

An explicit formula for the inverse of a factorial Hankel matrix

KAREN HABERMANN*

*Department of Statistics
University of Warwick
Coventry, CV4 7AL
United Kingdom*
karen.habermann@warwick.ac.uk

Abstract

We consider the $n \times n$ Hankel matrix H whose entries are defined by $H_{ij} = 1/s_{i+j}$ where $s_k = (k-1)!$ and prove that H is invertible for all $n \in \mathbb{N}$ by providing an explicit formula for its inverse matrix.

1 Introduction

Fix $n \in \mathbb{N}$ and let H be the $n \times n$ matrix given by, for $i, j \in \{1, \dots, n\}$,

$$H_{ij} = \frac{1}{(i+j-1)!}.$$

This defines a Hankel matrix because the entry H_{ij} depends only on the sum $i+j$. The factorial Hankel matrix H is used as a test matrix in numerical analysis and features as `gallery('ipjfact')` in the Matrix Computation Toolbox [5] by Nicholas Higham; also see [4] and [6]. Our interest in studying the matrix H is due to it arising in determining the covariance structure of an iterated Kolmogorov diffusion, that is, a Brownian motion together with a finite number of its iterated time integrals, see [2, Sec. 4.4] and [3, Sec. 3]. To find an explicit expression for a diffusion bridge associated with an iterated Kolmogorov diffusion, we need to invert its covariance matrix, which particularly requires us to invert the matrix H . It is therefore of interest, both from our point of view and for using H as a test matrix, to show that the matrix H is invertible and to obtain an explicit formula for its inverse. We use general binomial coefficients, which are discussed in more detail in Section 2.

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Theorem 1.1 *For all $n \in \mathbb{N}$, the inverse M of the Hankel matrix H exists and it is given by*

$$M_{ij} = (-1)^{n+i+j+1}(i-1)!j! \binom{n-1}{i-1} \binom{n+j-1}{j} \sum_{k=0}^{i-1} \binom{n-i+k}{j-1} \binom{n+k-1}{k}.$$

In particular, it immediately follows that all the entries of the inverse matrix M are integer-valued. An unpublished manuscript by Gover [1] already contains an explicit formula for the inverse of the factorial Hankel matrix H . However, our formula differs from the formula derived by Gover, and we employ a different proof technique. While Gover first determines expressions for the first row and last column of the inverse of H to then use a recursive procedure by Trench, see [9], to compute the remaining entries of the inverse matrix, we prove Theorem 1.1 directly by manipulating general binomial coefficients, and in particular without relying on any recursive procedures. For completeness, we add that the explicit formula [1, (3.17)] leads to

$$M_{ij} = n(-1)^{n-i-j-1} \sum_{k=\max(0,i+j-1-n)}^{i-1} \frac{(n+i+j-k-2)!(n+k-1)!(i+j-2k-1)}{(i+j-k-1)!k!(n+k-i-j+1)!(n-k)!},$$

which, for $m = i + j - 1$ and with binomial coefficients, Gover rewrites as

$$M_{ij} = (-1)^{n-m}n(m-1)! \sum_{k=\max(0,m-n)}^{i-1} \binom{n+m-k-1}{n-k} \binom{n+k-1}{n+k-m} \left(\binom{m-1}{k} - \binom{m-1}{k-1} \right).$$

We review two combinatorial identities in Section 2 which we frequently use in our manipulation of general binomial coefficients, before we give the proof of Theorem 1.1 in Section 3. Throughout, we use the convention that \mathbb{N} denotes the positive integers and \mathbb{N}_0 the non-negative integers.

2 Combinatorial identities

We use the notion of a general binomial coefficient which, for $t \in \mathbb{R}$ and $m \in \mathbb{N}_0$, is defined as

$$\binom{t}{m} = \prod_{i=1}^m \frac{t+1-i}{i} = \frac{t(t-1)\cdots(t-m+1)}{m!},$$

where it is understood that $\binom{t}{0} = 1$. Note that if $t \in \mathbb{N}_0$ and $t < m$ then

$$\binom{t}{m} = \left(\prod_{i=1}^t \frac{t+1-i}{i} \right) \frac{t+1-(t+1)}{t+1} \left(\prod_{i=t+2}^m \frac{t+1-i}{i} \right) = 0.$$

The first identity we frequently use in the proof of Theorem 1.1 is the reflection identity for general binomial coefficients; see Stanley [8, p. 26].

Proposition 2.1 (Reflection identity) For all $t \in \mathbb{R}$ and $m \in \mathbb{N}_0$, we have

$$\binom{t}{m} = (-1)^m \binom{m-t-1}{m}.$$

Secondly, we make use of the Chu-Vandermonde identity; e.g. see [7, Ch.3] or [8, p.22].

Proposition 2.2 (Chu-Vandermonde identity) For all $s, t \in \mathbb{R}$ and $m \in \mathbb{N}_0$, we have

$$\binom{s+t}{m} = \sum_{k=0}^m \binom{s}{k} \binom{t}{m-k}.$$

3 Inverse of a factorial Hankel matrix

To simplify the presentation of the proof of Theorem 1.1, we split up the analysis into two parts.

Lemma 3.1 For all $n \in \mathbb{N}$, we have, for $i, l \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $1 \leq i, l \leq n$ and $0 \leq k \leq i-1$,

$$\sum_{j=1}^n (-1)^j \frac{j!}{(l+j-1)!} \binom{n+j-1}{j} \binom{n-i+k}{j-1} = (-1)^{n+i+k+1} \frac{(n-l)!}{(n-1)!} \binom{n}{n+l-i+k}.$$

Proof: We observe that, for $j \in \{1, \dots, n\}$,

$$\frac{j!}{(l+j-1)!} \binom{n+j-1}{j} = \frac{j!}{(l+j-1)!} \frac{(n+j-1)!}{j!(n-1)!} = \frac{(n-l)!}{(n-1)!} \binom{n+j-1}{l+j-1}. \tag{3.1}$$

Moreover, the reflection identity for general binomial coefficients yields

$$\binom{n+j-1}{l+j-1} = (-1)^{l+j-1} \binom{l-n-1}{l+j-1}. \tag{3.2}$$

Using the equations (3.1) and (3.2), we obtain

$$\begin{aligned} & \sum_{j=1}^n (-1)^j \frac{j!}{(l+j-1)!} \binom{n+j-1}{j} \binom{n-i+k}{j-1} \\ &= \sum_{j=1}^n (-1)^j \frac{(n-l)!}{(n-1)!} \binom{n+j-1}{l+j-1} \binom{n-i+k}{j-1} \\ &= (-1)^{l-1} \frac{(n-l)!}{(n-1)!} \sum_{j=1}^n \binom{l-n-1}{l+j-1} \binom{n-i+k}{j-1}. \end{aligned}$$

If $j > n - i + k + 1$, that is, if $n - i + k < j - 1$, we have

$$\binom{n - i + k}{j - 1} = 0$$

since $i \leq n$ guarantees that $n - i + k \geq 0$. From $k \leq i - 1$, it also follows that $n - i + k + 1 \leq n$. The symmetry rule for binomial coefficients and reindexing the sum then give

$$\begin{aligned} \sum_{j=1}^n \binom{l - n - 1}{l + j - 1} \binom{n - i + k}{j - 1} &= \sum_{j=1}^{n-i+k+1} \binom{l - n - 1}{l + j - 1} \binom{n - i + k}{n - i + k - j + 1} \\ &= \sum_{a=l}^{n+l-i+k} \binom{l - n - 1}{a} \binom{n - i + k}{n + l - i + k - a}. \end{aligned}$$

By noting that for $a \in \mathbb{N}_0$ with $a < l$, we have $n - i + k < n + l - i + k - a$ and therefore,

$$\binom{n - i + k}{n + l - i + k - a} = 0,$$

and by applying the Chu-Vandermonde identity, we deduce that

$$\begin{aligned} \sum_{a=l}^{n+l-i+k} \binom{l - n - 1}{a} \binom{n - i + k}{n + l - i + k - a} &= \sum_{a=0}^{n+l-i+k} \binom{l - n - 1}{a} \binom{n - i + k}{n + l - i + k - a} \\ &= \binom{l - i + k - 1}{n + l - i + k}. \end{aligned}$$

Putting our conclusions together, and using the reflection identity for general binomial coefficients a second time, we obtain

$$\begin{aligned} \sum_{j=1}^n (-1)^j \frac{j!}{(l+j-1)!} \binom{n+j-1}{j} \binom{n-i+k}{j-1} &= (-1)^{l-1} \frac{(n-l)!}{(n-1)!} \binom{l-i+k-1}{n+l-i+k} \\ &= (-1)^{n+i+k+1} \frac{(n-l)!}{(n-1)!} \binom{n}{n+l-i+k}, \end{aligned}$$

as claimed. □

Let δ_{il} denote the Kronecker delta for $i, l \in \mathbb{N}$.

Lemma 3.2 *For all $n \in \mathbb{N}$ and all $i, l \in \mathbb{N}$ with $1 \leq i, l \leq n$, we have*

$$\sum_{k=0}^{i-1} (-1)^k \binom{n}{n+l-i+k} \binom{n+k-1}{k} = \delta_{il}.$$

Proof: For $k \in \mathbb{N}_0$, if $k > i - l$ then $n < n + l - i + k$ and therefore,

$$\binom{n}{n + l - i + k} = 0.$$

In particular, if $l > i$, that is, if $0 > i - l$, we immediately obtain

$$\sum_{k=0}^{i-1} (-1)^k \binom{n}{n + l - i + k} \binom{n + k - 1}{k} = 0.$$

Let us now suppose that $l \leq i$. By the reflection identity for general binomial coefficients, we know

$$(-1)^k \binom{n + k - 1}{k} = \binom{-n}{k},$$

and, by reindexing the sum, it follows that

$$\begin{aligned} \sum_{k=0}^{i-1} (-1)^k \binom{n}{n + l - i + k} \binom{n + k - 1}{k} &= \sum_{k=0}^{i-l} \binom{n}{n + l - i + k} \binom{-n}{k} \\ &= \sum_{b=0}^{i-l} \binom{n}{n - b} \binom{-n}{i - l - b}. \end{aligned}$$

Using the symmetry rule for binomial coefficients and the Chu-Vandermonde identity, we deduce

$$\sum_{b=0}^{i-l} \binom{n}{n - b} \binom{-n}{i - l - b} = \sum_{b=0}^{i-l} \binom{n}{b} \binom{-n}{i - l - b} = \binom{0}{i - l} = \delta_{il}.$$

Thus, we established the desired identity both for $l > i$ and for $l \leq i$. □

Combining both results gives the proof of Theorem 1.1.

Proof of Theorem 1.1: By first applying Lemma 3.1 and then Lemma 3.2, we conclude that, for all $n \in \mathbb{N}$ and all $i, l \in \{1, \dots, n\}$,

$$\begin{aligned} (MH)_{il} &= \sum_{j=1}^n M_{ij} H_{jl} \\ &= \sum_{j=1}^n \left((-1)^{n+i+j+1} \frac{(i-1)! j!}{(l+j-1)!} \binom{n-1}{i-1} \binom{n+j-1}{j} \sum_{k=0}^{i-1} \binom{n-i+k}{j-1} \binom{n+k-1}{k} \right) \\ &= (-1)^{n+i+1} (i-1)! \binom{n-1}{i-1} \sum_{k=0}^{i-1} (-1)^{n+i+k+1} \frac{(n-l)!}{(n-1)!} \binom{n}{n+l-i+k} \binom{n+k-1}{k} \\ &= (i-1)! \binom{n-1}{i-1} \frac{(n-l)!}{(n-1)!} \sum_{k=0}^{i-1} (-1)^k \binom{n}{n+l-i+k} \binom{n+k-1}{k} \\ &= \frac{(n-l)!}{(n-i)!} \delta_{il} = \delta_{il}. \end{aligned}$$

Hence, M is indeed the inverse matrix of the factorial Hankel matrix H . □

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References

- [1] M. J. C. Gover, *The Explicit Inverse of Factorial Hankel Matrices*, Department of Mathematics, University of Bradford, 1993.
- [2] K. Habermann, Small-time fluctuations for the bridge in a model class of hypoelliptic diffusions of weak Hörmander type, *Electron. J. Probab.* **24** (2019), Paper No. 11, 1–19.
- [3] K. Habermann, A semicircle law and decorrelation phenomena for iterated Kolmogorov loops, *J. Lond. Math. Soc.* (2020), OnlineOpen.
- [4] D. J. Higham and N. J. Higham, *MATLAB Guide*, SIAM, Philadelphia, third ed., 2017.
- [5] N. J. Higham, The Matrix Computation Toolbox, <http://www.ma.man.ac.uk/~higham/mctoolbox>.
- [6] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, SIAM, Philadelphia, second ed., 2002.
- [7] W. Koepf, *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities*, Advanced Lectures in Mathematics, Vieweg, Braunschweig, 1998.
- [8] R. P. Stanley, *Enumerative Combinatorics, Volume 1*, vol. 49 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, second ed., 2012.
- [9] W. F. Trench, An algorithm for the inversion of finite Toeplitz matrices, *J. Soc. Indust. Appl. Math.* **12** (1964), 515–522.

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