lfloor \log_2 s \rfloor + 3 & \text{if } s \geq 2, \end{cases}$$

and the bound is sharp.

1 Introduction

We use [2] for terminology and notation not defined here, and only consider finite and simple graphs unless otherwise noted (multiple edges appear in the proof of

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Lemma 4.4). In particular, $\kappa(G)$ and $\kappa'(G)$ represent the *connectivity* and *edge-connectivity* of a graph G, respectively; $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degrees of G, respectively. For two sets A and B, $A \triangle B$ denotes the symmetric difference of A and B. A graph is *trivial* if it contains no edges. An edge cut Y of G is *essential* if $G-Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$. We use $\kappa'_e(G)$ to denote the *essential edge connectivity* of a graph G. The degree sum of two end-vertices of an edge is closely related to $\kappa'_e(G)$.

Proposition 1.1 *(Shao, Proposition 2.1 of [13]) Let $n \geq 1$ be an integer and G be a graph which is not $K_{1,n-1}$ or K_3. Then the degree sum of any two adjacent vertices is at least $\kappa'_e(G) + 2$.*

Given any nonempty graph G, the *line graph* of G, denoted by $L(G)$ or $L^1(G)$, has the property that there exists a one-to-one correspondence between $E(G)$ and $V(L(G))$ such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have a common vertex. Iteratively, $L^n(G) = L(L^{n-1}(G))$ and $L^0(G) = G$. The following proposition reveals the relationship of connectivities of G and its line graph $L(G)$.

Proposition 1.2 *(Shao, Proposition 1.2 of [13]) Let $n \geq 1$ be an integer and G be a graph which is not K_3 or $K_{1,n-1}$. Then each of the following holds.*

(i) $\kappa'_e(G) \geq \kappa'(G)$.

(ii) $\kappa'_e(G) = \kappa(L(G))$.

(iii) $\kappa'_e(L(G)) \geq \kappa'_e(G)$.

(iv) $\kappa(L(G)) \geq \kappa(G)$.

Propositions 1.1 and 1.2 will be used in the proof of Lemma 4.4 and Theorem 4.3 in Section 4.

For a graph G and $v \in V(G)$, define $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$ and $E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}$. Note that the vertex subset in the line graph $L(G)$ corresponding to each $E_G(v)$ in G induces a complete subgraph in $L(G)$.

A cycle C in a graph G is *extendable* in G if there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. A graph G of order n is *cycle extendable* if G contains at least one cycle and every non-Hamiltonian cycle in G is extendable. A graph G of order n is *fully cycle extendable* if G is cycle extendable and every vertex of G lies in a triangle of G. If the removal of any s or fewer vertices in G results in a fully cycle extendable graph, we say G is an s-fully cycle extendable graph.

The concept of the Hamiltonian index of a graph G was first introduced by Chartrand and Wall [3] as the least nonnegative integer k such that $L^k(G)$ is Hamiltonian.

They showed that the Hamiltonian index exists as a finite number. In 1983, Clark and Wormald [4] extended this idea and introduced the Hamiltonian-like indices. Lai gave a best bound for Hamiltonian index $h(G)$ in [9]. For more results of Hamiltonian-like indices, see [5, 6, 8–12, 15–18]. Motivated by these results (Hamilton-connected index, pancyclic index etc.), we define the s-fully cycle extendable index, $\mathrm{fce}_s(G)$, of a simple connected graph G as the least nonnegative integer m such that $L^m(G)$ is s-fully cycle extendable.

For simple connected graphs that are neither paths nor cycles, we define $l(G) = \max\{m : G$ has a divalent path of length m that is not both of length 2 and in a $K_3\}$, where a divalent path is a path whose internal vertices have degree two in G. Note that an edge with both end-vertices of degree at least 3 is a divalent path of length 1. In this paper, we give a best bound for an iterated line graph to be s-fully cycle extendable as follows.

Theorem 1.3 *Let $s \geq 0$ be an integer and G be a simple connected graph that is not a path, a cycle or a $K_{1,3}$. Then*

$$fce_s(G) \leq \begin{cases} l(G) + s + 1 & \text{if } 0 \leq s \leq 1, \\ l(G) + \lfloor \log_2 s \rfloor + 3 & \text{if } s \geq 2, \end{cases}$$

and the bound is sharp.

In Section 2, we give some properties of line graphs which will be used in the proof of Theorem 1.3. In Section 3, we investigate quasi-trails and their extendability. The proof of our main result Theorem 1.3 lies in Section 4.

2 Properties of Line Graphs

In order to facilitate the proofs of our major results, we list some properties of line graphs as follows. Let G be a graph and $L(G)$ be its line graph. Proposition 2.1 states that deleting vertices in $L(G)$ corresponds to deleting edges in G. Proposition 2.2 provides us the minimum degree, connectivity and triangular properties of line graphs. Theorem 2.3 gives best bounds for the connectivity of line graphs.

Proposition 2.1 *(Zhang, Eschen, Lai and Shao, Proposition 3.1 of [17]) Let G be a simple graph with $|V(G)| = n$. Let $S' \subseteq E(G)$ and $S \subseteq V(L(G))$ be the corresponding vertex set of the edge set S'. Then $L(G) - S = L(G - S')$.*

A graph G is *k-triangular* if each edge of G lies in at least k triangles and G is *triangular* if it is 1-triangular. The following proposition describes a few properties of iterated line graphs.

Proposition 2.2 *(Zhang, Eschen, Lai and Shao, Lemma 3.2 of [17]) Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. Then each of the following holds:*

(i) For integers $m \geq 0$,

$$l(L^m(G)) = \begin{cases} l - m & \text{if } 0 \leq m < l, \\ 1 & \text{if } m \geq l. \end{cases}$$

(ii) For integers $k \geq 0$,

$$\delta(L^{l+k}(G)) \geq \begin{cases} 2 & \text{if } k = 0 \text{ or } k = 1, \\ 2^{k-2} + 2 & \text{if } k \geq 2. \end{cases}$$

(iii) $L^l(G)$, $L^{l+1}(G)$ and $L^{l+2}(G)$ are triangular. Moreover, $L^{l+k}(G)$ is 2^{k-3}-triangular when $k \geq 3$.

(iv) For integers $k \geq 0$, $\kappa(L^{l+k}(G)) \geq k + 1$.

The following theorem gives us best bounds for the connectivity of iterated line graphs.

Theorem 2.3 *(Shao, Theorem 1.5 of [14]) Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. Then each of the following holds:*

(i) For integers $s \geq 1$, $\kappa'_e(L^{l+s}(G)) \geq 2^s + 2$. The bound is best possible.

(ii) For integers $s \geq 2$, $\kappa(L^{l+s}(G)) \geq 2^{s-1} + 2$. The bound is best possible.

3 Quasi-trails in a graph G and Cycles in $L(G)$

In this section, Proposition 3.2 reveals a relationship between a cycle in $L(G)$ and a quasi-trail in G. It converts a cycle extendable question in $L(G)$ to an extension of a quasi trail in G. Theorem 3.3 and Corollary 3.4 serve as motivations for Theorem 3.6 and Corollary 3.7. As a corollary of Theorem 3.6, Corollary 3.7 provides us with a sufficient condition for the proof of our main result.

A *trail* in G is a sequence $v_1e_1v_2e_2 \dots e_{m-1}v_m$ whose terms are alternately vertices and edges of G such that e_i is the edge joining v_i and $v_{i+1} (1 \leq i \leq m - 1)$ and the edges are distinct. A trail is *closed* if $v_1 = v_m$ and *spanning* in G if it contains all vertices of G. A *dominating closed trail* T of G is a closed trail such that $G - V(T)$ is edgeless.

The following theorem reveals the relationship between a dominating closed trail in H and a Hamiltonian cycle in $L(H)$.

Theorem 3.1 *(Harary and Nash-Williams, [7]) The line graph $G = L(H)$ of a graph H is Hamiltonian if and only if H has a dominating closed trail.*

Motivated by Theorem 3.1 and the definition of dominating closed trail, we observe that any cycle in a line graph corresponds to a "spiky" trail that is the union of a closed trail and some edges with at least one end-vertex in the trail. If $P \subseteq G$ and P contains a trail T such that $P - V(T)$ is edgeless, then we call P a *quasi-trail* of G. Let e be an edge of the quasi-trail P. Then at least one end-vertex of e is in T. A *closed quasi-trail* is a quasi-trail containing a closed trail T such that $P - V(T)$ is edgeless. A quasi-trail P is *extendable* if there exists a quasi-trail P' in G such that $E(P) \subseteq E(P')$ and $|E(P')| = |E(P)| + 1$.

Using a similar argument as in [7], we have the following proposition.

Proposition 3.2 *Let G be a graph with $|E(G)| \geq 3$. Then G has a closed quasi-trail P if and only if $L(G)$ has a cycle C with $|V(C)| = |E(P)|$ (the vertex set $V(C)$ in $L(G)$ corresponds to the edge set $E(P)$ in G).*

Theorem 3.3 *(Zhang et al., Theorem 3.4 of [18]) Let G be a connected triangular graph. Let P be a closed quasi-trail of G with $E(G)-E(P) \neq \emptyset$ and T be a closed trail contained in P such that $P - V(T)$ is edgeless. Then there exist a closed quasi-trail P' and a closed trail T' contained in P' such that*

(i) $P' - V(T')$ *is edgeless,*

(ii) $E(P) \subseteq E(P')$ *with* $|E(P')| = |E(P)| + 1$,

(iii) $V(T) \subseteq V(T')$.

The following corollary follows from Proposition 3.2 and Theorem 3.3.

Corollary 3.4 *Let G be a connected triangular graph. Then $L(G)$ is fully cycle extendable.*

As defined in Section 2, a graph G is *triangular* or *edge triangular* if each edge of G lies in at least one triangle. In [18], we extended this concept to *an almost triangular* graph and defined a graph G as *an almost triangular* graph if there exists a connected triangular subgraph H such that $G - V(H)$ is edgeless. Note that a connected triangular graph must be almost triangular, but not vice versa.

A graph G is *vertex pancyclic* if for each vertex $v \in V(G)$, and for each integer k with $3 \leq k \leq |V(G)|$, G has a k-cycle C_k such that $v \in V(C_k)$. It is proved in [18] that the line graph of an almost triangular graph is vertex pancyclic.

However, the line graph of an almost triangular graph might not be fully cycle extendable. In Figure 1 below, let H be a connected triangular graph and v be an isolated vertex. We obtain G by joining v to three independent vertices in H. The claw with center v in G induces a triangle in $L(G)$, but there is no way to extend this triangle in $L(G)$ to a 4-cycle.

To exclude the example below, we define a *strongly almost triangular* graph as an almost triangular graph such that every vertex of degree at least three lies in a triangle. We prove the line graph of a strongly almost triangular graph is fully cycle extendable.

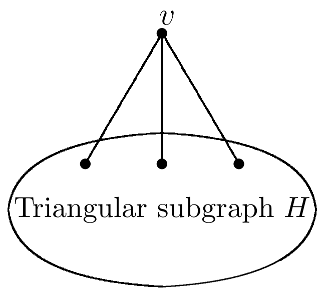


Figure 1: an almost triangular graph with a claw

Lemma 3.5 *Let G be a strongly almost triangular graph, P be a closed quasi-trail of G such that $E(G) - E(P) \neq \emptyset$, and T be a closed trail which is contained in P such that $P - V(T)$ is edgeless. If there exist $e \in E(P) - E(T)$ and $e' \in E(G) - E(P)$ such that e is incident with e' and e lies in a triangle of G, then P is extendable.*

Proof. In order to show P is extendable, we prove that there exist a closed quasi-trail P' and a closed trail T' contained in P' such that

(i) $P' - V(T')$ is edgeless,

(ii) $E(P) \subseteq E(P')$ with $|E(P')| = |E(P)| + 1$.

If there exists $e_1 \in E(G) - E(P)$ such that e_1 has at least one end-vertex in $V(T)$, then $P' = P \cup \{e_1\}$ and $T' = T$ satisfy (i) and (ii). So we may assume that

$$\text{if an edge has at least one end-vertex in } V(T)\text{, then it must be in } E(P). \quad (1)$$

Since e is incident with e', we let $e = uv$ and $e' = vv_1$. Since $e \in E(P) - E(T)$, we assume $u \in V(T)$. Since $e = uv$ is incident with e' and $e' \notin E(P)$, by (1),

$$v \notin V(T). \quad (2)$$

Since e lies in a triangle of G, we assume that e lies in a triangle uvw of G and consider the following two cases.

Case 1 $w \in V(T)$.

Let $T' = T \triangle \{uw, uv, vw\}$. By (2), $uv, vw \notin E(T)$, and so $uv, vw \in E(T')$. Then T' is a closed trail with $V(T') = V(T) \cup \{v\}$. By (1), $\{uw, vw, uv\} \subseteq E(P)$. So $E(T') \subseteq E(P)$. Let $P' = P \cup \{vv_1\}$. As $vv_1 \notin E(P)$, $P' - V(T')$ is edgeless and $E(P) \subseteq E(P')$ with $|E(P')| = |E(P)| + 1$, implying that P is extendable.

Case 2 $w \notin V(T)$.

Since $u \in V(T)$, by (1), $uv, uw \in E(P) - E(T)$ and by (2), $vw \notin E(P)$. Let $T' = T \cup \{uw, uv, vw\}$. Then T' is a closed trail with $V(T') = V(T) \cup \{v, w\}$. Let $P' = P \cup \{vw\}$. Since $E(T') \subseteq E(P) \cup \{vw\}$, $P' - V(T')$ is edgeless and $E(P) \subseteq E(P')$ with $|E(P')| = |E(P)| + 1$, implying that P is extendable. □

Theorem 3.6 *Let G be a strongly almost triangular graph, and P be a closed quasi-trail of G such that $E(G) - E(P) \neq \emptyset$. Then P is extendable.*

Proof. By the definition of quasi-trails, there exists a closed trail T which is contained in P such that $P - V(T)$ is edgeless. Since $E(G) - E(P) \neq \emptyset$ and G is connected, there exists an edge $xy \in E(G) - E(P)$ such that at least one vertex in $\{x, y\}$ is in $V(P)$. Without loss of generality, we assume that $x \in V(P)$. By (1), we may assume $\{x, y\} \cap V(T) = \emptyset$. Since $x \in V(P) - V(T)$, there exists $xz \in E(P)$ with $z \in V(T)$.

If xz lies in a triangle, then $e = xz$ and $e' = xy$ satisfy the conditions of Lemma 3.5, and so P is extendable. Next we assume that xz does not lie in any triangle of G.

Since G is strongly almost triangular, there exists a connected triangular subgraph H such that $G - V(H)$ is edgeless. Let H be a connected triangular subgraph H with the maximal number of edges such that $G - V(H)$ is edgeless. It implies that at least one of x, z is in $V(H)$ and we consider the following three cases.

Case 1 Both x and z are in $V(H)$.

Since H is connected and triangular, there exists a (x, z)-path in H such that every edge on the path lies in a triangle. Denote this (x, z)-path by $z_0 z_1 z_2 \ldots z_{m-1} z_m$ with $z_0 = x$ and $z_m = z$. Since xz doe not lie in any triangle, $m \geq 3$. Let z_t be the first vertex on the path from x to z such that $z_t \in V(T)$. Then $z_{t-1} \notin V(T)$ and $z_t z_{t-1}$ lies in a triangle. Then $e = z_t z_{t-1}$ and $e' = z_{t-1} z_{t-2}$ satisfy the conditions of Lemma 3.5, so P is extendable (if $z_{t-1} = x$, then let $z_{t-2} = y$).

Case 2 $z \in V(H)$ and $x \notin V(H)$.

Since $x \notin V(H)$, by the definition of a strongly almost triangular graph, all neighbors of x are in $V(H)$. So $y \in V(H)$. Since H is connected and triangular, there exists a (y, z)-path in H such that every edge on the path lies in a triangle. Using a similar argument as in Case 1, we can see that P is extendable.

Case 3 $x \in V(H)$ and $z \notin V(H)$.

Since $z \in V(T)$ and $zx \in E(P) - E(T)$, $d_G(z) \geq 3$. By the definition of a strongly almost triangular graph, z lies in a triangle. By the maximality of $E(H)$, $z \in V(H)$, a contradiction to the assumption of Case 3. □

The following corollary follows from Proposition 3.2 and Theorem 3.6.

Corollary 3.7 *If G is a strongly almost triangular graph, then $L(G)$ is fully cycle extendable.*

Proof. Let C be a cycle of $L(G)$ with $|V(C)| < |V(L(G))|$. By Proposition 3.2, there exists a closed quasi-trail P in G such that $|V(C)| = |E(P)|$ and the line graph of P is C in $L(G)$. Since G is a strongly almost triangular graph, P is extendable by Theorem 3.6. Let P' be an extension of P. By Proposition 3.2 again, the line graph of P' is a cycle, say C', in $L(G)$ with $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. □

4 Proof of Theorem 1.3

In the proof of Theorem 1.3, Theorem 4.3 and Lemma 4.4 are applied in the cases of $s \geq 2$ and $s = 1$ respectively. Proposition 4.1 and Theorem 4.2 are used in the proof of Theorem 4.3.

Proposition 4.1 *Let $m \geq 1$ be an integer and K_{m+2} be a complete graph of order $m+2$. Let $S \subseteq E(K_{m+2})$.*

(i) If $|S| \leq m-1$, then $K_{m+2} - S$ is triangular.

(i) If $K_{m+2} - S$ is not triangular, then $|S| \geq m$.

Proof. (i) Note that $d_{K_{m+2}}(v) = m+1$ for any $v \in V(K_{m+2})$. Denote $K_{m+2} - S$ by H. Let $uv \in E(H)$. Then $d_H(u)+d_H(v) \geq (m+1)+(m+1)-|S| \geq 2m+2-(m-1) = m+3 > m+2$, which implies that u and v must have at least one common neighbor in H. Since uv is arbitrary in H, it follows that H is triangular, i.e., $K_{m+2} - S$ is triangular.

(ii) follows immediately from (i). □

The following theorem characterizes a property of line graphs, which will be used in the proof of Theorem 4.3.

Theorem 4.2 *(Krausz, Theorem 10.2 of [1]) A nonempty graph H is a line graph if and only if $E(H)$ can be partitioned into subsets so that*

(i) the subgraph induced by each member of the partition is complete, and

(ii) no vertex of H lies in more than two of these induced subgraphs.

Theorem 4.3 *Let t, s be non-negative integers with $t \geq 1$. Let G be a graph with $\kappa(G) \geq t+2$. Then for each $S \subseteq E(L(G))$ with $|S| = s < 2t$, $L(G) - S$ is strongly almost triangular.*

Proof. By Theorem 4.2(i), we have the following:

(A) $E(L(G))$ can be partitioned into subsets so that the subgraph induced by each member of the partition is complete.

Let m be a natural number. Following the proof of Theorem 4.2 or by the definition of line graphs, the corresponding vertex set of the adjacent edges of each vertex of G induces a complete subgraph in $L(G)$. Since $\delta(G) \geq \kappa(G) \geq t+2$, by (A), we assume that

(B) $E(L(G)) = E_1 \cup E_2 \cup \cdots \cup E_m$, where each E_i induces a complete subgraph of order at least $t+2$ in $L(G)$ with $E_i \cap E_j = \emptyset$.

For convenience, we let $L[E_i]$ be the induced subgraph of E_i in $L(G)$. By Theorem 4.2(ii) and $\delta(G) \geq \kappa(G) \geq t+2 \geq 3$, we have the following:

(C) Every vertex of $L(G)$ lies in exactly two of $L[E_1], L[E_2], \ldots, L[E_m]$.

If $s \leq t-1$, then, by (A) and Proposition 4.1(i), $L(G) - S$ is triangular, and hence strongly almost triangular. Since $s < 2t$, we may assume that $s = t+a$, where a is an integer with $0 \leq a < t$. If $L[E_i] - S$ is not triangular in $L(G) - S$, then, by Proposition 4.1(ii), $|E_i \cap S| \geq t$.

We claim that only one of $L[E_1], L[E_2], \ldots, L[E_m]$ is not triangular after the deletion of S in G. Suppose, to the contrary, that at least two of $L[E_1], L[E_2], \ldots, L[E_m]$ are not triangular. Then, by (B) and Proposition 4.1(ii), two of $L[E_1], L[E_2], \ldots, L[E_m]$ contain at least $2t$ edges of S in total, contrary to the condition that $|S| = s < 2t$.

Without loss of generality, we assume that $L[E_1] - S$ is not triangular in $L(G) - S$. Then, by Proposition 4.1(ii), $|E_1 \cap S| \geq t$. Let $H = L[E_2 \cup E_3 \cup \cdots \cup E_m - S]$. Clearly H is triangular. By (C), every vertex of $L[E_1]$ lies in exactly one of $L[E_2], L[E_3], \ldots, L[E_m]$. So every vertex of $L(G) - S$ lies in a triangle and $V(H) = V(L(G) - S)$. It remains to show that H is connected.

Note that $L[E_1]$ is generated by the incident edges of a vertex in G, say v. Since $\kappa(G) \geq t+2 \geq 3$, $\kappa(G - \{v\}) \geq t+1$. By Proposition 1.2(iv), $\kappa(L(G - \{v\})) \geq t+1$. So $L[E_2 \cup E_3 \cup \cdots \cup E_m] - V(L[E_1]) = L(G - \{v\})$ is $(t+1)$-connected. Since $|E_1 \cap S| \geq t$ and $|S| < 2t$, $|(E_2 \cup E_3 \cup \cdots \cup E_m) \cap S| < t$. So $L[E_2 \cup E_3 \cup \cdots \cup E_m - S] - V(L[E_1])$ is connected. By (B) and (C), each vertex of $L[E_1]$ must have at least $t+1$ neighbors in $L[E_2 \cup E_3 \cup \cdots \cup E_m] - V(L[E_1])$, and thus have at least two neighbors in $L[E_2 \cup E_3 \cup \cdots \cup E_m - S] - V(L[E_1])$. So $H = L[E_2 \cup E_3 \cup \cdots \cup E_m - S]$ is still connected.

Hence H is a connected triangular subgraph of $L(G) - S$ and $L(G) - S - V(H) = \emptyset$, which implies that $L(G) - S$ is strongly almost triangular. □

Let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying two ends of each edge in X and then deleting the resulting loops.

Lemma 4.4 *Let G be a simple connected graph with $l(G) = l$, where G is not a path, cycle, or $K_{1,3}$. Then each of the following holds.*

(i) For any $e \in E(L^{l+1}(G))$, $L^{l+1}(G) - \{e\}$ is strongly almost triangular.

(ii) For any $v \in V(L^{l+2}(G))$, $L^{l+2}(G) - \{v\}$ is fully cycle extendable.

Proof (i) By Proposition 2.2(ii) and (iii),

$$L^{l+1}(G) \text{ is triangular with minimum degree at least 2.} \tag{3}$$

If $L^{l+1}(G) - \{e\}$ is still triangular, then we are done. So we assume that

$$L^{l+1}(G) - \{e\} \text{ is not triangular.} \tag{4}$$

By Proposition 2.2(iv) and Theorem 2.3(i),

$$\kappa(L^{l+1}(G)) \geq 2 \text{ and } \kappa'_e(L^{l+1}(G)) \geq 4. \tag{5}$$

Let $e = xy \in E(L^{l+1}(G))$. By (3), we consider the following three cases by showing $L^{l+1}(G) - \{xy\}$ has a connected triangular subgraph H such that $L^{l+1}(G) - \{xy\} - V(H)$ is edgeless, and every vertex of degree at least 3 of $L^{l+1}(G) - \{xy\}$ lies in a triangle of $L^{l+1}(G) - \{xy\}$.

Case 1 One of x, y has degree 2 in $L^{l+1}(G)$.

Without loss of generality we assume that $d_{L^{l+1}(G)}(x) = 2$. By (3), we assume that xyz is a triangle containing xy. Together with $d_{L^{l+1}(G)}(x) = 2$, xyz is the only triangle that contains the edge xy in $L^{l+1}(G)$. So xz, yz are the only edges possibly not lying in any triangle in $L^{l+1}(G) - \{xy\}$, i.e. every edge of $L^{l+1}(G) - \{xy, yz, xz\}$ lies in a triangle in $L^{l+1}(G) - \{xy\}$.

Case 1.1 yz lies in a triangle in $L^{l+1}(G) - \{xy\}$. Let $H = L^{l+1}(G) - \{x\}$. Then H is triangular. By (5), H is connected. Note that $L^{l+1}(G) - \{xy\} - V(H) = \{x\}$ is edgeless. So H is a connected triangular subgraph of $L^{l+1}(G) - \{xy\}$ such that $L^{l+1}(G) - \{xy\} - V(H)$ is edgeless.

Case 1.2 yz does not lie in any triangle in $L^{l+1}(G) - \{xy\}$. Let $H = L^{l+1}(G) - \{x\} - \{yz\}$. Then H is triangular.

By Proposition 1.1, $d_{L^{l+1}(G)}(y) \geq 4$ and $d_{L^{l+1}(G)}(z) \geq 4$. Then $y \in V(H)$, $z \in V(H)$. There exists at least one edge $yy_1 \in E(L^{l+1}(G)) - \{yx, yz\}$. Since yy_1 lies in a triangle, and $d_{L^{l+1}(G)}(x) = 2$, there exists a triangle yy_1y_2 with $y_2 \in V(L^{l+1}(G)) - \{x, y_1, z\}$. Similarly, z lies in a triangle not containing x and y. So both y and z lie in triangles in $L^{l+1}(G) - \{xy\}$ and $L^{l+1}(G) - \{xy\} - V(H) = \{x\}$ is edgeless. It remains to show H is connected.

Since the edge connectivity and essential edge connectivity does not decrease after contraction, by (5), $L^{l+1}(G)/\{xy\}$ is 2-edge-connected and essentially 4-edge-connected, which implies that the only 2-edge-cuts are the sets of edges incident with a vertex of degree two in $L^{l+1}(G)/\{xy\}$. As neither y ($y = x$ after contracting xy) nor z has degree two, $H = L^{l+1}(G) - \{x\} - \{yz\} = L^{l+1}(G)/\{xy\} - \{xz, yz\}$ is 2-edge-connected. So H is a connected triangular subgraph of $L^{l+1}(G) - \{xy\}$ and $L^{l+1}(G) - \{xy\} - V(H)$ is edgeless.

In either case, x is the only vertex not lying in any triangle of $L^{l+1}(G) - \{xy\}$. As x has degree 1 in $L^{l+1}(G) - \{xy\}$, every vertex of degree at least 3 of $L^{l+1}(G) - \{xy\}$ lies in a triangle of $L^{l+1}(G) - \{xy\}$.

Case 2 One of x, y has degree 3 in $L^{l+1}(G)$.

Without loss of generality we assume that $d_{L^{l+1}(G)}(x) = 3$. Assume that $N_{L^{l+1}(G)}(x) - \{y\} = \{z_1, z_2\}$. Since xy, xz_1, xz_2 lie in triangles, xy lies in at most two triangles in $L^{l+1}(G)$. Suppose xy lies in exactly one triangle, say xyz_2, in $L^{l+1}(G)$. Then, by (3), $z_1z_2 \in E(L^{l+1}(G))$, and so xz_1z_2 forms a triangle. Note that yz_2 is not lying in any triangle of $L^{l+1}(G) - \{xy\}$, otherwise $L^{l+1}(G) - \{xy\}$ is connected and triangular, contrary to (4). Let $H = L^{l+1}(G) - \{xy, yz_2\}$. Then H is triangular. By Proposition 1.1, $d_{L^{l+1}(G)}(y) \geq 3$, and together with (5), H is connected. So H is a connected triangular subgraph such that $L^{l+1}(G) - \{xy\} - V(H) = \emptyset$ is edgeless, and every vertex of $L^{l+1}(G) - \{xy\}$ lies in a triangle of $L^{l+1}(G) - \{xy\}$.

Now we assume that xy lies in exactly two triangles, xyz_1 and xyz_2, in $L^{l+1}(G)$. If $z_1z_2 \in E(L^{l+1}(G))$, then xz_1, yz_1, xz_2, yz_2 lie in triangles in $L^{l+1}(G) - \{xy\}$. By (5), $L^{l+1}(G) - \{xy\}$ is still connected and triangular, contrary to (4). Next we assume that $z_1z_2 \notin E(L^{l+1}(G))$.

By Proposition 1.1, $d_{L^{l+1}(G)}(y) \geq 3$ and $d_{L^{l+1}(G)}(z_i) \geq 3$ for $i = 1, 2$. There exists at least one edge $z_1w_1 \in E(L^{l+1}(G)) - \{xz_1, yz_1\}$ and $w_1 \neq z_2$. Since z_1w_1 lies in a triangle, $z_1z_2 \notin E(L^{l+1}(G))$ and $d_{L^{l+1}(G)}(x) = 3$, there exists a triangle $z_1w_1w_1'$ with $w_1' \notin \{x, z_2\}$. Similarly, z_2 lies in a triangle $z_2w_2w_2'$ with $w_2 \notin \{x, y, z_1\}$ and $w_2' \notin \{x, z_1\}$. So both z_1 and z_2 lie in triangles in $L^{l+1}(G) - \{xy\}$.

Case 2.1 $d_{L^{l+1}(G)}(y) = 3$. Then $w_1' \neq y$, $w_2' \neq y$, and both x and y have degree 2 in $L^{l+1}(G) - \{xy\}$. Let $H = L^{l+1}(G) - \{x, y\}$. Then H is triangular. By (5), $(L^{l+1}(G) - \{xy\})/\{xz_2, yz_2\}$ is 1-edge-connected and essentially 3-edge-connected, which implies that the only 1-edge-cuts (or 2 edge-cuts) are the sets of edges incident with a vertex of degree one (or two). So $H = L^{l+1}(G) - \{x, y\} = (L^{l+1}(G) - \{xy\}/\{xz_2, yz_2\}) - \{xz_1, yz_1\}$ is 1-edge-connected, $L^{l+1}(G) - \{xy\} - V(H) = \{x, y\}$ is edgeless, and every vertex of degree at least 3 of $L^{l+1}(G) - \{xy\}$ lies in a triangle of $L^{l+1}(G) - \{xy\}$.

Case 2.2 $d_{L^{l+1}(G)}(y) \geq 4$. Then let $yz_3 \in V(L^{l+1}(G)) - \{x, z_1, z_2\}$. As $L^{l+1}(G)$ is claw-free and $z_1z_2 \notin E(L^{l+1}(G))$, z_3 must be adjacent to one of z_1, z_2, say z_2. So yz_2 and yz_3 lie in a triangle of $L^{l+1}(G) - \{xy\}$. Similarly, every edge incident to y except yz_1 lies in a triangle of $L^{l+1}(G) - \{xy\}$.

If yz_1 also lies in a triangle of $L^{l+1}(G) - \{xy\}$, then $H = L^{l+1}(G) - \{x\}$. Then H is triangular, and by (5), H is connected. So $L^{l+1}(G) - \{xy\} - V(H) = \{x\}$ is edgeless, and every vertex of degree at least 3 of $L^{l+1}(G) - \{xy\}$ lies in a triangle of $L^{l+1}(G) - \{xy\}$.

Next we assume that yz_1 does not lie in any triangle of $L^{l+1}(G) - \{xy\}$. Let $H = L^{l+1}(G) - \{x\} - \{yz_1\}$. By (5), $(L^{l+1}(G) - \{xy\})/\{xz_2\}$ is 1-edge-connected and essentially 3-edge-connected, which implies that the only 1-edge-cuts (2 edge-cuts) are the sets of edges incident with a vertex of degree one (or two). As $d_{L^{l+1}(G)}(z_i) \geq 3$ for $i = 1, 2$, $H = (L^{l+1}(G) - \{xy\})/\{xz_2\} - \{xz_1, yz_1\}$ is 1-edge-connected. And $L^{l+1}(G) - \{xy\} - V(H) = \{x\}$ is edgeless, which implies that every vertex of degree at least 3 of $L^{l+1}(G) - \{xy\}$ lies in a triangle of $L^{l+1}(G) - \{xy\}$.

Case 3 Both x and y have degree at least 4 in $L^{l+1}(G)$.

By (4), $L^{l+1}(G) - \{xy\}$ has an edge not lying in any triangle. Then this edge must be incident with x or y. Without loss of generality, we assume xx_1 does not lie in any triangle in $L^{l+1}(G) - \{xy\}$. Then x_1 is not adjacent to any other neighbor of x in $L^{l+1}(G) - \{xy\}$, so by (3), x_1 must be adjacent to y in $L^{l+1}(G)$.

As $d_{L^{l+1}(G)-\{xy\}}(x) \geq 3$ and $L^{l+1}(G)$ is claw-free, $N_{L^{l+1}(G)}(x) - \{x_1, y\}$ induces a complete subgraph of order at least 2 in $L^{l+1}(G) - \{xy\}$. So every edge incident with x except xx_1 lies in a triangle in $L^{l+1}(G) - \{xy\}$.

Case 3.1 yx_1 does not lie in any triangle in $L^{l+1}(G) - \{xy\}$.

Then x_1 is not adjacent to any other neighbor of y in $L^{l+1}(G) - \{xy\}$. Since $L^{l+1}(G)$ is claw-free, $N_{L^{l+1}(G)}(y) - \{x, x_1\}$ induces a complete subgraph of order at least 2 in $L^{l+1}(G) - \{xy\}$. So every edge incident with y except yx_1 lies in a triangle in $L^{l+1}(G) - \{xy\}$.

If $d_{L^{l+1}(G)}(x_1) \geq 3$, then $H = L^{l+1}(G) - \{xy, xx_1, yy_1\}$. By (5), H is a connected triangular subgraph of $L^{l+1}(G) - \{xy\}$ and $L^{l+1}(G) - V(H) = \emptyset$ is edgeless, which implies that every vertex of $L^{l+1}(G) - \{xy\}$ lies in a triangle. If $d_{L^{l+1}(G)}(x_1) = 2$, then $H = L^{l+1}(G) - \{xy\} - \{x_1\}$. By (5), H is a connected triangular subgraph of $L^{l+1}(G) - \{xy\}$ and $L^{l+1}(G) - V(H) = \{x_1\}$ is edgeless, which implies every vertex of degree at least 3 of $L^{l+1}(G) - \{xy\}$ lies in a triangle.

Case 3.2 yx_1 lies in a triangle in $L^{l+1}(G) - \{xy\}$.

Then x_1 must be adjacent to another neighbor of y, say y_1, in $L^{l+1}(G) - \{xy\}$. As $d_{L^{l+1}(G)-\{e\}}(y) \geq 3$, $N_{L^{l+1}(G)}(y) - \{x_1, y_1, x\} \neq \emptyset$.

Case 3.2.1 If there exists $y_2 \in N_{L^{l+1}(G)}(y) - \{x_1, y_1, x\}$ such that yy_2 does not lie in any triangle in $L^{l+1}(G) - \{xy\}$, then y_2 is not adjacent to any other neighbor of y in $L^{l+1}(G) - \{xy\}$. In particular, y_2 is not adjacent to x_1. By (3), y_2 is adjacent to x. Let $x_2 \in N_{L^{l+1}(G)}(x) - \{x_1, y\}$. As x_1 is not adjacent to any vertex in $N_{L^{l+1}(G)}(x) - \{x_1, y\}$, $\{x, x_1, x_2, y_2\}$ induces a claw with center x in $L^{l+1}(G)$, contradicting the fact that a line graph is claw-free.

Case 3.2.2 Every edge incident with y in $L^{l+1}(G) - \{xy\}$ lies in a triangle. Let $H = L^{l+1}(G) - \{xy, xx_1\}$. By (5), H is a connected triangular subgraph of $L^{l+1}(G) - \{xy\}$ and $L^{l+1}(G) - V(H) = \emptyset$ is edgeless, which implies that every vertex of $L^{l+1}(G) - \{xy\}$ lies in a triangle.

(ii) Let $e_v \in E(L^{l+1}(G))$ be the edge corresponding to $v \in V(L^{l+2}(G))$. By (i), $L^{l+1}(G) - \{e_v\}$ is strongly almost triangular, and by Corollary 3.7 and Proposition 2.1, $L^{l+2}(G) - \{v\} = L(L^{l+1}(G) - \{e_v\})$ is fully cycle extendable. □

Proof of Theorem 1.3 For convenience, let $l(G) = l$.

If $s = 0$, by Proposition 2.2(iii), $L^l(G)$ is triangular. So $L^{l+s+1}(G) = L^{l+1}(G)$ is fully cycle extendable by Corollary 3.4.

If $s = 1$, by Lemma 4.4(ii), $L^{l+s+1}(G) = L^{l+2}(G)$ is 1-fully cycle extendable.

If $s \geq 2$, then $s = 2^k + a$ with $0 \leq a \leq 2^k - 1$ where k is a natural number. Then $l + \lfloor \log_2 s \rfloor + 3 = l + \lfloor \log_2(2^k + a) \rfloor + 3 = l + k + 3$.

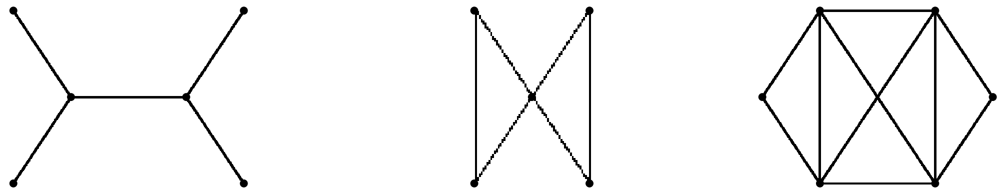
Let $S' \subseteq V(L^{l+k+3}(G))$ and $S \subseteq E(L^{l+k+2}(G))$, where S is the set of edges corresponding to vertices in S'. By Theorem 2.3(ii), $\kappa(L^{l+k+1}(G)) \geq 2^k + 2$ for

$k \geq 1$. Since $|S| \leq s = 2^k + a \leq 2^k + (2^k - 1) = 2^{k+1} - 1 < 2^{k+1}$, $L^{l+k+1}(G)$ satisfies the conditions of Theorem 4.3 with $t = 2^k$. So $L^{l+k+2}(G) - S$ is strongly almost triangular. By Corollary 3.7, $L^{l+k+3}(G) - S'$ is fully cycle extendable.

Now we show the bound is sharp.

(i) $s = 0$. Let G be the graph shown in Figure 2(a). Then $l(G) = 1$. If we reduce the bound to $l(G) + s = 1$, then $L(G)$ shown in Figure 2(b) is an hourglass and obviously not cycle extendable. So the bound cannot be reduced.

(ii) $s = 1$. Let G be the same graph shown in Figure 2(a). Then $l(G) = 1$. If we reduce the bound to $l(G) + s = 2$, then $L^2(G)$ shown in Figure 2(c) minus a vertex of degree 4 is not cycle extendable since it contains a vertex of degree one. So the bound cannot be reduced.

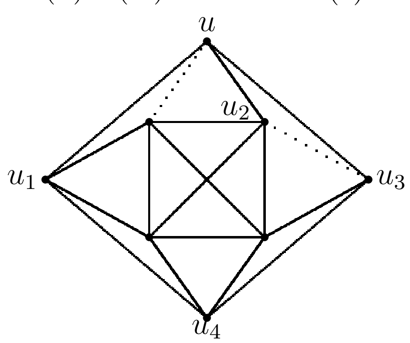


2(a) G 2(b) $L(G)$ 2(c) $L^2(G)$

Figure 2

(iii) $s \geq 2$. Let $s = 2^k + a$ and $0 \leq a \leq 2^k - 1$ where k is a natural number. We give an example to show $\mathrm{fce}_s(G)$ cannot be reduced to $l(G) + \lfloor \log_2 s \rfloor + 2 = l(G) + k + 2$.

3(a) G 3(b) $L(G)$ 3(c) $L^2(G)$ 3(d) $L^3(G)$



3(e) $L^4(G)$

Figure 3

By Theorem 2.3(ii), for $k \geq 1$, $\kappa(L^{l+k+1}(G)) \geq 2^k + 2$. First we show the graph G shown in Figure 3(a) satisfies $\delta(L^{l+k+1}(G)) = \kappa(L^{l+k+1}(G)) = 2^k + 2$ for any natural number k. Note $l(G) = l = 2$ and it suffices to show $\delta(L^{k+3}(G)) =$

$\kappa(L^{k+3}(G)) = 2^k + 2$. If $k = 1$, then the graph shown in Figure 3(e) satisfies $\delta(L^4(G)) = \kappa(L^4(G)) = 2^1 + 2 = 4$. Note that the 4-cycle $uu_3u_4u_1$ with every vertex of degree 4 in $L^4(G)$ generates a 4-cycle with each vertex of degree 6 in $L^5(G)$ by the definition of line graph. By induction, we have $\delta(L^{k+3}(G)) = \kappa(L^{k+3}(G)) = 2^k + 2$ for any natural number k.

Let $u \in V(L^{l+k+1}(G))$ of degree equal to $2^k + 2$ in $L^{l+k+1}(G)$. Let $e = xy \in E(L^{l+k}(G))$ be the edge corresponding to u. Note that u lies in exactly two complete subgraphs generated by $E_{L^{l+k}(G)}(x)$ and $E_{L^{l+k}(G)}(y)$ and we denote them by L_1 and L_2 respectively.

If $k = 1$, then $s = 2^1 = 2$ or $s = 2^1 + 1 = 3$. For the graph G shown in Figure 3(a), we have $l(G) = l = 2$. Then $l + k + 1 = 2 + 1 + 1 = 4$. As shown in Figure 3(e), deleting the two edges with dotted lines results in an induced claw $\{u, u_1, u_2, u_3\}$ with center u in $L^4(G) - S$, and the corresponding 3-cycle in $L^5(G) - S' = L(L^4(G) - S)$ is not extendable. So the bound cannot be reduced to $l + k + 2$.

Assume that $k \geq 2$. Since $\delta(L^{l+k}(G)) = \kappa(L^{l+k}(G)) = 2^{k-1} + 2$ and $d_{L^{l+k+1}(G)}(u) = 2^k + 2$, L_1 and L_2 must be complete subgraphs of order equal to $2^{k-1} + 2$, which implies $d_{L_1}(u) = d_{L_2}(u) = 2^{k-1} + 1$ for $k \geq 2$. Let $u_1 \in E_{L_1}(u)$. Since $L^{l+k}(G)$ is a simple graph, every edge in $E_{L^{l+k}(G)}(x) - \{xy\}$ can only be incident with at most one edge in $E_{L^{l+k}(G)}(y) - \{xy\}$. That means each vertex in $N_{L_1}(u)$ can only be adjacent to at most one vertex in $N_{L_2}(u)$. Without loss of generality, we assume that u_1 is adjacent to $w_1 \in N_{L_2}(u)$. Let $u_2 \in N_{L_2}(u) - \{w_1\}$. Let $S_1 = E_{L_1}(u) - \{uu_1\}$, $S_2 = E_{L_2}(u) - \{uu_2, uu_3\}$, $S_3 = \{u_2u_3\}$, and $S = S_1 \cup S_2 \cup S_3$. So $|S| = 2^{k-1} + (2^{k-1} - 1) + 1 = 2^k$ and $\{u, u_1, u_2, u_3\}$ induces a claw with center u in $L^{l+k+1}(G) - S$, and u does not lie in any triangle. So the corresponding 3-cycle in $L^{l+k+2}(G) - S'$ is not extendable. Thus we cannot reduce the bound to $l + k + 2$. □

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