AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **80(1)** (2021), Pages 143–156

Generalized path pairs and Fuss-Catalan triangles

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Abstract

Path pairs are a modification of parallelogram polyominoes that provide yet another combinatorial interpretation of the Catalan numbers. More specifically, the number of path pairs of length n and distance δ corresponds to the $(n-1,\delta-1)$ entry of Shapiro's so-called Catalan triangle. In this paper, we widen the notion of path pairs (γ_1,γ_2) to the situation where γ_1 and γ_2 may have different lengths, and then enforce divisibility conditions on runs of vertical steps in γ_2. This creates a two-parameter family of integer triangles that generalize the Catalan triangle and qualify as proper Riordan arrays for many choices of parameters. In particular, we use generalized path pairs to provide a new combinatorial interpretation for all entries in every proper Riordan array $\mathcal{R}(d(t),h(t))$ of the form $d(t)=C_k(t)^i$, $h(t)=tC_k(t)^k$, where $1\le i\le k$ and $C_k(t)$ is the generating function for some sequence of Fuss-Catalan numbers (some $k\ge 2$). Closed formulas are then provided for the number of generalized path pairs across an even broader range of parameters, as well as for the number of "weak" path pairs with a fixed number of non-initial intersections.

1 Introduction

The Catalan numbers are a seemingly ubiquitous sequence of positive integers whose n^{th} entry is $C_n=\frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers satisfy the recurrence $C_{n+1}=\sum_{i+j=n}C_iC_j$ for all $n\ge 0$, which translates to the ordinary generating function $C(t)=\sum_{n=0}^{\infty}C_nt^n$ as the relation $C(t)=tC(t)^2+1$. It follows that $C(t)=\frac{1-\sqrt{1-4t}}{2t}$.

Hundreds of combinatorial interpretations for the Catalan numbers have been compiled by Stanley [13]. One such interpretation identifies C_n with the number of parallelogram polyominoes with semiperimeter $n+1$. These are ordered pairs of lattice paths (γ_1,γ_2) that satisfy all of the following:

ISSN: 2202-3518

1. Both γ_1 and γ_2 are composed of $n+1$ steps from the step set $\{E = (1,0), N = (0,1)\}$, where γ_1 must begin with an N step and γ_2 must begin with an E step.
2. Both γ_1 and γ_2 begin at $(0,0)$ and end at the same point.
3. γ_1 and γ_2 only intersect at their initial and final points.

See Figure 1 for an illustration of all parallelogram polyominoes with semiperimeter 4, noting that the number of such paths is $C_3 = 5$.

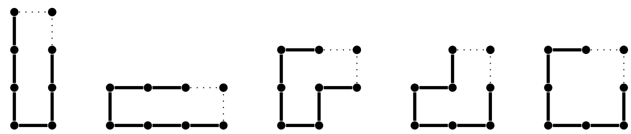

Figure 1: The $C_3 = 5$ parallelogram polyominoes with semiperimeter 4, with the corresponding path pairs of length 3 (and $\delta = 1$) appearing as the bold edges.

Generalizing the notion of parallelogram polyominoes are (fat) path pairs, as introduced by Shapiro [11] and developed by Deutsch and Shapiro [4]. A *path pair of length* n is an ordered pair (γ_1, γ_2) of lattice paths that satisfy all of the following:

1. Both γ_1 and γ_2 are composed of n steps from the step set $\{E = (1,0), N = (0,1)\}$.
2. Both γ_1 and γ_2 begin at $(0,0)$.
3. Apart from at $(0,0)$, γ_1 stays strongly above γ_2.

Now consider the path pair (γ_1, γ_2), and suppose that γ_1 terminates at (x_1, y_1) while γ_2 terminates at (x_2, y_2). Clearly $x_1 < x_2$ and $y_1 > y_2$. The path pair (γ_1, γ_2) is said to have *distance* δ if $x_2 - x_1 = \delta$, and in this case we write $|\gamma_2 - \gamma_1| = \delta$. We henceforth use $\mathcal{P}_{n,\delta}$ to denote the set of all path pairs of length n and distance δ.

There is a simple bijection between $\mathcal{P}_{n,1}$ and parallelogram polynomials of semiperimeter $n+1$, via a map that adds an E step to the end of γ_1 and an N step to the end of γ_2. See Figure 1 for an illustration of the $n = 3$ case. It follows that $\mathcal{P}_{n,1} = C_n$ for all $n \geq 0$.

Enumeration of $\mathcal{P}_{n,\delta}$ for all $\delta \geq 1$ and $n \geq 1$ was addressed by Shapiro [11], who identified $|\mathcal{P}_{n,\delta}| = \frac{2\delta}{2n}\binom{2n}{n-\delta}$ with the $(n-1, \delta-1)$ entry of his so-called Catalan triangle. See Figure 2 for the first five rows of Shapiro's Catalan triangle, an infinite lower-triangular matrix (with zero entries suppressed) whose entries $d_{i,j}$ are generated by the recurrence $d_{0,0} = 1$ and $d_{i,j} = d_{i-1,j-1} + 2d_{i-1,j} + d_{i-1,j+1}$ for all $i \geq 1, 0 \leq j \leq i$.[1]

[1]Shapiro's Catalan triangle should not be confused with the "Catalan triangle" whose (i,j) entry is the ballot number $d_{i,j} = \frac{j+1}{i+1}\binom{2i-j}{i}$. We alternatively refer to this second infinite lower-triangular matrix as the ballot triangle. See Aigner [1] for connections between the ballot triangle and the Catalan triangle.

1				
2	1			
5	4	1		
14	14	6	1	
42	48	27	8	1

Figure 2: The first five rows of Shapiro's Catalan triangle.

The Catalan triangle is a well-known example of a proper Riordan array. Given a pair of generating functions $d(t)$ and $h(t)$ such that $d(0) \neq 0$, $h(0) = 0$, and $h'(0) \neq 0$, the associated proper Riordan array $\mathcal{R}(d(t), h(t))$ is the infinite lower-triangular matrix whose (i, j) entry is $d_{i,j} = [t^i]d(t)h(t)^j$. Here we use the standard notation in which $[t^i]$ identifies the coefficient of t^i in a power series. It may be verified that Shapiro's Catalan triangle is the proper Riordan array with $d(t) = C(t)^2$ and $h(t) = t\,C(t)^2$.

For general information about Riordan arrays, see Rogers [10], Merlini et al. [9], or Shapiro et al. [12]. For a more focused discussion about how Riordan arrays similar to the Catalan triangle may be used to define so-called "Catalan-like numbers", see Aigner [2].

Central to our work is the fact that every proper Riordan array $\mathcal{R}(d(t), h(t))$ possesses sequences of integers $\{z_i\}_{i=0}^{\infty}$ and $\{a_i\}_{i=0}^{\infty}$ such that

$$d_{n,k} = \begin{cases} z_0\, d_{n-1,k} + z_1\, d_{n-1,k+1} + z_2\, d_{n-1,k+2} + \dots & \text{for } k = 0 \text{ and all } n \geq 1; \\ a_0\, d_{n-1,k-1} + a_1\, d_{n-1,k} + a_2\, d_{n-1,k+1} + \dots & \text{for all } k \geq 1 \text{ and } n \geq 1. \end{cases} \quad (1)$$

These sequences are referred to as the Z-sequence and the A-sequence of $\mathcal{R}(d(t), h(t))$, respectively. When represented as generating functions $Z(t) = \sum_i z_i t^i$ and $A(t) = \sum_i a_i t^i$, the Z- and A-sequences of a proper Riordan array are known to satisfy the relations

$$d(t) = \frac{d(0)}{1 - t\,Z(h(t))}, \qquad h(t) = t\,A(h(t)). \quad (2)$$

The defining recurrence of the Catalan triangle implies that it is a proper Riordan array with $Z(t) = 2 + t$ and $A(t) = 1 + 2t = t^2 = (1 + t)^2$.

We pause to recap a few facts about the one-parameter Fuss-Catalan numbers, also known as the k-Catalan numbers, since they will play a major role in what follows. For any $k \geq 2$, the k-Catalan numbers are an integer sequence whose n^{th} entry is $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$. Observe that the $k = 2$ case corresponds to the "original" Catalan numbers. For any $k \geq 2$, the k-Catalan numbers satisfy the recurrence $C_{n+1}^k = \sum_{i_1+\ldots+i_k} C_{i_1}^k \ldots C_{i_k}^k$ for all $n \geq 0$, implying that their generating functions $C_k(t) = \sum_{n=0}^{\infty} C_n^k t^n$ satisfy $C_k(t) = t\,C_k(t)^k + 1$. For an introduction to the k-Catalan numbers, see Hilton and Pederson [8]. For a list of combinatorial interpretations for the k-Catalan numbers, see Heubach, Li and Mansour [7].

The goal of this paper is to simultaneously explore several generalizations of path pairs. Firstly, we eliminate the requirement that the two paths of (γ_1, γ_2) have equal length, setting $\epsilon = |\gamma_2| - |\gamma_1|$ and examining the full range of differences $\epsilon \geq 0$ with $|\gamma_1| \geq 0$. We also enforce conditions on the N steps of γ_2 that are designed to mirror the generalization of the Catalan numbers to the k-Catalan numbers. We refer to the resulting combinatorial objects as *k-path pairs of length* $(n - \epsilon, n)$.

Section 2 focuses upon the enumeration of k-path pairs. In Subection 2.1, we construct a two-parameter collection of infinite lower-triangular arrays $A^{k,\epsilon}$, whose entries correspond to the number of k-path pairs of varying lengths and distances. For all $0 \leq \epsilon \leq k-1$, Theorem 2.2 identifies the triangle $A^{k,\epsilon}$ with the proper Riordan array $\mathcal{R}(d(t), h(t))$ where $d(t) = C_k(t)^{k-\epsilon}$ and $h(t) = tC_k(t)^k$. In Subsection 2.2, we directly enumerate sets of k-path pairs for all $k \geq 2$ and $\epsilon \leq 0$. Theorem 2.5 uses the results of Subsection 2.2 to derive a closed formula for the size of all such sets, and Theorem 2.6 provides a significantly simplified formula within the range of $0 \leq \epsilon \leq (k-1)\delta$.

Section 3 introduces a related generalization where we now allow the two paths (γ_1, γ_2) to intersect away from $(0, 0)$, so long as γ_1 stays weakly above γ_2 for the entirety of its length. Theorem 3.2 applies the techniques of Section 2 to derive a closed formula for the number of "weak k-path pairs" whose paths intersect precisely m times away from $(0, 0)$, assuming that we restrict ourselves to the range $0 \leq \epsilon \leq (k-1)\delta$.

2 Generalized k-Path Pairs

Take any pair of integers n, ϵ such that $0 \leq \epsilon < n$. Then define $\mathcal{P}^{\epsilon}_{n,\delta}$ to be the collection of ordered pairs (γ_1, γ_2) of lattice paths that satisfy all of the following:

1. Both γ_1 and γ_2 begin at $(0, 0)$ and use steps from $\{E = (1, 0), N = (0, 1)\}$.
2. γ_2 is composed of precisely n steps, the first of which is an E step.
3. γ_1 is composed of precisely $n - \epsilon$ steps, the first of which is a N step.
4. γ_1 and γ_2 do not intersect apart from at $(0, 0)$.
5. The difference between the terminal x coordinates of γ_1 and γ_2 is δ.

The case $\epsilon = 0$ obviously corresponds to the original notion of path pairs. If γ_2 terminates at (x_2, y_2), then γ_1 terminates at $(x_1, y_1) = (x_2 - \delta, y_2 + \delta - \epsilon)$. In particular, $y_1 - y_2 \geq 0$ precisely when $\delta \geq \epsilon$.

Now fix $k \geq 2$, and consider some $(\gamma_1, \gamma_2) \in \mathcal{P}^{\epsilon}_{n,\delta}$. The path pair (γ_1, γ_2) is said to be a *k-path pair of length* $(n - \epsilon, n)$ *and distance* δ if the bottom path $\gamma_2 = E^1 N^{b_1} E^1 N^{b_2} \ldots E^1 N^{b_m}$ satisfies $b_i = (k-2) \bmod (k-1)$ for all i. Clearly, 2-path pairs correspond to the notion of path pairs discussed above.

For any k-path pair (γ_1, γ_2), the bottom path γ_2 must decompose into a sequence of length-$(k-1)$ subpaths, each of which is either N^{k-1} or $E^1 N^{k-2}$. In particular, the length n of γ_2 must be divisible by $k-1$. To avoid a large number of empty sets, we define $\mathcal{P}^{k,\epsilon}_{n,\delta}$ to be the collection of all k-path pairs of length $((k-1)n-\epsilon,(k-1)n)$ and distance δ.

We continue to use the notation $\delta = |\gamma_2 - \gamma_1|$ for the distance of k-path pairs. For any $(\gamma_1,\gamma_2) \in \mathcal{P}^{k,\epsilon}_{n,\delta}$, it is always the case that $1 \leq \delta \leq n$, with the maximum distance of n only being obtained by the pair with $\gamma_1 = N^{n-\epsilon}$ and $\gamma_2 = (EN^{k-2})^n$. It follows that the sets $\mathcal{P}^{k,\epsilon}_{n,\delta}$ encompass all nonempty collections of k-path pairs if we range over $1 \leq \delta \leq n$ and $0 \leq \epsilon \leq (k-1)n$.

2.1 Generalized k-Path Pairs with $0 \leq \epsilon \leq k-1$

In order to enumerate arbitrary $\mathcal{P}^{k,\epsilon}_{n,\delta}$, we fix k,ϵ and define a recurrence with respect to n,δ. This recurrence will directly generalize Shapiro's original recurrence for the Catalan triangle [11]. We begin with the range $0 \leq \epsilon \leq k-1$, where the recursion will eventually correspond to the Z- and A-sequences of a proper Riordan array.

Theorem 2.1. *For any $k \geq 2$, $n \geq 1$, and $0 \leq \epsilon \leq k-1$,*

$$|\mathcal{P}^{k,\epsilon}_{n,\delta}| = \begin{cases} \displaystyle\sum_{j=1}^{k} \binom{k}{j} |\mathcal{P}^{k,\epsilon}_{n-1,j}| - \sum_{j=1}^{\epsilon} \binom{\epsilon}{j} |\mathcal{P}^{k,\epsilon}_{n-1,j}| & \text{for } \delta = 1, \text{ and} \\ \displaystyle\sum_{j=0}^{k} \binom{k}{j} |\mathcal{P}^{k,\epsilon}_{n-1,\delta-1+j}| & \text{for } \delta > 1. \end{cases}$$

Proof. For any length-$(k-1)$ word w in the alphabet $\{E,N\}$, define U_w to be the set of all $(\gamma_1,\gamma_2) \in \mathcal{P}^{k,\epsilon}_{n,\delta}$ such that γ_1 terminates with w and γ_2 terminates with N^{k-1}. If w contains precisely j instances of E, this implies $\gamma_1 = \eta_1 w$ and $\gamma_2 = \eta_2 N^{k-1}$ for some $(\eta_1,\eta_2) \in \mathcal{P}^{k,\epsilon}_{n-1,\delta+j}$. Similarly define V_w to be all $(\gamma_1,\gamma_2) \in \mathcal{P}^{k,\epsilon}_{n,\delta}$ such that γ_1 terminates with w and γ_2 terminates with EN^{k-2}. If w contains precisely j instances of E, then $\gamma_1 = \eta_1 w$ and $\gamma_2 = \eta_2 EN^{k-2}$ for some k-path pair $(\eta_1,\eta_2) \in \mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}$. By construction, $\mathcal{P}^{k,\epsilon}_{n,\delta} = (\bigcup_w U_w) \cup (\bigcup_w V_w)$.

See Figure 3 for the general form of terminal subpaths in an element (γ_1,γ_2) of U_w or V_w. In both diagrams, (a,b) is fixed as the terminal point of γ_1, whereas the final $k-1$ steps of γ_1 are determined by w and lie within the dotted triangle in the upper-left of each image.

Now take any length-$(k-1)$ word w with precisely j instances of E. Our strategy is to enumerate U_w and V_w via consideration of the injective maps $g_w : \mathcal{P}^{k,\epsilon}_{n-1,\delta+j} \to S$, $g_w(\eta_1,\eta_2) = (\eta_1 w, \eta_2 N^{k-1})$ and $h_w : \mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1} \to S$, $h_w(\eta_1,\eta_2) = (\eta_1 w, \eta_2 EN^{k-2})$. Here S denotes some collection of path-pairs whose elements may intersect apart from at $(0,0)$. We clearly have $U_w \subseteq \mathrm{Im}(g_w)$ and $V_w \subseteq \mathrm{Im}(h_w)$ for any word w. We also have $U_w = \mathrm{Im}(g_w)$ if and only if every path pair in $\mathrm{Im}(g_w)$ is non-intersecting apart from $(0,0)$, and $\mathrm{Im}(h_w) = V_w$ if and only if every path pair in $\mathrm{Im}(h_w)$ is non-intersecting apart from $(0,0)$.

Begin with g_w. The path pair $g(\eta_1, \eta_2) = (\eta_1 w, \eta_2 N^{k-1})$ can only feature an intersection away from $(0,0)$ if the final $k-1$ steps of $\eta_1 w$ pass through some northwest corner of $\eta_2 N^{k-1}$. As seen in Figure 3, the largest possible y-coordinate for a northwest corner of $\eta_2 N^{k-1}$ is $b - \delta + \epsilon - 2k + 3$, whereas the terminal point of η_1 has a y-coordinate of at least $b - k + 1$. Since we are assuming $\epsilon \leq k-1$, we have $\epsilon \leq \delta(k-1)$ for all $\delta \geq 1$. It follows that $b - \delta + \epsilon - 2k + 3 \leq b - k + 1$ for all $\delta \geq 1$, with the case of $b - d + \epsilon - 2k + 3 = b - k + 1$ being impossible because the input path (η_1, η_2) was assumed to be non-intersecting away from $(0,0)$. This implies that $\eta_1 w$ cannot intersect $\eta_2 N^{k-1}$ away from $(0,0)$ for any word w.

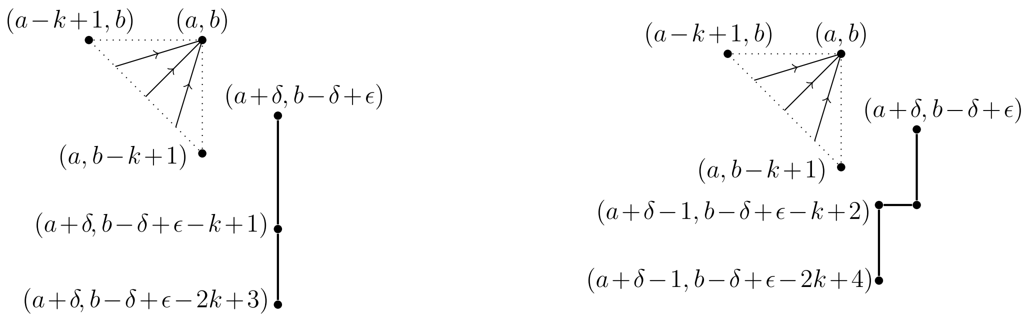

Figure 3: Terminal subpaths for arbitrary $(\gamma_1, \gamma_2) \in U_w$ (left side) and arbitrary $(\gamma_1, \gamma_2) \in V_w$ (right side), as referenced in the proof of Theorem 2.1.

It follows that g_w represents a bijection from $\mathcal{P}^{k,\epsilon}_{n-1,\delta+j}$ onto U_w for every word w when $\epsilon \leq k-1$. Since there are $\binom{k-1}{j}$ words w with precisely j instances of E, a total of $\binom{k-1}{j}$ sets U_w lie in bijection with $\mathcal{P}^{k,\epsilon}_{n-1,\delta+j}$ for each $0 \leq j \leq j-1$. This gives

$$\sum_w |U_w| = \sum_{j=0}^{k-1} \binom{k-1}{j} |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j}| = \sum_{j=1}^{k} \binom{k-1}{j-1} |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}|. \tag{3}$$

For h_w, we separately consider the cases of $\delta = 1$ and $\delta \geq 2$. Begin by assuming $\delta \geq 2$. We once again note that $h_w(\eta_1, \eta_2) = (\eta_1 w, \eta_2 E N^{k-2})$ has intersections away from $(0,0)$ only when the final $k-1$ steps of $\eta_1 w$ intersect some northwest corner of $\eta_2 E N^{k-2}$. From Figure 3, since $\delta \geq 2$ we see that the y-coordinate of such a corner can be at most $b - \delta + \epsilon - 2k + 4$. Our assumptions of $\epsilon \leq k-1$ and $\delta \leq 2$ together ensure $\epsilon \leq k - 3 + \delta$ and thus that $b - \delta + \epsilon - 2k + 4 \leq b - k + 1$, with the case of $b - \delta + \epsilon - 2k + 4 = b - k + 1$ being impossible because we've assumed that (η_1, η_2) lacks intersections away from $(0,0)$. This implies that $\eta_1 w$ cannot intersect $\eta_2 E N^{k-2}$ away from $(0,0)$ for any word w when $\delta \geq 2$, and thus that h_w is a bijection from $\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}$ onto V_w for every word w when $\delta \geq 2$.

When $\delta = 1$, the map h_w may introduce new intersections. Fixing w, either every image $h_w(\eta_1, \eta_2) = (\eta_1 w, \eta_2 E N^{k-2})$ will have an intersection away from $(0,0)$, or every image $h_w(\eta_1, \eta_2)$ will lack such an intersection. That first subcase implies that the corresponding set V_w is empty, whereas that second subcase implies that V_w is nonempty and in bijection with $\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}$. We only need to enumerate how many words w fall into each subcase (for each choice of $0 \leq j \leq k-1$).

As seen on the right side of Figure 3, when $\delta = 1$ the final northwest corner of $\eta_2 EN^{k-2}$ occurs at $(a, b+\epsilon-k+1)$. Fixing a word w with precisely j instances of E, we also see that η_1 terminates at $(a-j, b-k+j+1)$. This means that η_1 can only pass through $(a, b+\epsilon-k+1)$ if $j \leq \epsilon$. For any such $j \leq \epsilon$, there are precisely $\binom{\epsilon}{j}$ words w in which this additional intersection occurs. As there are $\binom{k-1}{j}$ words w with precisely j instances of E, if $\epsilon \leq k-1$ we know that V_w is nonempty for precisely $\binom{k-1}{j} - \binom{\epsilon}{j}$ choices of w. Combining our results for $\delta \geq 2$ and $\delta = 1$ gives

$$\sum_w |V_w| = \begin{cases} \displaystyle\sum_{j=0}^{k-1} \binom{k-1}{j} |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}| & \text{for } \delta \geq 2, \text{ and} \\ \displaystyle\sum_{j=0}^{k-1} \left(\binom{k-1}{j} - \binom{\epsilon}{j} \right) |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}| & \text{for } \delta = 1. \end{cases} \tag{4}$$

Once again noting that $\mathcal{P}^{k,\epsilon}_{n,\delta} = (\bigcup_w U_w) \cup (\bigcup_w V_w)$, for $\delta \geq 2$ we have

$$|\mathcal{P}^{k,\epsilon}_{n,\delta}| = \sum_w |U_w| + \sum_w |V_w| = \sum_{j=1}^{k} \binom{k-1}{j-1} |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}| + \sum_{j=0}^{k-1} \binom{k-1}{j} |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}|$$

$$= \sum_{j=0}^{k} \left(\binom{k-1}{j-1} + \binom{k-1}{j} \right) |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}| = \sum_{j=0}^{k} \binom{k}{j} |\mathcal{P}^{k,\epsilon}_{n-1,\delta+j-1}|.$$

For $\delta = 1$, the facts that $0 \leq \epsilon \leq k-1$ and $|\mathcal{P}^{k,\epsilon}_{n-1,0}| = 0$ prompt the similar result

$$\begin{aligned} |\mathcal{P}^{k,\epsilon}_{n,1}| &= \sum_w |U_w| + \sum_w |V_w| \\ &= \sum_{j=1}^{k} \binom{k-1}{j-1} |\mathcal{P}^{k,\epsilon}_{n-1,j}| + \sum_{j=0}^{k-1} \left(\binom{k-1}{j} - \binom{\epsilon}{j} \right) |\mathcal{P}^{k,\epsilon}_{n-1,j}| \\ &= \sum_{j=0}^{k} \left(\binom{k-1}{j-1} - \binom{k-1}{j} \right) |\mathcal{P}^{k,\epsilon}_{n-1,j}| - \sum_{j=0}^{k-1} \binom{\epsilon}{j} |\mathcal{P}^{k,\epsilon}_{n-1,j}| \\ &= \sum_{j=1}^{k} \binom{k}{j} |\mathcal{P}^{k,\epsilon}_{n-1,j}| - \sum_{j=1}^{\epsilon} \binom{\epsilon}{j} |\mathcal{P}^{k,\epsilon}_{n-1,j}|. \end{aligned}$$

□

It should be noted that the methods from Theorem 2.1 may be extended to a somewhat broader range of parameters than $\epsilon \leq k-1$. In particular, the summation of (3) may be shown to hold for all $\epsilon \leq (k-1)\delta$, whereas the $\delta \geq 2$ summation of

(4) may be shown to hold for all $\epsilon \leq (k-1)(\delta-1)$. Sadly, developing a general recursive relation for the full $\epsilon \leq \delta(k-1)$ range of Theorem 2.6 is extremely involved. The enumerative usage of those recursions is also limited when $\epsilon > k-1$, as they no longer qualify as the A- and Z-sequences of a proper Riordan array. As such, we delay the $\epsilon > k-1$ case until Subsection 2.2, where generating function techniques may be applied to directly derive closed formulas from pre-existing results for the general case.

For each choice of $k \geq 2$ and $0 \leq \epsilon \leq k-1$, the recursive relations of Theorem 2.1 may be used to generate an infinite lower-triangular matrix $A^{k,\epsilon}$ whose (i,j) entry is $a^{k,\epsilon}_{i,j} = |\mathcal{P}^{k,\epsilon}_{i+1,j+1}|$. These $A^{k,\epsilon}$ qualify as proper Riordan arrays:

Theorem 2.2. *For any $k \geq 2$ and $0 \leq \epsilon \leq k-1$, the integer triangle $A^{k,\epsilon}$ with (i,j) entry $|\mathcal{P}^{k,\epsilon}_{i+1,j+1}|$ is the proper Riordan array $\mathcal{R}(C_k(t)^{k-\epsilon}, tC_k(t)^k)$, where $C_k(t)$ is the generating function for the k-Catalan numbers.*

Proof. By Theorem 2.1, the array $A^{k,\epsilon}$ has A-sequence $A(t) = (1+t)^k$ and Z-sequence $Z(t) = \frac{(1+t)^k-(1+t)^\epsilon}{t}$. The k-Catalan relation $C_k(t) = tC_k(t)^k + 1$ may then be used to verify the identities of (2):

$$tA(h(t)) = t(1+tC_k(t)^k)^k = tC_k(t)^k = h(t),$$

$$\frac{d(0)}{1+tZ(h(t))} = \frac{1}{1-t\frac{(1+tC_k(t)^k)^k-(1+tC_k(t)^k)^\epsilon}{tC_k(t)^k}} = \frac{1}{1-\frac{C_k(t)^k-C_k(t)^\epsilon}{C_k(t)^k}} = \frac{C_k(t)^k}{C_k(t)^\epsilon} = d(t).$$

□

Every integer triangle $A^{k,\epsilon}$ is a Fuss-Catalan triangle of the type introduced by He and Shapiro [5], seeing as they all take the form $\mathcal{R}(C_k^i, C_k^j)$ for some $k \geq 2$ and some $i, j > 0$. Many specific triangles $A^{k,\epsilon}$ also correspond to Riordan arrays that are well-represented in the literature. The triangle $A^{2,0}$ is Shapiro's Catalan triangle, while $A^{2,0}$ and $A^{2,1}$ are two of the admissible matrices discussed by Aigner [1]. More generally, whenever $\epsilon = 0$ the triangle $A^{k,\epsilon}$ is a renewal array with "identical" A- and Z-sequences, as investigated by Cheon, Kim and Shapiro [3]. For additional results of this type, see He and Sprugnoli [6]

In a slight deviation from He and Shapiro [5], we refer to $A^{k,\epsilon}$ as the (k,ϵ)-*Catalan triangle*. See Figure 4 for all (k,ϵ)-Catalan triangles with $k = 2, 3, 4$.

One immediate consequence of Theorem 2.2 is a closed formula for the size of every set $\mathcal{P}^{k,\epsilon}_{n,\delta}$ when $0 \leq \epsilon \leq k-1$. Observe that every cardinality $|\mathcal{P}^{k,\epsilon}_{n,\delta}| = \frac{k\delta-\epsilon}{kn-\epsilon}\binom{kn-\epsilon}{n-\delta}$ from Corollary 2.3 is the Raney number $R_{k,k\delta-\epsilon}(n-\delta)$. As defined by Hilton and Pedersen [8], the Raney numbers (two-parameter Fuss-Catalan numbers) are defined to be $R_{k,r}(n) = [t^n]C_k(t)^r$, with the original k-Catalan numbers corresponding to $C^k_n = R_{k,1}(n) = R_{k,k}(n-1)$.

Corollary 2.3. *For any $k \geq 2$ and $0 \leq \epsilon \leq k-1$,*

$$|\mathcal{P}^{k,\epsilon}_{n,\delta}| = [t^{n-\delta}]C_k(t)^{k\delta-\epsilon} = \frac{k\delta-\epsilon}{kn-\epsilon}\binom{kn-\epsilon}{n-\delta}.$$

	$\epsilon = \mathbf{0}$	$\epsilon = \mathbf{1}$	$\epsilon = \mathbf{2}$	$\epsilon = \mathbf{3}$
$\mathbf{k = 2}$	1 2 1 5 4 1 14 14 6 1 42 48 27 8 1	1 1 1 2 3 1 5 9 5 1 14 28 20 7 1		
$\mathbf{k = 3}$	1 3 1 12 6 1 55 33 9 1 273 182 63 12 1	1 2 1 7 5 1 30 25 8 1 143 130 52 11 1	1 1 1 3 4 1 12 18 7 1 55 88 42 10 1	
$\mathbf{k = 4}$	1 4 1 22 8 1 140 60 12 1 969 456 114 16 1	1 3 1 15 7 1 91 49 11 1 612 357 99 15 1	1 2 1 9 6 1 52 39 10 1 340 272 85 14 1	1 1 1 4 5 1 22 30 9 1 140 200 72 13 1

Figure 4: Top five rows for all (k, ϵ)-Catalan triangles $A^{k,\epsilon}$ with $k = 2, 3, 4$.

Proof. By the definition of $A^{k,\epsilon}$ we have

$$a^{k,\epsilon}_{i,j} = [t^i]C_k(t)^{k-\epsilon}(tC_k(t)^k)^j = [t^{i-j}]C_k(t)^{k-\epsilon+kj}.$$

The corollary then follows from the fact that $|\mathcal{P}^{k,\epsilon}_{n,\delta}| = a^{k,\epsilon}_{n-1,\delta-1}$. □

2.2 Generalized k-Path Pairs, all $\epsilon \geq 0$

If $\epsilon > k - 1$, there need not be a bijection between $\mathcal{P}^{k,\epsilon}_{n,\delta}$ and some Raney number $R_{k,r}(n) = [t^n]C_k(t)^r$. This implies that the cardinalities $|\mathcal{P}^{k,\epsilon}_{n,\delta}|$ cannot be organized into any Fuss-Catalan triangle. One may still define an infinite lower-triangular array $A^{k,\epsilon}$ whose (i, j) entry is $a^{k,\epsilon}_{i,j} = |\mathcal{P}^{k,\epsilon}_{i+1,j+1}|$, but for $\epsilon > k - 1$ we always have $a^{k,\epsilon}_{0,0} = 0$ and the resulting arrays never qualify as a proper Riordan array.

For general ϵ, we still have the following decomposition for $|\mathcal{P}^{k,\epsilon}_{n,\delta}|$:

Proposition 2.4. *Fix $n \geq 1$, $1 \leq \delta \leq n$, and $0 \leq \epsilon \leq (k - 1)n$. For any pair of non-negative integers ϵ_1, ϵ_2 such that $\epsilon = (k - 1)\epsilon_1 + \epsilon_2$,*

$$|\mathcal{P}^{k,\epsilon}_{n,\delta}| = \sum_{i=1}^{\delta} \binom{\epsilon_1}{\delta - i} |\mathcal{P}^{k,\epsilon_2}_{n-\epsilon_1,i}|.$$

Proof. As seen in Figure 5, for any $(\gamma_1, \gamma_2) \in \mathcal{P}^{k,\epsilon}_{n,\delta}$ we may divide γ_2 into an initial subpath η_1 of length $n - (k - 1)\epsilon_1$ and a terminal subpath η_2 of length $(k - 1)\epsilon_1$. As the length of η_1 is divisible by $k - 1$, it is always the case that $(\gamma_1, \eta_1) \in \mathcal{P}^{k,\epsilon_2}_{n-\epsilon_1,i}$ for some $1 \leq i \leq \delta$.

Then consider the map $f : \mathcal{P}^{k,\epsilon}_{n,\delta} \to \bigcup_{i=1}^{\delta} \mathcal{P}^{k,\epsilon_2}_{n-\epsilon_1,i}$ where $f(\gamma_1, \gamma_2) = (\gamma_1, \eta_1)$. This map is clearly surjective. For any $1 \leq i \leq \delta$ and any $(\gamma_1, \eta_1) \in \mathcal{P}^{k,\epsilon_2}_{n-\epsilon_1,i}$, every way of appending precisely $\delta - i$ copies of E^1N^{k-2} and $\epsilon_1 - \delta + i$ copies of N^{k-1} to the end of η_1 (in any order) produces an element of $\mathcal{P}^{k,\epsilon}_{n,\delta}$. It follows that the inverse image $f^{-1}(\gamma'_1, \gamma'_2)$ of every $(\gamma'_1, \gamma'_2) \in \mathcal{P}^{k,\epsilon_2}_{n-\epsilon_1,i}$ has size $\binom{\epsilon_1}{\delta-i}$. Ranging over $1 \le i \le \delta$ gives the required summation. □

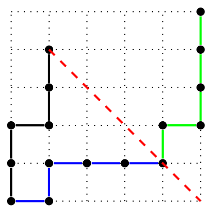

Figure 5: The decomposition of γ_2 for some $(\gamma_1, \gamma_2) \in \mathcal{P}^{2,5}_{10,4}$, as in the proof to Proposition 2.4. If $k > 2$, note that the initial subpath of γ_2 extends beyond the dotted diagonal line, until its length is divisible by $k - 1$.

The summation on the right side of Proposition 2.4 may feature fewer than δ nonzero terms, as $|P^{k,\epsilon_2}_{n-\epsilon_1,i}| = 0$ when $n - \epsilon_1 < i$. The decomposition $\epsilon = (k-1)\epsilon_1 + \epsilon_2$ also fails be be unique when $\epsilon \ge k - 1$. However, there always exists at least one decomposition of ϵ in which $\epsilon_2 \le k - 1$.

When $\epsilon \le k - 1$, this preferred decomposition of ϵ with $\epsilon_2 \le k - 1$ corresponds to $\epsilon_1 = 0$ and reduces the summation of Proposition 2.4 to the single term $|\mathcal{P}^{k,\epsilon}_{n,\delta}|$. When $\epsilon > k - 1$, choosing ϵ_1 so that $\epsilon \le k - 1$ allows us to apply Corollary 2.3 to each term in the summation:

Theorem 2.5. *Fix $n \ge 1$, $1 \le \delta \le n$, and $0 \le \epsilon \le (k-1)n$. For any pair of non-negative integers ϵ_1, ϵ_2 such that $\epsilon = (k-1)\epsilon_1 + \epsilon_2$ and $0 \le \epsilon_2 \le k - 1$,*

$$\begin{aligned} |\mathcal{P}^{k,\epsilon}_{n,\delta}| &= [t^{n-\epsilon_1}] \sum_{i=1}^{\delta} \binom{\epsilon_1}{\delta - i} t^i C_k(t)^{ki-\epsilon_2} \\ &= \sum_{i=1}^{\delta} \frac{ki - \epsilon_2}{k(n-\epsilon_1) - \epsilon_2} \binom{\epsilon_1}{\delta - i} \binom{k(n-\epsilon_1) - \epsilon_2}{n - \epsilon_1 - i}. \end{aligned}$$

Beyond the $\epsilon \le k - 1$ case of Subsection 2.1, there are several situations where the general identity of Theorem 2.5 simplifies to give an enumeration equivalent to Corollary 2.3.

Theorem 2.6. *Fix $n \ge 1$ and $0 \le \epsilon \le (k-1)n$, and take any pair of non-negative integers ϵ_1, ϵ_2 such that $\epsilon = (k-1)\epsilon_1 + \epsilon_2$ and $0 \le \epsilon_2 \le k - 1$. For all $\delta > \epsilon_1$, as well as for all $0 \le \epsilon \le (k-1)\delta$, we have*

$$|\mathcal{P}^{k,\epsilon}_{n,\delta}| = [t^{n-\delta}] C_k(t)^{k\delta - \epsilon} = \frac{k\delta - \epsilon}{kn - \epsilon} \binom{kn - \epsilon}{n - \delta}.$$

Proof. Beginning with Theorem 2.5, when $\delta - \epsilon_1 > 0$ we may rewrite the bounds of the summation and then perform the change of variables $j = \epsilon_1 - \delta + i$ to give

$$\begin{aligned}
|\mathcal{P}_{n,\delta}^{k,\epsilon}| &= [t^{n-\epsilon_1}] \sum_{i=1}^{\delta} \binom{\epsilon_1}{\delta - i} t^i C_k(t)^{ki-\epsilon_2} = [t^{n-\epsilon_1}] \sum_{i=\delta-\epsilon_1}^{\delta} \binom{\epsilon_1}{\delta - i} t^i C_k(t)^{ki-\epsilon_2} \\
&= [t^{n-\epsilon_1}] \sum_{j=0}^{\epsilon_1} \binom{\epsilon_1}{j} t^{j+\delta-\epsilon_1} C_k(t)^{k(j+\delta-\epsilon_1)-\epsilon_2} \\
&= [t^{n-\epsilon_1}] t^{\delta-\epsilon_1} C_k(t)^{k\delta-k\epsilon_1-\epsilon_2} \sum_{j=0}^{\epsilon_1} \binom{\epsilon_1}{j} (tC_k(t)^k)^j.
\end{aligned}$$

Recognizing the binomial expansion and applying the identity $C_k(t) = tC_k(t)^k + 1$ yields

$$\begin{aligned}
|\mathcal{P}_{n,\delta}^{k,\epsilon}| &= [t^{n-\delta}] C_k(t)^{k\delta-k\epsilon_1-\epsilon_2} (1 + tC_k(t)^k)^{\epsilon_1} \\
&= [t^{n-\delta}] C_k(t)^{k\delta-k\epsilon_1-\epsilon_2} C_k(t)^{\epsilon_1} = [t^{n-\delta}] C_k(t)^{k\delta-\epsilon}.
\end{aligned}$$

For the second range of parameters given, we separately consider $\epsilon < (k-1)\delta$ and $\epsilon = (k-1)\delta$. For the first subcase we always have $\epsilon < (k-1)\delta \leq (k-1)\delta + \epsilon_2$ and $\epsilon - \epsilon_2 = (k-1)\epsilon_1 < (k-1)\delta$, which implies $\epsilon_1 < \delta$ and allows us to apply our first result. When $\epsilon = (k-1)\delta$ we may choose $\epsilon_1 = \delta - 1$ and $\epsilon_2 = k - 1$, which again implies $\epsilon_1 < \delta$. □

3 Weak k-Path Pairs

In this section, we loosen our restriction that generalized k-path pairs (γ_1, γ_2) cannot intersect apart from $(0,0)$ and merely require that γ_1 stays weakly above γ_2. Formally, for any $k \geq 2$ and any set of non-negative integers n, ϵ, δ such that $0 \leq \epsilon \leq (k-1)n$ and $0 \leq \delta \leq n$, we define $\widetilde{\mathcal{P}}_{n,\delta}^{k,\epsilon}$ to be the collection of ordered pairs (γ_1, γ_2) of lattice paths that satisfy all of the following:

1. Both γ_1 and γ_2 begin at $(0,0)$ and use steps from $\{E = (1,0), N = (0,1)\}$.
2. γ_2 is composed of precisely $(k-1)n$ steps, the first of which is an E step.
3. γ_1 is composed of precisely $(k-1)n - \epsilon$ steps, the first of which is an N step.
4. γ_1 stays weakly above γ_2.
5. The difference between the terminal x coordinates of γ_1 and γ_2 is δ.
6. $\gamma_2 = E^1 N^{b_1} E^1 N^{b_2} \ldots E^1 N^{b_m}$ satisfies $b_i = (k-2) \bmod (k-1)$ for all i.

We refer to any element $(\gamma_1, \gamma_2) \in \widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta}$ as a *weak k-path pair of distance δ*. Notice that $\delta = 0$ is now possible when we also have $\epsilon = 0$, corresponding to the case where γ_1 and γ_2 terminate at the same point. We refer to this special case of $\delta = \epsilon = 0$ as a *closed (weak) k-path pair*. All nonempty sets $\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta}$ fall within the ranges $0 \leq \delta \leq n$ and $0 \leq \epsilon \leq (k-1)n$.

Elements of $(\gamma_1, \gamma_2) \in \widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta}$ may then be subdivided according to the number of intersections between γ_1 and γ_2. We let $\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta,m}$ denote the collection of $(\gamma_1, \gamma_2) \in \widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta}$ where γ_1 and γ_2 intersect precisely m times away from $(0, 0)$, and we define such path pairs to be weak k-path pairs with *m returns*. It is easy to show that $\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta,m}$ is empty unless $0 \leq m \leq n$, and that ϵ places further restrictions on which m are possible. For example, $m = n$ is only possible when $\epsilon = 0$.

We henceforth call a closed k-path pair with only $m = 1$ return as an *irreducible (closed) k-path pair*. Any weak k-path pair $(\gamma_1, \gamma_2) \in \widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta,m}$ with precisely m returns may be uniquely decomposed into a sequence of subpath pairs $(\gamma_{1,1}, \gamma_{2,1}), \ldots, (\gamma_{1,m+1}, \gamma_{2,m+1})$ such that $(\gamma_{1,i}, \gamma_{2,i})$ corresponds to an irreducible k-path pair for each $1 \leq i \leq m$ (after translating each subpath pair so that it begins at the origin). If (γ_1, γ_2) is a closed k-path pair, then the final subpath pair $(\gamma_{1,m+1}, \gamma_{2,m+1})$ is empty. Otherwise, that final subpath pair corresponds to some k-path pair $(\gamma_1', \gamma_2') \in \mathcal{P}^{k,\epsilon}_{n',\delta}$ for some $n' > 0$.

To enumerate $\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta}$ and the $\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta,m}$, we begin by enumerating irreducible k-path pairs:

Proposition 3.1. *Fix $k \geq 2$. For any $n \geq 1$,*

$$|\widetilde{\mathcal{P}}^{k,0}_{n,0,1}| = [t^{n-1}]C_k(t)^{k-1} = \frac{k-1}{kn-1}\binom{kn-1}{n-1}.$$

Proof. For any $(\gamma_1, \gamma_2) \in \widetilde{\mathcal{P}}^{k,0}_{n,0,1}$, observe that the final step of γ_1 must be an E step. This means that $\widetilde{\mathcal{P}}^{k,0}_{n,0,1}$ lies in bijection with $\mathcal{P}^{k,1}_{n,1}$, via the map the deletes the final step of γ_1. The result then follows from Corollary 2.3. □

Observe that $\widetilde{\mathcal{P}}^{2,0}_{n,0,1}$ is equivalent to the original notion of parallelogram polynominoes with semiperimeter n. Proposition 3.1 recovers this preexisting combinatorial interpretation of the Catalan numbers as $|\widetilde{\mathcal{P}}^{2,0}_{n,0,1}| = [t^{n-1}]C(t) = C_{n-1}$. For any $k \geq 2$, one could define the elements of $\widetilde{\mathcal{P}}^{k,0}_{n,0,1}$ as k-parallelogram polyominoes with semiperimeter $(k-1)n$, although for $k > 2$ these objects do not provide a combinatorial interpretation for the k-Catalan numbers.

The primary application of Proposition 3.1 is that it may be used to quickly enumerate any collection $\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta,m}$, assuming ϵ and δ fall within the range proscribed by Theorem 2.6:

Theorem 3.2. *Fix $n \geq 1$ and $k \geq 2$. For any non-negative integers δ, ϵ, m such that $\epsilon = \delta = 0$ or $0 \leq \epsilon \leq (k-1)\delta$,*

$$|\widetilde{\mathcal{P}}^{k,\epsilon}_{n,\delta,m}| \;=\; [t^{n-\delta-m}]C_k(t)^{k\delta-\epsilon+(k-1)m} \;=\; \frac{k\delta-\epsilon+(k-1)m}{kn-\epsilon-m}\binom{kn-\epsilon-m}{n-m-\delta}.$$

Proof. By Proposition 3.1, for any $k \geq 2$ the generating function of irreducible k-path pairs is $\sum_{i=0}^{\infty} |\widetilde{\mathcal{P}}_{n,0,1}^{k,0}| t^i = tC_k(t)^{k-1}$. From Theorem 2.6, when $0 \leq \epsilon < (k-1)\delta$ we also have the generating function $\sum_{i=0}^{\infty} |\mathcal{P}_{n,\delta}^{k,\epsilon}| t^i = t^\delta C_k(t)^{k\delta-\epsilon}$. We treat the two cases of the theorem statement separately.

For the $\epsilon = \delta = 0$ case, every element of $\widetilde{\mathcal{P}}_{n,0,m}^{k,0}$ may be uniquely decomposed into a sequence of m non-empty irreducible k-path pairs. It follows that

$$\sum_{i=0}^{\infty} |\widetilde{\mathcal{P}}_{i,0,m}^{k,0}| t^i = (tC_k(t)^{k-1})^m = t^m C_k(t)^{(k-1)m}.$$

In this case we then have

$$|\widetilde{\mathcal{P}}_{n,0,m}^{k,0}| = [t^n] t^m C_k(t)^{(k-1)m} = [t^{n-m}] C_k(t)^{(k-1)m}.$$

For the $0 \leq \epsilon < (k-1)\delta$ case, every element of $\widetilde{\mathcal{P}}_{n,\epsilon,m}^{k,\delta}$ may be uniquely decomposed into a sequence of m non-empty irreducible k-path pairs and an element of $\mathcal{P}_{n',\delta}^{k,\epsilon}$ for some $0 < n' < n - m$. Here we have

$$\sum_{i=0}^{\infty} |\widetilde{\mathcal{P}}_{i,\epsilon,m}^{k,\delta}| t^i = (tC_k(t)^{k-1})^m \, t^\delta C_k(t)^{k\delta-\epsilon} = t^{\delta+m} C_k(t)^{k\delta-\epsilon+(k-1)m}.$$

For this second case we then have

$$|\widetilde{\mathcal{P}}_{n,\epsilon,m}^{k,\delta}| = [t^n] t^{\delta+m} C_k(t)^{k\delta-\epsilon+(k-1)m} = [t^{n-\delta-m}] C_k(t)^{k\delta-\epsilon+(k-1)m}.$$

□

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(Received 7 July 2020; revised 1 Feb 2021)