

The watchman's walk problem on directed graphs

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Abstract

In a graph, a watchman's walk is a minimum closed dominating walk. Given a graph G and a single watchman, the length of a watchman's walk in G (the watchman number) is denoted by $w(G)$ and the typical goals of the watchman's walk problem are to determine $w(G)$ and find a watchman's walk in G . In this paper, we extend the watchman's walk problem to directed graphs. The directed graphs considered in this paper are families of tournaments and orientations of complete multipartite graphs. We give bounds on the watchman number and discuss its relationship to variants of the domination number.

1 Introduction

Given a graph G and a single watchman, the watchman's walk problem looks at the scenario of that watchman traversing the graph in such a way that the route is a minimum closed dominating walk. Any such walk in G is a *watchman's walk* of G . The length of a watchman's walk in G is denoted by $w(G)$; this is called the *watchman number* of G . The typical goals of the watchman's walk problem are to

determine $w(G)$ and find a closed dominating walk in G that achieves this number. The watchman's walk was introduced by Hartnell, Rall, and Whitehead in 1998 in [14].

Throughout this paper, we follow the notation of [24], with the exception of our use of $\deg(v)$ for the degree of a vertex v ; basic definitions and notation can be found there.

The watchman's walk problem is a variation of the domination problem. Domination in graphs was formally introduced in 1958 by Berge in [2]. The domination problem in graphs aims to find a minimum dominating set; a set of vertices in a graph such that every vertex of the graph is either in that set or a neighbour of a vertex in that set. Variations of this problem include finding a minimum total dominating set, or a minimum connected dominating set. Directed domination in tournaments was first considered by Erdős in [11]. In [21], Reid et al. consider the domination number and irredundance number of tournaments. They give upper bounds on the domination number of tournaments of small order. A survey of results on the domination number of graphs can be found in [15].

Research in the watchman problem has been pursued in several different directions, though until now, only for (undirected) graphs. In [6], the authors considered the structure of watchman's walks and then gave results in enumerating watchman's walks. Hartnell and Whitehead considered the Cartesian product in [13], particularly of trees, and these results were expanded in [8]. The watchman's walk has also been studied in the block intersection graphs of Steiner Triple Systems [7]. A heavily related problem has been to use multiple watchmen to limit the amount of time each vertex is unguarded in [1, 4, 10].

If a dominating set induces a weakly connected subdigraph, we call it a *weakly connected dominating set*. The size of a minimum weakly connected dominating set is denoted by $\gamma_{wc}(D)$. If a dominating set induces a strongly connected subdigraph, we call it a *strongly connected dominating set*. The minimum size of such a set is denoted by $\gamma_{sc}(D)$. We call a minimum weakly connected dominating set a γ_{wc} -set, and a minimum strongly connected dominating set a γ_{sc} -set.

Finding a connected dominating set in an undirected graph, and a weakly or strongly connected dominating set in a directed graph is a problem similar to finding a watchman's walk. In a directed graph, we say that the watchman can only move to and see the vertices that are adjacent to him relative to outgoing arcs. That is, a watchman's walk is oriented and domination occurs in the direction of the arcs. The vertices of a minimum closed dominating walk in an undirected graph are always a connected dominating set. In a directed graph, the vertices of a watchman's walk will be a strongly connected dominating set. However, in many directed graphs, these parameters and sets of vertices are not equal. In fact, a digraph may not contain any watchman's walks. If we consider the directed path in Figure 1, the set of vertices $\{v_1, v_2, v_3, v_4, v_5\}$ is a minimum weakly connected dominating set, and $\gamma_{wc}(D) = 5$. However, there is no strongly connected dominating set or watchman's walk.

In directed graphs that have strongly connected dominating sets and watchman's walks, $\gamma_{sc}(D)$ and $w(D)$ are not necessarily equal. Consider the directed graph D in Figure 2. In any γ_{sc} -set, each vertex in the set is dominated by another vertex in

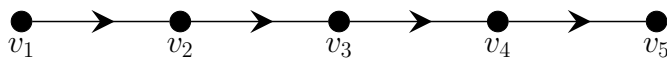


Figure 1: A digraph with no watchman's walk.

the set. Hence, each vertex v in a γ_{sc} -set dominates at most $|N^+(v)|$ other vertices. It follows that a strongly connected subset of D containing six vertices dominates at most fifteen vertices where v_7 is in any strongly connected dominating set and u_7 is not, due to their number of out-neighbours. It can be checked that no dominating set of size six induces a strongly connected subset. Thus, $\gamma_{sc}(D) = 7$, and the set $S = \{v_1, v_2, \dots, v_7\}$ is a γ_{sc} -set. The shortest closed walk containing all of these vertices has length 9. The walk of length eight illustrated by dashed arcs is a closed dominating walk in the graph, and hence $w(D) \leq 8$. Moreover, it can be shown computationally that $w(D) = 8$. Thus, no watchman's walk in this graph contains the γ_{sc} -set S as a subset of its vertices and, in fact, the watchman's walk illustrated by the dashed arcs is completely disjoint from the γ_{sc} -set S . Additionally, $\{v_7, u_1, u_3, u_5\}$ is a γ -set, so $\gamma(G) = 4$.

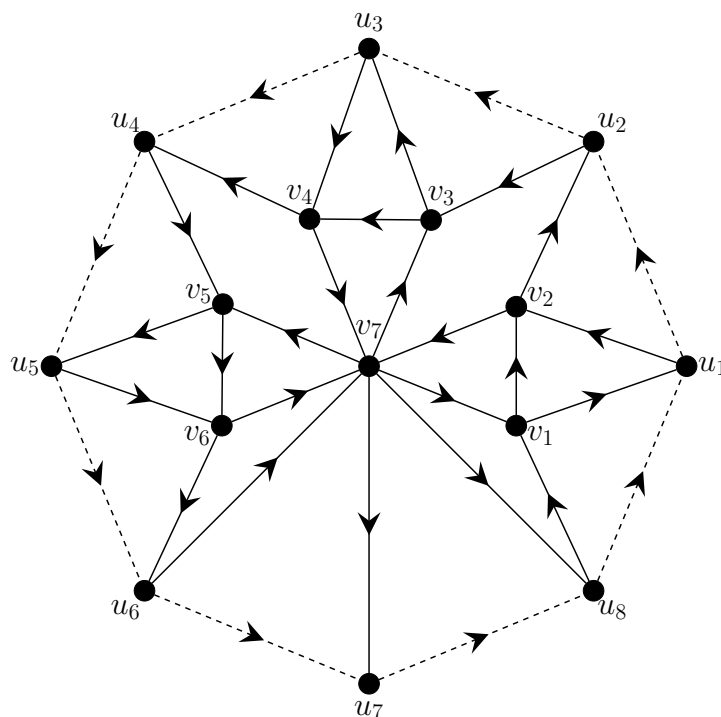


Figure 2: A digraph with disjoint γ_{sc} -set and watchman's walk

A directed graph can be formed from an undirected simple graph G by assigning a direction to each edge in G . The resulting graph is called an *orientation* of G . A *tournament* is an orientation of a complete graph. A *strongly connected component* of a digraph D is a subdigraph of D that is also strongly connected. The *condensation*

of a digraph D , denoted by D^* , is the digraph found by contracting each maximal strongly connected component to a single vertex. In this digraph, there is an arc from vertex W to vertex U if all arcs between strongly connected components W and U in T are directed from a vertex in W to a vertex in U .

The results in this paper originally appeared as part of [18].

2 Watching directed graphs

Unlike an undirected graph, a digraph may not have a watchman's walk. Thus, we begin with results on the existence of watchman's walks in digraphs, before moving on to families where we can guarantee a watchman's walk exists.

2.1 Watchman's walks in general digraphs

In the following theorem, we generalize the result in Corollary 3.6 for tournaments to general digraphs.

Theorem 2.1. *If D is a digraph with a strongly connected subdigraph D' such that $V(D')$ is a dominating set in D , then D has a watchman's walk.*

Proof. Let D' be a strongly connected subdigraph of D with its vertices forming a dominating set in D . It follows that there exists a directed path between any pair of vertices in D' . This implies there exists a closed walk W that passes through every vertex in D' , possibly repeating some vertices. Clearly W is a dominating walk in D' and since the vertices of D' dominate D , W is also a dominating walk in D . W is not necessarily of minimal length but is a closed dominating walk, and therefore D has a minimum closed dominating walk. \square

We now know that any strongly connected digraph has a watchman's walk, regardless of its structure. For the remainder of this section, we consider digraphs that are not necessarily strongly connected. Recall that a source vertex in a digraph is a vertex with no in-neighbours. Any nontrivial digraph with a source vertex is not strongly connected, as there is no nontrivial walk that ends at a source vertex. It is easy to see that a digraph that has exactly one source vertex has a watchman's walk if and only if that source vertex dominates all the other vertices. Also, if a digraph has more than one source vertex it cannot have a watchman's walk. Thus, we obtain the following theorem.

Theorem 2.2. *A digraph D of order n with $k \geq 1$ source vertices has a watchman's walk if and only if D has exactly one source vertex v , where $\deg^+(v) = n - 1$.*

If we take an orientation of any path on at least four vertices, we obtain a digraph that is not strongly connected and, in fact, no such orientation has a watchman's walk. However, there does exist an orientation of a n -cycle which will have a watchman's walk; one in which all the arcs are oriented as a directed n -cycle. Thus there are only two such orientations.

2.2 Orientations of complete multipartite graphs

In this section, we will consider the existence of watchman's walks in orientations of complete bipartite and multipartite graphs. Unlike in tournaments, these do not always have a watchman's walk. Consider, for example, a complete multipartite graph with more than one source vertex. In the following observation, we consider the existence of watchman's walks in multipartite graphs with at least one source vertex.

Observation. *Let $D = (X_1 \cup \dots \cup X_k, A)$ be an orientation of a complete multipartite graph for $k \geq 2$, where X_1, \dots, X_k are the vertex sets of the partition. If $v \in X_i$ is a source vertex for any $1 \leq i \leq k$, then D has a watchman's walk if and only if $|X_i| = 1$.*

We move on to orientations that have no source vertices. We begin by considering orientations of complete bipartite graphs.

Theorem 2.3. *Let D be an orientation of a complete bipartite graph with partition (A, B) such that $\delta^-(D) \geq 1$. If there exists a $U \subseteq B$ such that $D[A \cup U]$ is strongly connected, then D has a watchman's walk.*

Proof. First note that since $\delta^-(D) \geq 1$, A dominates every vertex in B . This means that $D[A \cup U]$ is strongly connected and dominating. By Theorem 2.1, D has a watchman's walk. \square

Corollary 2.4. *Let D be an orientation of a complete bipartite graph with partition (A, B) such that $\delta^-(D) \geq 1$ and $|A| \leq |B|$. If there is a set of vertices $U \subseteq B$ such that each vertex in A is dominated by exactly one vertex in U , then D has a watchman's walk. Moreover, $w(D) \leq 2|A|$.*

Proof. Let D be an orientation of a $K_{m,n}$ such that each vertex has at least one in-neighbour. Let $V(D) = A \cup B$, such that $|A| = m$ and $|B| = n$. Let $A = \{v_1, v_2, \dots, v_m\}$. Suppose U is a set as defined in the statement of the corollary. Clearly $D[A \cup U]$ is strongly connected and dominating, so by Theorem 2.3, D has a watchman's walk. We will exhibit a closed dominating walk of length $2|A|$.

By definition, each vertex in U has a private out-neighbour in A . Thus, $|U| = m$; let $U = \{u_1, u_2, \dots, u_m\}$. Label the vertices of U such that vertex u_i dominates v_i . It follows that v_i dominates each vertex in $U \setminus \{u_i\}$. Thus, we can consider the closed walk $W = \{u_1, v_1, u_2, v_2, \dots, u_m, v_m, u_1\}$. The walk W uses every vertex in A exactly once, and hence, W is a closed dominating walk of length $2|A|$ in D . \square

Theorem 2.3 and Corollary 2.4 tell us that there exists orientations of complete bipartite graphs that have a watchman's walk. We generalize this result to orientations of complete multipartite graphs that do not contain any source vertices.

Theorem 2.5. *Let D be an orientation of a complete k -partite graph for $k > 1$. If $\delta^-(D) \geq 1$, then the condensation of D has a source vertex, and D has a watchman's walk.*

Proof. Let D be an orientation of a complete multipartite graph such that $\delta^-(D) \geq 1$. That is, each vertex in D has at least one in-neighbour. Let $\{T_1, T_2, \dots, T_k\}$ be the maximal strongly connected components of D . Consider the condensation, D^* . Let the vertices of D^* be $\{t_1, t_2, \dots, t_k\}$, where vertex t_i in D^* corresponds to the component T_i in D . Let t be any vertex in D^* , and let P be a maximal path ending at t . Consider the start vertex u of P . Any cycle in D^* would correspond to a larger strongly connected component in D^* , contradicting the maximality of each T_i . Hence, u has no in-neighbours on P . Since P was maximal, u also does not have any in-neighbours in the subdigraph $D^* \setminus P$. This means that, in D , any vertex in the component U corresponding to the vertex $u \in D^*$ has no in-neighbours from any vertex in another strongly connected component. As each vertex has at least one in-neighbour, U contains more than one vertex. Hence, U contains vertices from more than one set in the partition of V . As D is an orientation of a complete multipartite graph, it follows that there is at least one arc between a vertex in U and a vertex in each set in the partition of V , and hence an arc between U and every other maximal strong component. This guarantees that there is an arc between u in D^* and each other vertex in D^* . Since u is a source in D^* , the component U in D is dominating. As U is a strongly connected component, there is a closed walk containing all of the vertices in U . As U is a dominating set, this walk is a closed dominating walk in D . Therefore, there is a watchman's walk for D . \square

3 Watchman's walks in tournaments

If a tournament is strongly connected, we call it a strong tournament. In this section, we refer to classical results, primarily those found in [20] and [22], to prove the existence of a watchman's walk in strong tournaments, and later, we generalize our result for tournaments that are not necessarily strong. We also provide bounds on the length of a watchman's walk in tournaments. We begin by stating a theorem that will be useful in the proof of subsequent results.

Theorem 3.1. [3] *A tournament is Hamiltonian if and only if it is strongly connected.*

Theorem 3.2. *If T is a strong tournament, then T has a watchman's walk.*

Proof. If T is a strong tournament then, by Theorem 3.1, T has a Hamilton cycle H . Since every vertex is on this cycle, this is a closed dominating walk. This means that the set of closed dominating walks of T is non-empty. Thus, a minimum closed dominating walk of T exists. \square

Theorem 3.2 implies that any strong tournament on n vertices has a closed dominating walk of length n . The following theorem draws a connection between the watchman's walk of a spanning subdigraph and the tournament itself.

Theorem 3.3. *Let D be a digraph and D' be a spanning subdigraph of D . If D' has a watchman's walk, then $w(D) \leq w(D')$.*

If we have a spanning subdigraph T' in T , we cannot assume that T' has a watchman's walk, since T' is a digraph. However, if T' has a watchman's walk, Theorem 3.3 shows that the length of a watchman's walk in T is bounded above by $w(T')$. Moreover, if we have a (not necessarily spanning) subdigraph T' such that the vertices of T' are a dominating set in T , we know that $w(T)$ is at most $w(T')$. This fact is a strengthening of Theorem 3.3, and is proved in the following theorems.

Note that a tournament $T = (V, A)$ is transitive if for any vertices a, b , and c , $(a, b) \in A$, and $(b, c) \in A$ implies that $(a, c) \in A$. From [12], we obtain the following important theorem.

Theorem 3.4. [12] *The condensation of any tournament is a transitive tournament.*

We show below that Theorem 3.4 limits the possible vertices that may occur in a watchman's walk. Given a tournament T , where T' is the maximal strong component that corresponds to the dominating vertex in the condensation of T , we call T' the *dominating strong component*. Since any transitive tournament has a dominating vertex, and the condensation of a tournament is a transitive tournament, the dominating strong component in a tournament always exists.

Theorem 3.5. *Let T be a tournament. If T' is the dominating strong component in T , then $w(T) = w(T')$.*

Proof. Let T be a tournament. If T is a strong tournament, then T^* has exactly one component, and this component is T itself. In this case, the result is clear.

Now suppose that T is not strong. This means, by Theorem 3.4, T^* is transitive with more than one component. Let t' be the dominating vertex of T^* , which corresponds to the dominating strong component of T . Since T^* is transitive, t' is a dominating vertex. This means in T , there is no arc from $V(T) \setminus V(T')$ to $V(T')$. Thus, a watchman's walk in T' is a closed dominating walk in T . It is also minimum as any shorter walk that included a vertex from $V(T) \setminus V(T')$ would not be closed, and a shorter walk completed inside T' would not be dominating. Thus $w(T) = w(T')$. \square

From the proof of Theorem 3.4, it is clear that the watchman's walk in a tournament is always contained in the dominating strong component. This result is stated below in Corollary 3.6. It also leads us to an important result in Theorem 3.7.

Corollary 3.6. *If T is a tournament, and T' is the dominating component in the T^* , then all watchman's walks for T are contained in T' .*

It is not obvious whether a given tournament T contains a watchman's walk when T is not strong, as many digraphs that are not strongly connected do not have a watchman's walk. In fact, there are many infinite families of digraphs that have no watchman's walk, including the family of orientations of paths on $n \geq 4$ vertices.

Theorem 3.7. *If T is a tournament, then T has a watchman's walk.*

A *semicomplete digraph* is a digraph in which there is at least one arc between any pair of vertices. The previous theorem can be extended to semicomplete digraphs with only minor changes to the proof given above.

Corollary 3.8. *If D is a semicomplete digraph, then D has a watchman's walk.*

4 Domination number

4.1 Watchman's walk in relation to domination number

In general, for both undirected and directed graphs, there can be a large difference between the domination number and the length of a watchman's walk in a graph. In this section, we show that this is not the case for tournaments. We begin by proving a relationship between the domination number and size of a watchman's walk in a tournament. We later give further results relating to domination number and its variants.

Theorem 4.1. [19] *Every tournament contains a Hamilton path.*

Theorem 4.2. *Let T be a tournament of order $n \geq 3$. If $\gamma(T) > 1$, then $w(T) = \gamma(T)$ or $w(T) = \gamma(T) + 1$.*

Proof. Let T be a tournament of order n . Let $\gamma(T) = k$ for some $1 < k < n$. From Theorem 4.1, any tournament contains a Hamilton path. Suppose that there is some minimum dominating set $D = \{v_1, v_2, v_3, \dots, v_k\}$ of T , with a Hamilton path $H = v_1, v_2, v_3, \dots, v_k$ in the subtournament induced by D such that $v_1 \in N_T^+(v_k)$. In this case (v_k, v_1) is an arc, so $W = v_1, v_2, v_3, \dots, v_k, v_1$ is a closed walk in T . Since D was a minimum dominating set in T , and W is a closed walk that uses exactly the vertices of D , W is a watchman's walk for T . This walk has length k . Hence, $w(T) = k = \gamma(T)$.

Now suppose that every minimum dominating set in T induces a subtournament that does not have a Hamilton cycle. It follows that we cannot construct a closed dominating walk in T that is of length $\gamma(T) = k$, as this walk uses only the vertices of a minimum dominating set, and there is no closed walk using exactly the vertices of a minimum dominating set. This means that $w(T) > k = \gamma(T)$. Let $D = \{v_1, v_2, v_3, \dots, v_k\}$ be a minimum dominating set in T , and let $P = v_1, v_2, v_3, \dots, v_k$ be a Hamilton path in $H = T[D]$. Consider $N_T^+(v_k) \cap N_T^-(v_1)$. If this is empty, every vertex outside of D that is dominated by v_k is also dominated by v_1 , and v_k is dominated by v_1 . It follows that $D \setminus \{v_k\}$ is a dominating set, contradicting the minimality of D . Hence, there is some vertex u in T , such that $u \notin D$, and $u \in N_T^+(v_k) \cap N_T^-(v_1)$. Thus, $W = v_1, v_2, v_3, \dots, v_k, u, v_1$ is a closed dominating walk of length $k+1 = \gamma(T)+1$. Since W has length $k+1$ it is a minimum closed dominating walk and so, $w(T) = \gamma(T) + 1$. Therefore, for any tournament T , $w(T) = \gamma(T)$ or $\gamma(T) + 1$. \square

In the tournament in Figure 3a, $\gamma(T) = 3$, and $\{v_1, v_2, v_6\}$ is a minimum dominating set. Since $W = v_1, v_2, v_6, v_1$ is a closed walk through this set, $w(T) = 3 = \gamma(T)$. In the tournament in Figure 3b, $\gamma(T) = 2$, $\{v_1, v_5\}$ is a minimum dominating set, and $W = v_1, v_2, v_5, v_1$ is a watchman's walk, so $w(T) = 3 = \gamma(T) + 1$.

The result in Theorem 4.2 can also be extended to semicomplete digraphs.

Corollary 4.3. *Let D be a semicomplete digraph of order $n \geq 2$. If $\gamma(D)$ is the size of a minimum dominating set and $\gamma(D) > 1$, then $w(D) = \gamma(D)$ or $\gamma(D) + 1$.*

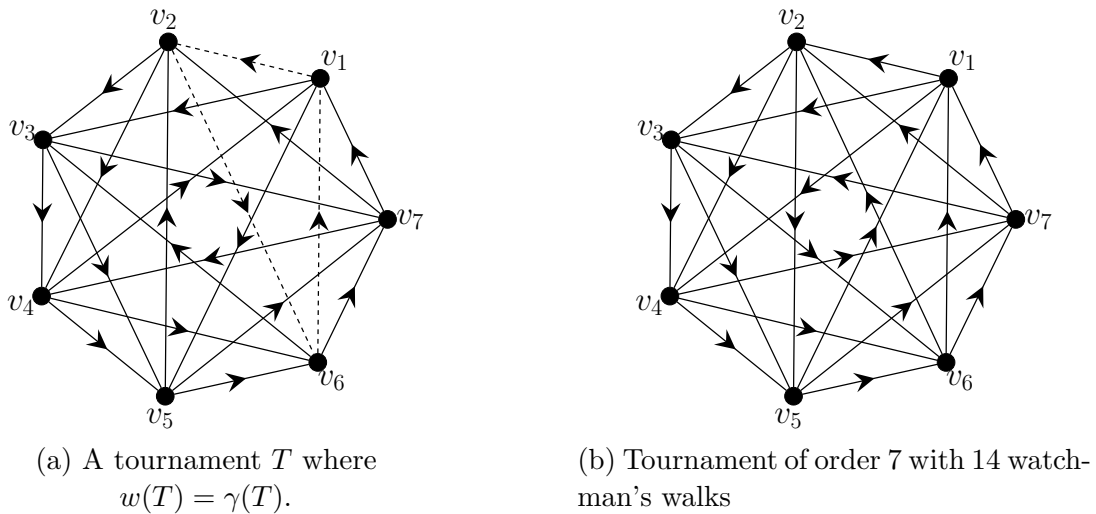


Figure 3: Tournaments of order 7.

From Theorem 4.2, we also obtain the following two corollaries.

Corollary 4.4. *If T is a tournament, then no watchman's walk in T repeats a vertex.*

Proof. Let T be a tournament. Suppose there is some minimum dominating set D such that the subtournament induced by D , $T[D]$, contains a Hamilton cycle. In this case, $w(T) = |D| = \gamma(T)$. This means that any watchman's walk contains the vertices of a minimum dominating set, since no other set of size $\gamma(T)$ will be dominating, and any larger dominating set will not give us a closed walk of length $\gamma(T)$. Since any minimum dominating set contains only distinct vertices, each watchman's walk for T does not repeat any vertices.

Now suppose that there is no minimum dominating set such that the subtournament it induces contains a Hamilton cycle. In this case, $w(T) = \gamma(T) + 1$, as we have no closed dominating walk containing exactly the vertices of a minimum dominating set. Let W be any watchman's walk of T , and let D be the set of vertices in the walk. Since D is a dominating set of T , we have that D is either a minimal dominating set of size $\gamma(T) + 1$, or a minimum dominating set with an additional vertex. Note that it cannot be a dominating set of size $\gamma(T)$ with one of those vertices used twice as that would mean the dominating set was strongly connected and hence would have a Hamilton cycle. In the first case, each vertex is distinct in the walk as the length of the walk equals the number of vertices. In the second case, each of the vertices from the dominating set occurs once and the additional vertex is from outside the dominating set, thus the length of the walk equals the number of vertices and the result holds. □

Corollary 4.5. *If T is a tournament, then $\gamma(T) = w(T)$ exactly when there exists some minimum dominating set D in T such that $T[D]$ is itself a strong tournament.*

Proof. Let T be a tournament and suppose that for some minimum dominating set D , $T[D]$ is strongly connected. It follows, by Theorem 3.1, that $T[D]$ has a Hamilton

cycle H and this is also a closed dominating walk in T . Since $w(T) \geq \gamma(T)$, by Theorem 4.2, this value of $w(T)$ is minimum.

Now suppose that for every minimum dominating set D in T , $T[D]$ is not a strong subtournament. This means there is no Hamilton cycle in $T[D]$. Thus, if $\gamma(T) = k$ then there is no k -cycle in T containing exactly the vertices of a minimum dominating set. Hence, any cycle of length k does not dominate T . Thus, any closed walk of length k is not dominating, and $w(T) \neq k$. By Theorem 4.2, $w(T) = k$ or $k + 1$, so in this case, $w(T) = k + 1 = \gamma(T) + 1$. Therefore, $w(T) = \gamma(T)$ exactly when $T[D]$ is strong for some minimum dominating set D . \square

The corollaries given of Theorem 4.2 can also be generalized to semicomplete digraphs. In particular, we have the following corollary, the proof of which can be inferred from Theorem 4.6.

Theorem 4.6. [3] *Every strongly connected semicomplete digraph has a Hamilton cycle.*

Corollary 4.7. *If T is a semicomplete digraph, then no watchman's walk repeats a vertex.*

The following result was stated by Moon in [17] and is attributed to Erdős. A proof of this theorem can be found in [21].

Theorem 4.8. [17] *If T is a tournament with $n \geq 2$ vertices, then $\gamma(T) \leq \lceil \log_2(n) \rceil$.*

The question arises, do all vertices in a tournament need to be included in a watchman's walk? Theorem 4.9 below asserts that we can always find a closed dominating walk with no more than $n - 2$ vertices for tournaments of order at least 5.

Theorem 4.9. *If T is a tournament of order $n \geq 5$, then $w(T) \leq n - 2$.*

The cycle domination number of a digraph D , $\gamma_{\text{cyc}}(D)$, is the length of the shortest directed cycle C in D such that the vertices in C form a dominating set in D . By Corollary 4.4, a watchman's walk in a tournament never repeats a vertex and hence is a dominating cycle of minimum length. We obtain the following theorem.

Theorem 4.10. *If T is a tournament, then $\gamma_{\text{cyc}}(T) = w(T)$.*

The total domination number of a digraph D , denoted $\gamma_t(D)$, is the size of the smallest set S such that for every vertex v in D , there is a vertex $u \neq v$ in S such that (u, v) is an arc.

Theorem 4.11. *If T is a tournament, then $\gamma(T) \leq \gamma_t(T) \leq w(T)$.*

Proof. Any minimum total dominating set is also a dominating set, so $\gamma(T) \leq \gamma_t(T)$. If W is a watchman's walk, each vertex in W is dominated by at least one other vertex in the walk, and by Corollary 4.4, the vertices of W are unique. The vertices of W form a dominating set in T , so the vertices of W are also a total dominating set in T . Therefore, $\gamma(T) \leq \gamma_t(T) \leq w(T)$. \square

Note that the total domination number is not necessarily equal to either the domination number or the watchman number. In Figure 3b, $w(T) = \gamma_t(T) = 3 = \gamma(T) + 1$. The smallest example the authors could find with $w(T) > \gamma_t(T) = \gamma(T)$ has $w(T) = 5$, $\gamma_t(T) = \gamma(T) = 4$, and has 272 vertices. We give a construction for such a tournament at the end of the next section.

5 Small and fixed watchman number

For some families of tournaments, particularly those that are highly structured or of small order, the watchman number is fixed or bounded. The following theorem, from [21], provides bounds on the domination number for small tournaments.

Theorem 5.1. [21] *If T is a tournament on n vertices, then $\gamma(T) \leq 2$ if $n < 7$, and $\gamma(T) \leq 3$ if $n < 19$.*

By using these upper bounds on the domination number of small tournaments, we obtain the following upper bounds on the watchman's walk number.

Corollary 5.2. *If T is a tournament on n vertices, then $w(T) \leq 3$ if $n < 7$, and $w(T) \leq 4$ if $n < 19$.*

We know that any watchman's walk for a tournament will be completely contained in the dominating strong component. From this, it follows that we can also apply these upper bounds to some larger tournaments. Using Theorem 3.5, we obtain the following corollary.

Corollary 5.3. *If T is a tournament such that the dominating strong component contains fewer than 7 vertices, then $w(T) \leq 3$.*

Similarly, from the second upper bound, we obtain the following corollary.

Corollary 5.4. *If T is a tournament such that the dominating strong component contains fewer than 19 vertices, then $w(T) \leq 4$.*

From [5], we know that there exists a tournament with domination number k for any arbitrarily large value of k . This paper gives a construction, which when given the inheritance graph of a tournament with domination number larger than k , and any tournament X such that $X - M$ dominates M for a specified subset M , produces a tournament with domination number k and $|M|$ minimum dominating sets. For many uses and constructions, it is more helpful to have a strong tournament over one that is not strong. While [5] gives us a tournament T with domination number k , this theorem does not guarantee to give a strong tournament. We give the following construction to build a strong tournament from any tournament that is not already strong, while preserving the domination number. This can be used to obtain a strong tournament with domination number k for any integer k .

Lemma 5.5. *Let T be any tournament of order n that is not strongly connected. If $\gamma(T) = k \geq 3$, then we can construct a strongly connected tournament, T_s , of order $n + 1$ such that $\gamma(T_s) = k$ and T is a subtournament of T_s .*

Proof. Let $T = (V, A)$ be a tournament that is not strongly connected. Since T is a tournament, we know from Theorem 4.1 that T has at least one Hamilton path. Let $H = u, v_1, v_2, \dots, v_{n-1}$ be a Hamilton path in T . Consider the tournament T_s , where $V(T_s) = V(T) \cup \{w\}$, and $A(T_s) = A(T) \cup \{(x, w) \mid x \in V(T) \setminus \{u\}\} \cup \{(w, u)\}$ for some $w \notin V(T)$. It follows that $H_s = u, v_1, v_2, \dots, v_{n-1}, w, u$ is a closed walk in T_s . Since H uses each vertex in T exactly once, H_s uses each vertex in T_s exactly once, except u , which is both the start-vertex and end-vertex of the walk. Thus, H_s is a Hamilton cycle in T_s . Hence, by Theorem 3.1, T_s is strongly connected.

Now suppose D is a minimum dominating set in T . Since $\gamma(T) = k \geq 3$, we know $|D| = k \geq 3$. As w is dominated in T_s by all but one vertex from T , at least two vertices in D dominate w in T_s . Also, $A(T) \subset A(T_s)$, so D is also a dominating set in T_s . Thus, $\gamma(T_s) \leq k$.

Consider a minimum dominating set D_s in T_s . We know $|D_s| \leq k$. Suppose to the contrary that $|D_s| < k$. Suppose that $D_s \subset V(T)$. Since $A(T) \subset A(T_s)$, D_s would be a dominating set in T . However, $|D_s| \leq k - 1$ and D_s does not dominate T . Thus, it also does not dominate T_s . This means that $D_s \not\subset V(T)$ and $w \in D_s$. By the construction of T_s , vertex w only dominates w and u in T_s . Thus, $D_s \setminus \{w\}$ dominates $T \setminus \{u\}$. It follows that $D_s \setminus \{w\} \cup \{u\}$ dominates T . However, $|D_s \setminus \{w\} \cup \{u\}| \leq k - 1$. This contradicts the fact that $\gamma(T) = k$. Thus, $\gamma(T_s) \geq k$. Therefore $\gamma(T_s) = k$. \square

Theorem 5.6. *For every integer $k \geq 2$, there exists a strong tournament T such that $\gamma(T) = k$.*

Proof. Let $k = 2$. The directed 3-cycle is a strong tournament of order 3 with $\gamma(T) = 2$. Now let k be any integer such that $k \geq 3$. From [5], there exists a tournament T such that $\gamma(T) = k$. If T is strongly connected, we have the desired tournament. However, if T is not strong, by Theorem 5.5, there exists a tournament T_s from T such that T is strong and $\gamma(T_s) = k$ as required. \square

We now use the above, along with the construction in [5], to obtain a tournament with any watchman number greater than 3.

Theorem 5.7. *If $k \geq 3$, then there exists a tournament T such that $w(T) = k$.*

Proof. Let k be an integer such that $k \geq 3$. We will construct a tournament T with $w(T) = k$ using the construction given in Theorem 2 in [5] for a tournament with $\gamma(T) = k$.

Let T_p be the tournament on p vertices, $\{0, 1, \dots, p - 1\}$, such that (i, j) is an arc if and only if $i - j$ is a quadratic residue of p . Let p be a prime congruent to 3 modulo 4 such that $\gamma(T_p) > k$.

Now, let $J_k(T_p)$ be the tournament constructed as follows:

- Relabel the vertices of T_p as $\{(0, 0), (1, 0), \dots, (p - 1, 0)\}$. Take k copies of T_p , labelled T_1, T_2, \dots, T_k . Let the vertices of T_i be $\{(0, i), (1, i), \dots, (p - 1, i)\}$.
- $((a, i), (b, j))$ is an arc in $J_k(T_p)$ if and only if $((a, 0), (b, 0))$ is an arc in T_p or $a = b$ and $i < j$.

Let X be a tournament on $k+m-1$ vertices, where $m \geq 1$ is any natural number. Let $V(X) = \{x_{1,1}, x_{1,2}, \dots, x_{1,m}, x_2, \dots, x_k\}$. For each j such that $2 \leq j \leq k$, we will also let x_j be denoted as $x_{j,r}$ for every r such that $1 \leq r \leq m$. For each r such that $1 \leq r \leq m$, $X_r = \{x_{1,r}, x_2, \dots, x_k\} = \{x_{1,r}, x_{2,r}, \dots, x_{k,r}\}$. Let $M = \{x_{1,1}, x_{1,2}, \dots, x_{1,m}\}$. We want to choose the arcs of X such that $X \setminus M$ dominates each vertex in M . To do this, we can let $(x_3, x_{1,r})$ be an arc for all r such that $1 \leq r \leq m$. Now, since $x_3 \notin M$, $X \setminus M$ dominates each vertex in M . We can now define these arcs to give a strong tournament for the subdigraph induced by X_r for each r . To do this, we can let (x_i, x_{i+1}) be an arc in X for each i such that $2 \leq i \leq k-1$, and $(x_k, x_{1,r})$ be an arc. Now, $X[X_r]$ has a Hamilton cycle. Hence, by Theorem 3.1, it is strong. From the construction given in Theorem 2 in [5], we know that the remaining arcs can be chosen randomly.

We can now construct T . First, let $V(T) = V(J_k(T_p)) \cup V(X)$. Define the arc set of T such that any arc in $J_k(T_p)$ or in X is also an arc in T . For the remaining arcs, $(x_{i,r}, y)$ is an arc for $x_{i,r} \in X$ and $y \in J_k(T_p)$ if $y \in T_i$, otherwise $(y, x_{i,r})$ is an arc. Each X_r now dominates the vertices of $T_k(T_p)$ since $x_{i,r}$ dominates the vertices of T_i for $q \leq i \leq k$, and X_r dominates each vertex in X . Hence, X_r is a dominating set in T . Now we must show that X_r is a minimum dominating set.

Suppose D is a dominating set that does not contain any X_r . In this case, either $D \cap M = \emptyset$ or $D \cap M \neq \emptyset$. If $D \cap M = \emptyset$, then no $x_{1,i}$ is in D , so no element of T_1 is dominated by $D \cap X$. If we have that $D \cap M \neq \emptyset$, then there is some j where $2 \leq j \leq k$ such that x_j is not in D . Otherwise, we would have that $X_r \subseteq D$. Thus, T_j is not dominated by $D \cap X$. It follows that there is some t such that $1 \leq t \leq k$ where no element of T_t is dominated by $D \cap X$. Hence, T_t is dominated by elements of $J_k(T_p)$. However, $|J_k(T_p) \cap D| \geq \gamma(T) > k$, which implies that $|D| > k$. Hence, any D is a minimum dominating set if and only if it is X_r for some r such that $1 \leq r \leq m$. Since each X_r induces a strong tournament, and $|X_r| = k$, by Theorem 4.5, $w(T) = k = \gamma(T)$. □

In the construction given in the proof of Theorem 5.7, the choice of m determines the number of distinct watchman’s walks in T . From this, we obtain the following corollary.

Corollary 5.8. *For any integers $k \geq 3$, and $m \geq 1$ there exists a tournament T such that $w(T) = k$ and T has exactly m watchman’s walks. Moreover, there exists a tournament T such that T has a unique watchman’s walk of length k .*

To construct the tournament with $w(T) = 5$, and $\gamma_t(T) = \gamma(T) = 4$, we use the construction given in [5]. Using $k = 4$ and $m = 1$, where in the construction the tournament X , and hence the unique γ set, is taken to be the tournament formed from adding a sink to a three-cycle and the ingredient tournament is the quadratic residue tournament on 67 vertices. This guarantees $w(T) = 5$ and $\gamma_t(T) = 4$.

5.1 Computational results

Even when considering only those of small order, the number of non-isomorphic tournaments on a given number of vertices can be very large. As a result, the length,

structure, and multiplicity of watchman's walks can vary greatly, even between tournaments on the same number of vertices. For further results on multiplicity of watchman's walks of undirected graphs, see [6]. In this section, we present and summarize computer compiled data on the watchman's walks and domination number in tournaments of order up to 10. The table in Appendix A presents a summarized collection of this data. In this table, we specify the order, length of watchman's walks, and domination number, and give the number of tournaments with the stated parameters. This table contains computational data regarding the watchman's walks in all tournaments of order at most 10, using the collection of adjacency matrices given in [16].

If we consider the table in Appendix A, we can easily identify some interesting tournaments. There is exactly one tournament of order 7 with domination number 3. This is the Paley tournament of order 7. Let $q \equiv 3 \pmod{4}$ be a prime power. Consider the finite field of order q , F_q . The *Paley tournament* is the digraph with vertex set $V = F_q$, where (a, b) is an arc if $b - a \in (F_q)^2$.

There is also a unique tournament T on seven vertices with domination number two such that T has fourteen watchman's walks. This graph is illustrated in Figure 3b.

We let $T(n)$ be the set of non-isomorphic tournaments on n vertices. Consider the set of tournaments, $T(n)$. If we add a source vertex to each tournament in this set, we obtain $T(n)$ non-isomorphic tournaments of order $n + 1$, each having a domination number of 1 and a watchman's walk of length 0. The total number of tournaments of order n is equal to the number of tournaments of order $n + 1$ that have a dominating vertex. Similarly, we can consider a tournament on $n - 1$ vertices with domination number γ , watchman number w , and watchman multiplicity m , we can add a sink vertex to the tournament to get a new tournament on n vertices with the same parameters. It follows that there are $|T(n)|$ many tournaments on $n + 1$ vertices with a watchman number of 0. If we instead add a sink vertex to a tournament on n vertices, we obtain a tournament on $n + 1$ vertices with the same watchman number, domination number, and watchman multiplicity. Thus, the number of tournaments with order n , watchman number w , multiplicity m , and domination number γ is less than or equal to the number of tournaments with order $n + 1$ with watchman number w , multiplicity m , and domination number γ .

6 Families of tournaments

As noted previously, there is a large number of non-isomorphic tournaments on a given number of vertices, and the Table in Appendix A highlights that these tournaments are structurally very different, and their domination or watchman numbers can vary greatly. For many tournaments, however, we may expect their watchman numbers to be low. For many families, we can prove this to be the case. In this section, we consider families of tournaments or tournaments having certain characteristics, in order to offer more precise results on their watchman number.

6.1 Simple tournaments

The *score sequence* of a tournament is the sequence of the vertex out-degrees, typically listed in non-decreasing order. A tournament with score sequence S is said to be *simple* if there are no other non-isomorphic tournaments with score sequence S . That is, T is the unique tournament with score sequence S . The following theorems demonstrate that any simple tournament has a small watchman number.

Theorem 6.1. [23] *A score sequence S is the score sequence of exactly one tournament T if and only if every strong component has score sequence (0) , $(1, 1, 1)$, $(1, 1, 2, 2)$, or $(2, 2, 2, 2, 2)$.*

Theorem 6.2. *If a score sequence S is the score sequence of a unique tournament T , then $w(T) = 0$ or 3 .*

Proof. Let T be a tournament with score sequence S such that for any other tournament T_2 with score sequence S , $T \simeq T_2$. By Theorem 6.1, the dominating strong component of T has score sequence (0) , $(1, 1, 1)$, $(1, 1, 2, 2)$, or $(2, 2, 2, 2, 2)$. It is clear that the tournament defined by (0) has a dominating vertex, and has a watchman's walk of length 0. None of the score sequences $(1, 1, 1)$, $(1, 1, 2, 2)$, or $(2, 2, 2, 2, 2)$ define a subtournament with a dominating vertex, and each define a subtournament with a dominating walk of length 3. Hence T has a watchman's walk of length 3 and, by Theorem 3.5, $w(T) = 3$. \square

6.2 Transitive tournaments

Recall that a tournament $T = (V, A)$ is transitive if there exists an ordering of the vertices of T , (v_1, v_2, \dots, v_n) such that $(v_i, v_j) \in A(T)$ if and only if $i < j$. These tournaments are always acyclic. We can also have tournaments whose structure closely resembles that of a transitive tournament, but do not have such an ordering. Three such families of tournaments are locally transitive, locally-in-transitive, and locally-out-transitive tournaments. A tournament T is *locally transitive* (*locally-in-transitive*, *locally-out-transitive*) if for each vertex v in T , the subdigraph induced by in- and out-neighbourhoods (in-neighbourhood, out-neighbourhood) of v is a transitive sub-tournament.

Recall that any transitive tournament has a dominating vertex. This is not necessarily the case for tournaments that are locally-in-transitive or locally-out-transitive. However, due to their ordering, or similar structure to transitive tournaments, we would expect these tournaments to have small dominating sets, and hence, small watchman's walks. In this section, we show this is the case.

Theorem 6.3. *If T is a locally-in-transitive or locally-out-transitive tournament, then $\gamma(T) \leq 3$ and $w(T) \leq 3$.*

Proof. Let T be a tournament, and let $\gamma = \gamma(T)$. By Theorem 4.2, $w(T) = \gamma$ or $\gamma + 1$. Suppose to the contrary that $\gamma \geq 4$, and $w(T) = \gamma$. Let $W = v_1, v_2, \dots, v_\gamma, v_1$ be a watchman's walk in T , and let D be the set of vertices in W . Since $w(T) = \gamma(T)$,

D is a minimum dominating set of T . Each vertex in D is dominated by at least one other vertex in D , namely the vertex that precedes it in the closed walk W . Thus, each vertex in D has at least one private out-neighbour in $V(T \setminus D)$ with respect to the dominating set. Let p_i be a private neighbour of v_i for each $1 \leq i \leq \gamma$. It follows that v_1, p_1 , and $p_{\gamma-1}$ are all in-neighbours of v_2 . Hence, if T is a locally-in-transitive tournament, $T[\{v_1, p_1, p_{\gamma-1}\}]$ is a transitive subtournament of T . Since $(p_{\gamma-1}, v_1)$ and (v_1, p_1) are both arcs, $(p_{\gamma-1}, p_1)$ is an arc in T . Now, $v_{\gamma-1}, p_{\gamma-1}$, and p_1 are all in-neighbours of v_γ , but $v_{\gamma-1}, p_{\gamma-1}, p_1, v_{\gamma-1}$ is a cycle in the in-neighbourhood of v_γ . However, this cannot happen since T is locally-in-transitive. Thus, if T is locally-in-transitive, we cannot have that $\gamma \geq 4$ when $w(T) = \gamma$. Similarly, v_2, \dots, v_γ are all out-neighbours of p_1 . Thus, if T is locally-out-transitive, $T[\{v_2, \dots, v_\gamma\}]$ is a transitive subtournament of T . Since $v_2, v_3, \dots, v_\gamma$ is a path in this subtournament, (v_2, v_γ) is an arc. Now, v_1, v_2 and v_γ are all out-neighbours of p_2 . However, since (v_2, v_γ) and (v_γ, v_1) are both arcs, we have that $v_\gamma, v_1, v_2, v_\gamma$ is a cycle. However, this cannot happen since T is locally-out-transitive. Thus, we have a contradiction and if T is locally-out-transitive then $\gamma(T) < 4$ when $w(T) = \gamma(T)$.

Now suppose $\gamma \geq 3$ and $w(T) = \gamma + 1$ in T . Let W be a watchman's walk in T , and let D denote the set of vertices in W . Since W is a dominating walk, D is either a minimal dominating set of size $\gamma + 1$, or D contains as a proper subset a minimum dominating set.

First consider the case where D is a minimal dominating set. Similar to the case when $w(T) = \gamma$, each vertex in D has at least one private out-neighbour in $V(T \setminus D)$ with respect to the dominating set. Let p_i be a private neighbour of v_i for each $1 \leq i \leq \gamma + 1$. Also, v_2, \dots, v_γ are all out-neighbours of p_1 . It follows that if T is locally-out-transitive, $T[\{v_2, \dots, v_\gamma\}]$ is a transitive subtournament of T . Since $v_2, v_3, \dots, v_\gamma$ is a path in this acyclic subtournament, (v_2, v_γ) is an arc. Now, v_1, v_2 and v_γ are all out-neighbours of p_2 . However, we have that $v_\gamma, v_1, v_2, v_\gamma$ is a cycle. This is a contradiction. Similarly, v_1, p_1 , and p_γ are all in-neighbours of v_2 . This implies that if T is a locally-in-transitive tournament, $T[\{v_1, p_1, p_\gamma\}]$ is a transitive subtournament of T . Since (p_γ, v_1) and (v_1, p_1) are both arcs, $(p_{\gamma-1}, p_1)$ is an arc in T . Now, $v_{\gamma-1}, p_{\gamma-1}$, and p_1 are all in-neighbours of v_γ , but $v_{\gamma-1}, p_{\gamma-1}, p_1, v_{\gamma-1}$ is a cycle in the in-neighbourhood of v_γ . This is a contradiction, since T is locally-in-transitive.

Now suppose that D contains as a subset a minimum dominating set D' . Let $H = v_1, v_2, \dots, v_{\gamma-1}, v_\gamma$ be a Hamilton path in $T[D']$; note that $T[D']$ does not have a Hamilton cycle so (v_1, v_γ) is an arc. Each vertex $v_i \in D \setminus \{v_1\}$ is dominated by v_{i-1} , so each vertex in $D' \setminus \{v_1\}$ has at least one private out-neighbour when considering D . If p_γ is a private out-neighbour of v_γ , then p_γ dominates all vertices in D' except for v_γ . In particular, it dominates v_1 and v_2 . If v_1 does not have a private out-neighbour, then $\{v_2, v_3, \dots, v_\gamma, p_\gamma, v_2\}$ is a closed dominating walk of length γ . This is a contradiction, as $w(T) = \gamma + 1$. Hence v_1 has at least one private out-neighbour. Let p_i be a private out-neighbour of v_i for each $1 \leq i \leq \gamma - 1$. Vertices $p_{\gamma-1}, v_1$, and $v_{\gamma-1}$ are all in-neighbours of v_γ , since (v_1, v_γ) is an arc. If T is locally-in-transitive, we know $T[\{v_{\gamma-1}, p_{\gamma-1}, v_1\}]$ is a transitive subtournament of T , and $(v_{\gamma-1}, p_{\gamma-1})$ and $(p_{\gamma-1}, v_1)$ are both arcs and $(v_{\gamma-1}, v_1)$ is an arc. However, p_1 is also an in-neighbour of v_γ , so $T[\{v_{\gamma-1}, p_1, v_1\}]$ is also a transitive subtournament, and $v_{\gamma-1}, p_1, v_1, v_{\gamma-1}$ is

a cycle. Thus, if T is locally-in-transitive and $\gamma(T) \geq 3$, $w(T) \neq \gamma(T) + 1$.

Similarly, p_i dominates v_j for all $v_j \in D \setminus v_i$. Thus, we can see that $T[\{v_2, \dots, v_\gamma\}]$ is a transitive subtournament if T is locally-out-transitive, since v_2, \dots, v_γ are all out-neighbours of p_1 . Thus, we know that (v_i, v_j) is an arc for all $1 < i < j \leq \gamma$. By considering the out-neighbours of p_γ , which include $v_1, \dots, v_{\gamma(T)-1}$, it is clear that (v_1, v_i) is also an arc for $2 \leq i \leq \gamma(T) - 1$. If (p_2, p_3) is an arc, then v_1, p_3 , and v_3 are out-neighbours of p_2 , and v_1, v_3, p_3, v_1 is a cycle in the out-neighbourhood of p_2 . However, this out-neighbourhood induces a transitive subtournament. Thus, (p_3, p_2) is an arc. However, p_2, v_2 , and v_1 would be out-neighbours of p_3 , and v_1, v_2, p_2, v_1 is a cycle. Hence, (p_3, p_2) also cannot be an arc if T is locally-out-transitive. This is a contradiction, since T is a tournament. Therefore, we cannot have that $\gamma(T) \geq 3$ when $w(T) = \gamma(T) + 1$ if T is locally-out-transitive. \square

The tournament in Figure 3b is locally-transitive. However, it is not a transitive tournament as it contains 3-cycles. Since each locally-transitive tournament is both locally-in-transitive and locally-out-transitive, we obtain the following corollary.

Corollary 6.4. *If T is a locally-transitive tournament, then $\gamma(T) \leq 3$, and $w(T) = 0$ or 3.*

7 Conclusion

There are many further directions that this research could take in the future. In particular, variations of the watchman's walk problem on directed graphs, such as allowing multiple watchmen, could be considered. The most efficient route for a given number of guards is also not currently known in directed graphs. Additionally, for families of directed graphs, the minimum number of guards needed to achieve a given specified maximum unseen time is also not currently known.

There are many other digraphs that lend themselves to the watchman's walk problem. In particular, de Bruijn graphs [9] and other digraphs with Hamilton cycles, as they are guaranteed to have a watchman's walk.

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Appendix

The following table specifies the number of tournaments (N) with a given watchman's walk number (w), domination number (γ), and multiplicity of watchman's walks (m) for each order (n) up to 10.

n	w	γ	(m, N)
2	0	1	(1, 1)
3	0	1	(1, 1)
	3	2	(1, 1)
4	0	1	(1, 2)
	3	2	(1, 1), (2, 1)
5	0	1	(1, 4)
	3	2	(1, 1), (2, 2), (3, 3), (4, 1), (5, 1)
6	0	1	(1, 12)
	3	2	(1, 2), (2, 4), (3, 10), (4, 12), (5, 6), (6, 8), (8, 2)
7	0	1	(1, 56)
	3	2	(1, 4), (2, 12), (3, 38), (4, 74), (5, 69), (6, 63), (8, 40), (7, 53), (9, 26), (11, 4), (10, 11), (12, 3), (13, 1), (14, 1)
	3	3	(7, 1)
8	0	1	(1, 456)
	3	2	(1, 12), (2, 48), (3, 208), (4, 544), (5, 770), (6, 820), (7, 788), (8, 892), (9, 704), (10, 657), (11, 387), (12, 294), (13, 114), (14, 99), (15, 36), (16, 27), (17, 8), (18, 9), (20, 2)
	3	3	(7, 2), (8, 1), (9, 1), (10, 1)
9	0	1	(1, 6880)
	3	2	(1, 56), (2, 296), (3, 1648), (4, 5684), (5, 11125), (6, 14911), (8, 18889), (7, 15929), (9, 20493), (10, 21489), (11, 19734), (12, 17157), (13, 12413), (14, 8912), (15, 6108), (16, 3884), (17, 2319), (18, 1519), (19, 801), (20, 461), (21, 286), (22, 147), (23, 72), (24, 60), (25, 19), (26, 4), (27, 8), (28, 4), (29, 1), (30, 1)
	3	3	(7, 6), (8, 19), (9, 48), (10, 65), (11, 46), (12, 22), (13, 14), (14, 3), (15, 1), (18, 1), (27, 1)
10	0	1	(1, 191536)
	3	2	(1, 456), (2, 3040), (3, 20808), (4, 90528), (5, 232866), (6, 395927), (7, 493369), (8, 590172), (9, 714023), (10, 874685), (11, 952415), (12, 1013385), (13, 933658), (14, 823741), (15, 665467), (16, 527268), (17, 377567), (18, 277459), (19, 184796), (20, 126674), (21, 78776), (22, 52721), (23, 31016), (24, 21213), (25, 11643), (26, 7727), (27, 4137), (28, 2622), (29, 1437), (30, 1015), (31, 400), (32, 367), (33, 112), (34, 121), (35, 24), (36, 47), (38, 18), (40, 4)
	3	3	(5, 1), (7, 45), (8, 360), (9, 1603), (10, 3933), (11, 5672), (12, 5752), (13, 4869), (14, 3298), (15, 2015), (16, 1176), (17, 585), (18, 255), (19, 127), (20, 58), (21, 22), (22, 19), (23, 7), (24, 2), (25, 3), (26, 3), (27, 5), (29, 1), (30, 2), (31, 1), (36, 2)

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