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Signed Langford sequences and directed cyclic cycle systems

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Abstract

For positive integers d and t, a Langford sequence of order t and defect d is a sequence $\mathcal{L}_d^t = (s_1, \ldots, s_{2t})$ of length $2t$ that satisfies (i) for every $k \in \{d, d+1, \ldots, t+d-1\}$, there are exactly two elements $s_i, s_j \in \mathcal{L}_d^t$ such that $s_i = s_j = k$ and (ii) if $s_i = s_j = k$ with $i < j$, then $j - i = k$. Note that (ii) could be written as $j - i - k = 0$ or $i + k - j = 0$. Hence, one extension of a Langford sequence is as follows. For positive integers d and t, a signed Langford sequence of order t and defect d is a sequence $\pm\mathcal{L}_d^t = (s_{-2t}, s_{-2t+1}, \ldots, s_{-1}, *, s_1, \ldots, s_{2t})$ of length $4t+1$ that satisfies (i) for every $k \in \{\pm d, \pm(d+1), \ldots \pm (t+d-1)\}$, there are exactly two elements $s_i, s_j \in \pm\mathcal{L}_d^t$ such that $s_i = s_j = k$ and (ii) if $s_i = s_j = k$ with $i < 0 < j$, then $i + j + k = 0$. Here we give necessary and sufficient conditions for the existence of a signed Langford sequence of order t and defect d for $d \in \{1, 2, 3\}$. We also use these sequences to find cyclic decompositions of circulant digraphs into directed m-cycles for $m \geq 3$. In particular, we find a cyclic m-cycle decomposition of the complete symmetric digraph K_{2m+1}^*.

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1 Introduction

For integers a and b, the notation $[a, b]$ denotes the set $\{a, a+1, \ldots, b\}$ and $\pm[a, b]$ denotes the set $\{\pm a, \pm(a+1), \ldots, \pm b\}$.

A *Langford sequence of order t and defect d* is a sequence $\mathcal{L}_d^t = (\ell_1, \ell_2, \ldots, \ell_{2t})$ of $2t$ integers that satisfies

(L1) for every $k \in [d, d+t-1]$ there are exactly two elements $\ell_i, \ell_j \in \mathcal{L}_d^t$ such that $\ell_i = \ell_j = k$; and

(L2) if $\ell_i = \ell_j = k$ with $i < j$, then $j - i = k$.

A Langford sequence with defect $d = 1$ is called a *Skolem sequence*, and necessary and sufficient conditions for the existence of Skolem sequences are well known.

Theorem 1.1 (Skolem [18]) *For a positive integer t, a Skolem sequence of order t exists if and only if $t \equiv 0, 1 \pmod 4$.*

Necessary and sufficient conditions for the existence of Langford sequences are also known. In [9], Davies handled the case in which $d = 2$ while Bermond, Brouwer and Germa handled the cases in which $d = 3$ and $d = 4$ in [2]. For any $d \geq 5$, the case in which t is odd was also handled in [2] while the case in which t is even was handled by Simpson in [17].

Theorem 1.2 (Davies [9], Bermond, Brouwer, Germa [2], Simpson [17]) *There exists a Langford sequence of order t and defect d if and only if*

1. $t \geq 2d - 1$, *and*
2. $t \equiv 0, 1 \pmod 4$ *and d is odd, or* $t \equiv 0, 3 \pmod 4$ *and d is even.*

Skolem sequences and their generalizations have been used widely in the construction of combinatorial designs and a survey on Skolem sequences by Francetić and Mendelsohn can be found in [11]. Note that in (L2) above, we may write $j - i - k = 0$ or $i + k - j = 0$. In this paper, we are interested in a generalization of Langford sequences, called *signed* Langford sequences, in which both positive and negative integers appear. In Section 3, we give necessary and sufficient conditions for the existence of signed Langford sequences for some small values of d.

In combinatorial design theory, a well-studied problem is decomposing graphs into cycles (see the survey [8] by Bryant and Rodger), and in particular, decompositions that behave nicely from an algebraic point-of-view, the so-called *cyclic* decomposition (see, for example, [3, 4, 5, 6, 7, 10, 12, 14, 15, 16, 19, 20]). In fact, an application of Skolem sequences gives cyclic 3-cycle systems of complete graphs. Here we are interested in using signed Langford sequences to find directed cyclic cycle decompositions. Necessary and sufficient conditions for a directed m-cycle system of the complete symmetric digraph were given by Alspach, Šajna, Verrall and the second author in [1]; however, very little is known about directed cyclic cycle decompositions. In fact the only directed cyclic m-cycle systems known to exist are the ones in which m is as large as possible, i.e., directed cyclic hamiltonian cycle systems. Necessary and sufficient conditions for directed cyclic hamiltonian cycle systems were given by Morris and the second author in [13].

In Section 4, we extend the results of Section 3 to construct difference sets of m-tuples for $m \geq 3$ for use in Section 5 where cyclic m-cycle decompositions of circulant digraphs, including cyclic m-cycle decompositions of complete symmetric digraphs, are given.

2 Definitions and Preliminaries

In a Langford sequence $\mathcal{L}_d^t = (\ell_1, \ell_2, \dots, \ell_{2t})$, we know that whenever $\ell_i = \ell_j = k$, then $j - i = k$ where necessarily i, j, and k are all positive integers. Note that this equation could be written as $j - i - k = 0$ (or $i + k - j = 0$) so that one might consider introducing negative integers in such a sequence.

Definition 2.1 A *signed Langford sequence of order t and defect d* is a sequence $\pm\mathcal{L}_d^t = (\ell_{-2t}, \ell_{-2t+1}, \dots, \ell_{-1}, *, \ell_1, \ell_2, \dots, \ell_{2t})$ of length $4t + 1$ that satisfies

(S1) for every $k \in \pm[d, t + d - 1]$ there are exactly two elements $\ell_i, \ell_j \in \pm\mathcal{L}_d^t$ such that $\ell_i = \ell_j = k$, and

(S2) if $\ell_i = \ell_j = k$ with $i < 0 < j$, then $i + j + k = 0$.

For $t = 5$ and defect $d = 2$, one such sequence is:

$$(5, 6, 4, -2, -4, 3, -3, 2, -6, -5, *, 2, 3, 6, 4, 5, -5, -3, -6, -2, -4). \tag{1}$$

A signed Langford sequence of order t and defect $d = 1$ will be called a *signed Skolem sequence of order t*. For example, a signed Skolem sequence of order 3 is

$$(3, -1, 2, -2, 1, -3, *, 1, 2, 3, -3, -2, -1). \tag{2}$$

Signed Langford sequences also have the very nice property that if

$$(\ell_{-2t}, \ell_{-2t+1}, \dots, \ell_{-1}, *, \ell_1, \ell_2, \dots, \ell_{2t})$$

is a signed Langford sequence of order t and defect d, then

$$(-\ell_{2t}, -\ell_{2t-1}, \dots, -\ell_1, *, -\ell_{-1}, -\ell_{-2}, \dots, -\ell_{-2t})$$

is also a signed Langford sequence. So,

$$(1, 2, 3, -3, -2, -1, *, 3, -1, 2, -2, 1, -3)$$

is also a signed Skolem sequence of order 3.

A signed Langford sequence of order t and defect d provides a partition of the set $\pm[d, 3t+d-1]$ into $2t$ triples (a_i, b_i, c_i) such that $a_i + b_i + c_i = 0$; for example, if $\pm\mathcal{L}_d^t = (\ell_{-2t}, \ldots, \ell_{2t})$, then $\{(k, i-(t+d-1), j+t+d-1) \mid 1 \leq k \leq t$ or $-t \leq k \leq -1,\ \ell_i = \ell_j = k$ with $i < j\}$ is a partition of $\pm[d, 3t+d-1]$ into $2t$ such triples. The 10 such triples from the signed Langford sequence of order $t = 5$ and defect $d = 2$ in (1) are

$$\begin{aligned}\{&(-2,-13,15),(2,-9,7),(-3,-10,13),(3,-11,8),(-4,-12,16),\\&(4,-14,10),(-5,-7,12),(5,-16,11),(-6,-8,14),(6,-15,9)\}.\end{aligned} \tag{3}$$

We will use such a partition to show that for a signed Langford sequence of order t and defect d to exist, as in the case of Langford sequences, it must be the case that $t \geq 2d-1$.

Lemma 2.2 *If a signed Langford sequence of order t and defect d exists, then $t \geq 2d-1$.*

Proof. Suppose a $\pm\mathcal{L}_d^t$ of order t and defect d exists. Then, we have a partition of $\pm[d, d+3t-1]$ into $2t$ 3-tuples $a_i+b_i+c_i = 0$ with $|a_i| < |b_i| < |c_i|$ for $i = 1, 2, \ldots, 2t$. Note that we may write each 3-tuple as $|a_i| + |b_i| = |c_i|$ for $i = 1, 2, \ldots, 2t$ and that every integer in the set $[d, d+3t-1]$ appears twice. Hence,

$$\sum_{i=1}^{2t}(|a_i| + |b_i| + |c_i|) \;=\; 2(d + (d+1) + \cdots + (d+3t-1)) = 9t^2 - 3t + 6td.$$

Next, since $|a_i| + |b_i| = |c_i|$ for $i = 1, 2, \ldots, 2t$,

$$\sum_{i=1}^{2t}(|a_i| + |b_i|) = \sum_{i=1}^{2t}|c_i| = \frac{9t^2 - 3t + 6td}{2}$$

and

$$\sum_{i=1}^{2t}|c_i| \;\leq\; 2[(d+2t) + (d+2t+1) + \cdots + (d+3t-1)] = 5t^2 - t + 2td$$

so that $(9t^2 - 3t + 6td)/2 \leq 5t^2 - t + 2td$ and hence $2d - 1 \leq t$. □

Next we obtain signed Langford sequences of order t and defect d from Langford sequences of order t and defect d.

Lemma 2.3 *If a Langford sequence $\mathcal{L}_d^t$ of order t and defect d exists, then a signed Langford sequence $\pm\mathcal{L}_d^t$ of order t and defect d also exists.*

Proof. Let $\mathcal{L}_d^t = (\ell_1, \ell_2, \ldots, \ell_{2t})$ be a Langford sequence of order t and defect d. Then, a signed Langford sequence

$$\pm\mathcal{L}_d^t = (s_{-2t}, s_{-2t+1}, \ldots, s_{-1}, *, s_1, s_2, \ldots, s_{2t})$$

of order t and defect d can be found by defining $s_j = s_{-i} = -k$ and $s_{-j} = s_i = k$ if $\ell_i = \ell_j = k$ with $i < 0 < j$, for each $k = d, d+1, \ldots, d+t-1$. □

Yet another way to obtain a signed Langford sequence is to *compose* two of them in the following fashion.

Definition 2.4 Let d, t and s be positive integers. The *composition* of two signed Langford sequences

$$\pm\mathcal{L}_d^t = (\ell_{-2t}, \ell_{-2t+1}, \ldots, \ell_{-1}, *, \ell_1, \ell_2, \ldots, \ell_{2t})$$

and

$$\pm\mathcal{L}_{d+t}^s = (a_{-2s}, a_{-2s+1}, \ldots, a_{-1}, *, a_1, a_2, \ldots, a_{2s})$$

is the signed Langford sequence

$$\pm\mathcal{L}_d^{t+s} = (b_{-2(t+s)}, b_{-2(t+s)+1}, \ldots, b_{-1}, *, b_1, b_2, \ldots, b_{2(t+s)})$$

whose entries are given by

$$b_k = \begin{cases} \ell_k & \text{if } 1 \le |k| \le 2t \\ a_{k-2t} & \text{if } 2t < k \le 2(t+s) \\ a_{k+2t} & \text{if } -2(t+s) \le k < -2t \end{cases}.$$

3 Signed Langford Sequences for Small Values of d

In this section, we give necessary and sufficient conditions for the existence of signed Langford sequences of order t and defect d for $d \in \{1, 2, 3\}$. We begin by showing that, for every positive integer t, there exists a signed Skolem sequence of order t, in contrast to Skolem sequences which only exist for $t \equiv 0, 1 \pmod 4$.

Theorem 3.1 *For every positive integer t, there exists a signed Skolem sequence of order t.*

Proof. Let $t \geq 1$ be an integer. Then

$$(t, -1, t-1, -2, \ldots, 1, -t, *, 1, 2, \ldots, t, -t, -(t-1), \ldots, -1)$$

is a signed Skolem sequence of order t. □

An example with $t = 3$ of the signed Skolem sequence given in the above proof can be found in (2).

We now consider signed Langford sequences of order t and defect $d = 2$. By Lemma 2.2, we may assume $t \geq 3$. In order to show that signed Langford sequences exists for all $t \geq 3$ with defect $d = 2$, we need to construct such sequences for some small values of t. For a signed Langford sequence $\pm\mathcal{L}_d^t$, let $^+\mathcal{L}_d^t$ denote the part of $\pm\mathcal{L}_d^t$ with positive subscripts while $^-\mathcal{L}_d^t$ will denote the part of $\pm\mathcal{L}_d^t$ with negative subscripts. Hence in what follows, we need give only $^+\mathcal{L}_d^t = (s_1, s_2, \ldots, s_{2t})$ as determining $^-\mathcal{L}_d^t$ can be done as follows: if $s_j = k$ for $1 \leq j \leq 2t$, then $s_{-(j+k)} = k$.

Lemma 3.2 *For every $t \in \{6, 9, 10, 13, 14, 17\}$, there exists a signed Langford sequence of order t and defect $d = 2$.*

Proof. For each value of $t \in \{6, 9, 10, 13, 14, 17\}$, the sequence $^+\mathcal{L}_2^t$ is given below:

$$\begin{aligned}
^+\mathcal{L}_2^6 &= (3,4,2,7,5,6,-5,-7,-6,-3,-2,-4),\\
^+\mathcal{L}_2^9 &= (2,5,3,4,6,7,9,10,8,-8,-10,-7,-9,-4,-6,-2,-5,-3),\\
^+\mathcal{L}_2^{10} &= (3,4,2,6,8,5,7,11,9,10,-9,-11,-10,-6,-8,-7,-5,-3,-2,-4),\\
^+\mathcal{L}_2^{13} &= (2,5,3,4,6,9,7,8,10,11,13,14,12,-12,-14,-11,-13,-8,-10,-7,-9,\\
&\quad -4,-6,-2,-5,-3),\\
^+\mathcal{L}_2^{14} &= (2,5,3,4,6,8,10,7,9,11,12,14,15,13,-13,-15,-12,-14,-9,-11,-8,\\
&\quad -10,-7,-4,-6,-2,-5,-3), \text{ and}\\
^+\mathcal{L}_2^{17} &= (2,5,3,4,6,9,7,8,10,13,11,12,14,15,17,18,16,-16,-18,-15,-17,\\
&\quad -12,-14,-11,-13,-8,-10,-7,-9,-4,-6,-2,-5,-3).
\end{aligned}$$

□

We now show that a signed Langford sequence exists for every $t \geq 3$ and defect $d = 2$.

Theorem 3.3 *For every positive integer $t \geq 3$, there exists a signed Langford sequence of order t and defect $d = 2$.*

Proof. Let $t \geq 3$ be an integer. By Theorem 1.2, for defect $d = 2$, an $\mathcal{L}_2^t$ exists if and only if $t \equiv 0, 3 \pmod 4$ and hence by Lemma 2.3, a $\pm\mathcal{L}_2^t$ exists for $t \equiv 0, 3 \pmod 4$. Hence, we need only consider values of t with $t \equiv 1, 2 \pmod 4$. Also, by Theorem 1.2, there exists an $\mathcal{L}_7^s$ for any $s \equiv 0, 1 \pmod 4$ with $s \geq 13$. Hence, by Lemma 2.3, there exists a $\pm\mathcal{L}_7^s$ for any $s \equiv 0, 1 \pmod 4$ with $s \geq 13$. Now, a $\pm\mathcal{L}_2^5$ was given in (1). Thus, from Definition 2.4, composing a $\pm\mathcal{L}_2^5$ with a $\pm\mathcal{L}_7^s$ for $s \geq 13$ with $s \equiv 0, 1 \pmod 4$ gives a $\pm\mathcal{L}_2^t$ for all $t \equiv 1, 2 \pmod 4$ with $t \geq 18$. Hence, it remains to construct a $\pm\mathcal{L}_2^t$ for $t \in \{6, 9, 10, 13, 14, 17\}$. These exist by Lemma 3.2. □

We now consider signed Langford sequences with defect $d = 3$. By Lemma 2.2, we may assume $t \geq 5$. Again, we begin by constructing such sequences for some small values of t.

Lemma 3.4 *For every $t \in \{6, 7, 10, 11, 14, 15, 18, 19, 22\}$, there exists a signed Langford sequence of order t and defect $d = 2$.*

Proof. For each value of $t \in \{6, 7, 10, 11, 14, 15, 18, 19, 22\}$, the sequence ${}^{+}\mathcal{L}_2^t$ is given below:

$$
\begin{aligned}
{}^{+}\mathcal{L}_3^6 &= (3, 7, 5, 8, 6, 4, -6, -5, -3, -8, -4, -7)\\
{}^{+}\mathcal{L}_3^7 &= (3, 7, 5, 8, 9, 4, 6, -6, -4, -9, -8, -5, -7, -3),\\
{}^{+}\mathcal{L}_3^{10} &= (5, 3, 4, 8, 9, 12, 6, 11, 7, 10, -7, -11, -10, -12, -6, -5, -9, -8, -4, -3),\\
{}^{+}\mathcal{L}_3^{11} &= (4, 10, 6, 3, 5, 9, 7, 12, 13, 11, 8, -7, -12, -11, -13, -10, -9, -7, -6, -4,\\
&\quad -3, -5),\\
{}^{+}\mathcal{L}_3^{14} &= (3, 5, 8, 4, 11, 7, 10, 6, 9, 15, 13, 16, 14, 12, -10, -14, -16, -15, -13, -11,\\
&\quad -9, -12, -8, -3, -6, -4, -7, -5),\\
{}^{+}\mathcal{L}_3^{15} &= (3, 8, 6, 4, 9, 10, 5, 7, 11, 15, 13, 16, 17, 12, 14, -14, -12, -17, -16, -13, -15,\\
&\quad -11, -10, -6, -8, -5, -4, -9, -7, -3),\\
{}^{+}\mathcal{L}_3^{18} &= (3, 5, 8, 4, 11, 7, 10, 6, 9, 15, 13, 19, 14, 12, 17, 20, 18, 16, -14, -18, -20, -19,\\
&\quad -17, -15, -13, -16, -12, -7, -9, -11, -8, -10, -4, -6, -5, -3),\\
{}^{+}\mathcal{L}_3^{19} &= (4, 10, 6, 3, 5, 9, 7, 12, 13, 11, 8, 16, 17, 21, 14, 15, 20, 18, 19, -16, -19, -21,\\
&\quad -20, -18, -17, -15, -14, -12, -11, -13, -7, -6, -10, -9, -8, -4, -3, -5),\\
&\quad \text{and}\\
{}^{+}\mathcal{L}_3^{22} &= (3, 5, 8, 4, 11, 7, 10, 6, 9, 12, 13, 17, 15, 18, 19, 14, 16, 23, 21, 24, 22, 20, -18, -22,\\
&\quad -24, -23, -21, -19, -17, -20, -16, -11, -14, -9, -15, -13, -10, -12, -8,\\
&\quad -3, -6, -4, -7, -5).
\end{aligned}
$$

□

We now show that a signed Langford sequence exists for every $t \geq 3$ and defect $d = 3$.

Theorem 3.5 *For every positive integer $t \geq 5$, there exists a signed Langford sequence of order t and defect $d = 3$.*

Proof. Let $t \geq 5$ be an integer. By Theorem 1.2, for defect $d = 3$, an $\mathcal{L}_3^t$ exists if and only if $t \equiv 0, 1 \pmod 4$ and hence by Lemma 2.3, there exists a $\pm\mathcal{L}_3^t$ for $t \equiv 0, 1 \pmod 4$. Hence, we need only consider values of t with $t \equiv 2, 3 \pmod 4$. Also, by Theorem 1.2, there exists an $\mathcal{L}_9^s$ for any $s \equiv 0, 1 \pmod 4$ with $s \geq 17$. Hence, by Lemma 2.3, there exists a $\pm\mathcal{L}_9^s$ for any $s \equiv 0, 1 \pmod 4$ with $s \geq 17$. Now, a $\pm\mathcal{L}_3^6$ of order 6 and defect 3 exists by Lemma 3.4. Thus, from Definition 2.4, composing a $\pm\mathcal{L}_3^6$ with a $\pm\mathcal{L}_9^s$ for $s \geq 17$ with $s \equiv 0, 1 \pmod 4$ gives a $\pm\mathcal{L}_3^t$ for all $t \equiv 2, 3 \pmod 4$ with $t \geq 23$. Hence, it remains to construct a $\pm\mathcal{L}_3^t$ for $t \in \{7, 10, 11, 14, 15, 18, 19, 22\}$. These exist by Lemma 3.4. □

4 Signed Langford m-tuple Difference Sets

As we are interested in m-cycle decompositions, our interest is in the triples, or m-tuples, obtained from a (signed) Langford sequence and hence we need the following definitions from [7], modified here to include both positive and negative integers.

Definition 4.1 An m-tuple $(d_1, d_2, \ldots, d_m)$ is of *Skolem-type* if $d_1+d_2+\cdots+d_m = 0$. A set $\{(d_{i,1}, d_{i,2}, \ldots, d_{i,m}) \mid i = 1, 2, \ldots, 2t\}$ of $2t$ Skolem-type m-tuples such that $\{d_{i,j} \mid 1 \le i \le 2t, 1 \le j \le m\} = \pm[d, mt + d - 1]$ is called a *signed Langford m-tuple difference set of order t and defect d.*

The results of Section 3 give signed Langford 3-tuple difference sets of order t and defect $d \in \{1, 2, 3\}$ and hence signed Langford 3-tuple difference sets. For example, in Section 2, the 3-tuple difference set given in (3) is a signed Langford 3-tuple difference set of order $t = 5$ and defect $d = 2$. We now wish to find signed Langford m-tuple difference sets for $m > 3$. The following array $Y(r, n, t)$ will play a crucial role in finding these difference sets, given in [7].

Definition 4.2 Let $Y'(r, n, t)$ be the $t \times 4r$ matrix

$$\begin{bmatrix} 1 & 2 & 2t+1 & 2t+2 & & (4r-2)t+1 & (4r-2)t+2 \\ 3 & 4 & 2t+3 & 2t+4 & & (4r-2)t+3 & (4r-2)t+4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 2t-3 & 2t-2 & 4t-3 & 4t-2 & & 4rt-3 & 4rt-2 \\ 2t-1 & 2t & 4t-1 & 4t & & 4rt-1 & 4rt \end{bmatrix} + \begin{bmatrix} n & \cdots & n \\ & & \\ \vdots & \ddots & \vdots \\ & & \\ n & \cdots & n \end{bmatrix}$$

and let $Y(r, n, t) = [y_{i,j}]$ be the $t \times 4r$ matrix obtained from $Y'(r, n, t)$ by multiplying each entry in column j by -1 for all $j \equiv 2, 3 \pmod 4$. Note that $\{|y_{i,j}| \mid 1 \le i \le t, 1 \le j \le 4r\} = [n + 1, n + 4rt]$, the sum of the entries in each row of $Y(r, n, t)$ is zero, and $|y_{i,1}| < |y_{i,2}| < \ldots < |y_{i,4r}|$ for $i = 1, 2, \ldots, t$.

As a signed Langford $4r$-tuple difference set of order t and defect d can be constructed from the $2t$ rows of the $t \times 4r$ array $Y(r, d, t)$ followed by the $t \times 4r$ array $-Y(r, d, t)$, we have the following result.

Lemma 4.3 *For positive integers d, r and t, there exists a signed Langford $4r$-tuple difference set of order t and defect d.*

Thus, for all positive integers d and t, there exists a signed Langford m-tuple difference set of order t and defect d with $m \ge 4$ and $m \equiv 0 \pmod 4$. We now handle the case in which $m \equiv 2 \pmod 4$.

Lemma 4.4 *For positive integers d, m and t with $m \ge 6$ and $m \equiv 2 \pmod 4$, there exists a signed Langford m-tuple difference set of order t and defect d.*

Proof. Let d, m and t be positive integers such that $m \geq 6$ and $m \equiv 2 \pmod 4$. Let

$$Z = \left[\begin{array}{llllllc} 1 & -2 & 3 & -4 & -5 & 7 & \\ 6 & -8 & 10 & -9 & -11 & 12 & \\ 13 & -14 & 15 & -16 & -17 & 19 & \\ 18 & -20 & 22 & -21 & -23 & 24 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & Y(\frac{m-6}{4}, 6t, t) \\ 6t-11 & -(6t-10) & 6t-9 & -(6t-8) & -(6t-7) & 6t-5 & \\ 6t-6 & -(6t-4) & 6t-2 & -(6t-3) & -(6t-1) & 6t & \end{array}\right]$$

where $Y(\frac{m-6}{4}, 6t, t)$ is the $t \times \frac{m-6}{4}$ matrix given in Definition 4.2. Then, the $2t$ rows of Z followed by $-Z$ give a signed Langford m-tuple difference set of order t and defect 1. To construct a signed Langford m-tuple difference set of order t and defect d, create the $t \times m$ array Z' by adding $d-1$ to every positive entry and subtracting $d-1$ from every negative entry of Z. Then, since each row of Z has $m/2$ positive entries and $m/2$ negative entries, the sum of each row is still 0. Hence, the $2t$ rows of Z' followed by $-Z'$ give a signed Langford m-tuple difference set of order t and defect d. □

Hence, signed Langford m-tuple difference sets of order t and defect d exist for all positive integers t and d and positive even integers $m \geq 4$. We now consider the case when m is odd and begin with the case in which $m \equiv 3 \pmod 4$.

Lemma 4.5 *For all positive integers d and t and for every positive integer $m \equiv 3 \pmod 4$, if there exists a signed Langford 3-tuple difference set of order t and defect d, then there exists a signed Langford m-tuple difference set of order t and defect d.*

Proof. Let d and t be positive integers, and let m be a positive integer such that $m \equiv 3 \pmod 4$. Assume there exists signed Langford 3-tuple difference set of order t and defect d. These $2t$ difference 3-tuples will form the rows of a $2t \times 3$ array X' such that entries in each row sum to zero and are from the set $\pm[d, 3t+d-1]$. Augment the columns of X' with the $2t \times (m-3)$ array

$$\left[\begin{array}{r} Y(\frac{m-3}{4}, 3t+d-1, t) \\ -Y(\frac{m-3}{4}, 3t+d-1, t) \end{array}\right]$$

where $Y(\frac{m-3}{4}, 3t+d-1, t)$ is the $t \times \frac{m-3}{4}$ array given in Definition 4.2 to obtain a $2t \times m$ array X. Note again that every integer in the set $\pm[d, mt+d-1]$ appears in X and that for each $i = 1, 2, \ldots, 2t$, we have $x_{i,1} + x_{i,2} + \cdots + x_{i,m} = 0$. Thus, the $2t$ rows of X give a signed Langford m-tuple difference set of order t and defect d. □

Therefore, by Theorems 1.2, 3.1, 3.3, and 3.5, signed Langford m-tuple difference sets of order t and defect $d = 1$, $d = 2$ with $t \geq 3$, and $d = 3$ with $t \geq 5$ exist for all positive integers $m \equiv 3 \pmod 4$. We now consider the case in which $m \equiv 1 \pmod 4$ and find signed Langford m-tuple difference sets of order t and defect d for all positive integers $m \equiv 1 \pmod 4$ for which there exists a signed Langford 3-tuple difference set of order t and defect d.

Lemma 4.6 *For all positive integers d and t and for every positive integer $m \geq 5$ with $m \equiv 1 \pmod 4$, if there exists a signed Langford 3-tuple difference set of order t and defect d such that no two elements of the set $[t+d, 3t+d-1]$ belong to the same 3-tuple, then there exists a signed Langford m-tuple difference set of order t and defect d.*

Proof. Let $t \geq 1$ be an integer, and let $m \geq 5$ be a positive integer such that $m \equiv 1 \pmod 4$. Assume there exists a signed Langford 3-tuple difference set of order t and defect d such that no two elements of the set $[t+d, 3t+d-1]$ belong to the same 3-tuple. These $2t$ difference 3-tuples will form the rows of a $2t \times 3$ array $X' = [x_{i,j}]$ such that entries in each row sum to zero and are from the set $\pm[d, 3t+d-1]$. Furthermore, interchanging rows as necessary, we may assume that the first column of X' is $\begin{bmatrix} 3t+d-1 & 3t+d-2 & \cdots & t+d \end{bmatrix}^T$. Let X be the $2t \times m$ array whose first 5 columns are defined as follows:

$$\begin{bmatrix} x_{2,1} & x_{1,2} & x_{1,3} & -(3t+d) & 3t+d+1 \\ x_{1,1} & x_{2,2} & x_{2,3} & 3t+d & -(3t+d+1) \\ x_{4,1} & x_{3,2} & x_{3,3} & -(3t+d+2) & 3t+d+3 \\ x_{3,1} & x_{4,2} & x_{4,3} & 3t+d+2 & -(3t+d+3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{2t,1} & x_{2t-1,2} & x_{2t-1,3} & -(5t+d-2) & 5t+d-1 \\ x_{2t-1,1} & x_{2t,2} & x_{2t,3} & 5t+d-2 & -(5t+d-1) \end{bmatrix}.$$

Although we could augment X in a similar fashion as in Lemma 4.5, for use in the next section, we will augment the odd rows of X with the $t \times (m-5)$ array $Y(\frac{m-5}{4}, 5t+d-1, t)$ and the even rows of X with $-Y(\frac{m-5}{4}, 5t+d-1, t)$ where $Y(\frac{m-5}{4}, 5t+d-1, t)$ is the $t \times \frac{m-5}{4}$ array given in Definition 4.2. Note again that every integer in the set $\pm[d, mt+d-1]$ appears exactly once in X and that for each $i = 1, 2, \ldots, 2t$, we have $x_{i,1} + x_{i,2} + \cdots + x_{i,m} = 0$. Thus, the $2t$ rows of X give a signed Langford m-tuple difference set of order t and defect d. ☐

For $m = 9$, $t = 5$ and $d = 2$, the 10×9 array X in the proof of Lemma 4.6 is given below using the 3-tuple difference set given in (3):

$$X = \begin{bmatrix} 15 & -12 & -4 & -17 & 18 & 27 & -28 & -37 & 38 \\ 16 & -13 & -2 & 17 & -18 & -27 & 28 & 37 & -38 \\ 13 & -8 & -6 & -19 & 20 & 29 & -30 & -39 & 40 \\ 14 & -10 & -3 & 19 & -20 & -29 & 30 & 39 & -40 \\ 11 & -7 & -5 & -21 & 22 & 31 & -32 & -41 & 42 \\ 12 & -16 & 5 & 21 & -22 & -31 & 32 & 41 & -42 \\ 9 & -14 & 4 & -23 & 24 & 33 & -34 & -43 & 44 \\ 10 & -15 & 6 & 23 & -24 & -33 & 34 & 43 & -44 \\ 7 & -11 & 3 & -25 & 26 & 35 & -36 & -45 & 46 \\ 8 & -9 & 2 & 25 & -26 & -35 & 36 & 45 & -46 \end{bmatrix}. \tag{4}$$

Hence, the 10 rows of X give a signed Langford 9-tuple difference set of order $t = 5$ and defect $d = 2$.

Therefore, by Theorems 1.2, 3.1, 3.3, and 3.5, since a signed Langford 3-tuple difference set of order t and defect $d = 1$, $d = 2$ with $t \geq 3$, or $d = 3$ with $t \geq 5$ exists such that no two elements of the set $[t + d, 3t + d - 1]$ belong to the same 3-tuple, we have that signed Langford m-tuple difference sets of order t and defect $d = 1$, $d = 2$ with $t \geq 3$ or $d = 3$ with $t \geq 5$ exist for all positive integers $m \geq 5$ with $m \equiv 1 \pmod 4$.

5 Directed Cyclic m-Cycle Systems of Circulant Digraphs

An *m-cycle system* of a graph G is a decomposition of G into m-cycles. Let ρ denote the permutation $(0\,1\,\ldots\,n-1)$, so $\langle\rho\rangle = \mathbb{Z}_n$, the additive group of integers modulo n. An m-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is *cyclic* if, for every m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. Cyclic m-cycle systems of graphs have been investigated (see [3, 4, 5, 6, 7, 10, 12, 14, 15, 16, 19, 20]) but very little is known about directed *cyclic* m-cycle systems. However, necessary and sufficient conditions for directed m-cycle systems are known to exist. In [1], it was shown that for positive integers m and n with $2 \leq m \leq n$, the complete symmetric digraph K_n^* can be decomposed into directed cycles of length m if and only if m divides the number of arcs in K_n^* and $(n, m) \neq (4, 4), (6, 3), (6, 6)$. However, these constructions are not cyclic. The only directed cyclic m-cycle systems known to exist are the ones in which m is as large as possible, that is, directed cyclic hamiltonian cycle systems. In [13], it was shown that, for n odd, there exists a directed cyclic n-cycle system of K_n^* if and only if $n \neq 15$ and $n \neq p^\alpha$ where p is an odd prime and $\alpha \geq 2$, and for n even, there exists a directed cyclic n-cycle system of K_n^* if and only if $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$.

We are interested in directed cyclic cycle decompositions of digraphs for other values of m. Notice that in order for a digraph D to admit a directed cyclic m-cycle system, D must be a circulant digraph, so circulant digraphs provide a natural setting in which to construct directed cyclic m-cycle systems. Let $n \geq 2$ be an integer and let $S \subseteq [1, n-1]$. We will often use -1 for $n-1$ and thus we may assume $S \subseteq \pm[1, \lfloor n/2 \rfloor]$. The *circulant digraph* $\overrightarrow{X}(n; S)$ is defined to be that digraph whose vertices are the elements of $\mathbb{Z}_n$, with an arc from vertex g to vertex h if and only if $h = g + \ell$ for some $\ell \in S$; the *length* of the arc (g, h) is ℓ in this case. The digraph K_n^* is a circulant digraph since $K_n^* = \overrightarrow{X}(n; \pm[1, \lfloor n/2 \rfloor])$.

In this section, we will use the signed Langford m-tuple difference sets constructed in Section 4 to find directed cyclic m-cycle systems of circulant digraphs. However, it is not necessarily the case that each m-tuple will give rise to an m-cycle without reordering its elements. For example, the 9-tuple $(15, -12, -4, -17, 18, 27, -28, -37, 38)$ corresponding to the first row of the 10×9 array X given in (4) gives rise to a subdigraph of $\overrightarrow{X}(n; \{-37, -28, -17, -12, -4, 15, 18, 27, 38\})$, $n \geq 77$, consisting of a 5-cycle and a 4-cycle with one vertex in common when the arcs are added in the order given in the 9-tuple, starting from vertex 0. Hence, we have the following definitions.

Definition 5.1 Let $n > 0$ be an integer and suppose there exists an ordered m-tuple $(d_1, d_2, \ldots, d_m)$ satisfying each of the following:

(i) $d_i \in \pm[1, \lfloor n/2 \rfloor]$ $i = 1, 2, \ldots, m$;

(ii) $d_i \neq d_j$ for $1 \leq i < j \leq m$;

(iii) $d_1 + d_2 + \cdots + d_m = 0 \pmod n$; and

(iv) $d_1 + d_2 + \cdots + d_r \not\equiv d_1 + d_2 + \cdots + d_s \pmod n$ for $1 \leq r < s \leq m$.

Then $(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1})$ *generates* a cyclic m-cycle system of the digraph $\overrightarrow{X}(n; \{d_1, d_2, \ldots, d_m\})$. An m-tuple satisfying (i)-(iv) is called a *difference m-tuple*, it *corresponds* to the starter m-cycle $(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1})$, and it *uses* arcs of lengths $d_1, d_2, \ldots, d_m$.

An *m-cycle difference set of size t*, when the value of n is understood, is a set consisting of t difference m-tuples that use arcs of distinct lengths $\ell_1, \ell_2, \ldots, \ell_{tm}$; the m-cycles corresponding to the difference m-tuples generate a directed cyclic m-cycle system $\mathcal{C}$ of $\overrightarrow{X}(n; \{\ell_1, \ell_2, \ldots, \ell_{tm}\})$.

For $3 \leq m \leq 5$, note that a signed Langford m-tuple difference set of order t and defect d generates a directed cyclic m-cycle system of $\overrightarrow{X}(n; \pm[d, mt + d - 1])$ for all $n \geq 2(mt + d - 1) + 1$ since each m-tuple is a difference m-tuple. However, for $m \geq 6$, some reordering of the elements in each m-tuple of the signed Langford m-tuple difference set is necessary. For the 9-tuple $(15, -12, -4, -17, 18, 27, -28, -37, 38)$ given above, the reordering $(-4, -12, 18, -28, -37, 27, -17, 15, 38)$ will produce a 9-cycle in the appropriate circulant digraph. Hence, if each m-tuple has been reordered in a signed Langford m-tuple difference set of order t and defect d so that it is now a difference m-tuple, we will call the signed Langford m-tuple difference set a *signed Langford m-cycle difference set of order t and defect d*. If $d = 1$, then such a difference set will be called a *signed Skolem m-cycle difference set of order t*.

In [7], for positive integers m and t with $m \geq 3$, Bryant, Ling and the second author showed that there exists a cyclic m-cycle system of $X(n; [1, mt])$ for $mt \equiv 0, 3 \pmod 4$ and all $n \geq 2mt + 1$ by constructing Skolem m-cycle difference sets of order t. Here we use a similar, although necessarily different, approach.

Theorem 5.2 *Let $m \geq 3$ be an integer.*

- *For every integer $t \geq 1$, there exists a signed Skolem m-cycle difference set of size t.*
- *For every integer $t \geq 3$, there exists a signed Langford m-cycle difference set of size t and defect $d = 2$.*
- *For every integer $t \geq 5$, there exists a signed Langford m-cycle difference set of size t and defect $d = 3$.*

Proof. The proof splits into four cases depending on the congruence class of m modulo 4. For each case we use a previously constructed $2t \times m$ array $X = [x_{i,j}]$ whose entries are $\pm[d, mt + d - 1]$ such that for each $i = 1, 2, \ldots, 2t$, we have

$$\sum_{j=1}^{m} x_{i,j} = 0.$$

The entries in each row of our arrays will also satisfy various inequalities which will allow us to arrange them so that for $1 \le r < s \le m$ and $n \ge 2mt + 1$, we have $d_1 + d_2 + \ldots, d_r \not\equiv d_1 + d_2 + \ldots, d_s \pmod n$; hence a signed Skolem or Langford m-cycle difference set of size t can be obtained.

In what follows, to find signed Skolem m-cycle difference set of size t, we let $d = 1$.

Case 1. *Suppose that* $m \equiv 0 \pmod 4$. Let $X = [x_{i,j}]$ be the $2t \times m$ array constructed from the $t \times m$ array $Y(\frac{m}{4}, d-1, t)$ given by Definition 4.2 followed by the $t \times m$ array $-Y(\frac{m}{4}, d-1, t)$. For $i = 1, 2, \ldots, 2t$, we have $|x_{i,1}| < |x_{i,2}| < \cdots < |x_{i,m}|$ and $x_{i,j} < 0$ precisely when $j \equiv 2, 3 \pmod 4$ in the first t rows and $x_{i,j} < 0$ precisely when $j \equiv 0, 1 \pmod 4$ in the last t rows. Hence the required set of difference m-tuples can be constructed from the rows of X by using the following reordering:

$$(x_{i,1}, x_{i,3}, x_{i,5}, x_{i,7}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,6}, x_{i,4}, x_{i,2}, x_{i,m})$$

for $i = 1, 2, \ldots, 2t$.

Case 2. *Suppose that* $m \equiv 2 \pmod 4$. Let $X = [x_{i,j}]$ be the $2t \times m$ array constructed from the $t \times m$ array Z' given in the proof of Lemma 4.4 followed by the $t \times m$ array $-Z'$. For $i = 1, 2, \ldots, 2t$, we have $|x_{i,1}| < |x_{i,2}| < |x_{i,4}| < |x_{i,5}| < |x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,2}| < |x_{i,3}| < |x_{i,5}|$, and $x_{i,j} < 0$ precisely when $j = 2$ and when $j \equiv 0, 1 \pmod 4$ for $j \ge 4$ in the first t rows of X and $x_{i,j} < 0$ precisely when $j = 1$ and when $j \equiv 2, 3 \pmod 4$ for $j \ge 3$ in the last t rows of X. Hence, the required set of difference m-tuples can be constructed from the rows of X by using the following reordering:

$$(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,5}, x_{i,7} \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,6}, x_{i,4}, x_{i,m})$$

for $i = 1, 2, \ldots, 2t$.

Case 3. *Suppose that* $m \equiv 3 \pmod 4$. Let $X = [x_{i,j}]$ be the $2t \times m$ array constructed in the proof of Lemma 4.5 with $t \ge 3$ if $d = 2$ or $t \ge 5$ if $d = 3$. For $i = 1, 2, \ldots, 2t$, we have $|x_{i,3}| < |x_{i,2}| < |x_{i,4}| < |x_{i,5}| < |x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,1}| < |x_{i,4}|$, and $x_{i,j} < 0$ precisely when $j = 2$, $j = 3$ and $j \equiv 1, 2 \pmod 4$ for $j \ge 4$ in the first t rows and $x_{i,j} < 0$ precisely when $j = 2$ and $j \equiv 0, 3 \pmod 4$ for $j \ge 4$ in the last t rows of X. Hence, the required set of difference m-tuples can be constructed from the rows of X by using the following reordering:

$$(x_{i,3}, x_{i,2}, x_{i,4}, x_{i,6}, x_{i,8}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,5}, x_{i,1}, x_{i,m})$$

for $i = 1, 2, \ldots, 2t$.

CASE 4. *Suppose that* $m \equiv 1 \pmod 4$. Let $X = [x_{i,j}]$ be the $2t \times m$ array constructed in the proof of Lemma 4.6 with $t \geq 3$ if $d = 2$ or $t \geq 5$ if $d = 3$. For $i = 1, 2, \ldots, 2t$, we have $|x_{i,3}| < |x_{i,2}| < |x_{i,4}| < |x_{i,6}| < |x_{i,7}| < \cdots < |x_{i,m}|$, $|x_{i,1}| < |x_{i,5}| < |x_{i,6}|$, $|x_{i,1}|+|x_{i,3}| < |x_{i,5}|$ for $t+1 \leq i \leq 2t$, and $|x_{i,2}|+|x_{i,3}| < |x_{i,5}|$ for $1 \leq i \leq t$. Note also that $x_{i,j} < 0$ precisely when $j = 2$, $j = 3$ and $1 \leq i \leq t$, and $j \equiv 0, 3 \pmod 4$ with $j \geq 4$ and i odd or $j \equiv 1, 2 \pmod 4$ with $j \geq 5$ and i even. Hence, the required set of difference m-tuples can be constructed from the rows of X by using the following reordering:

- $(x_{i,3}, x_{i,2}, x_{i,5}, x_{i,7}, \ldots, x_{i,m-2}, x_{i,m-1}, x_{i,m-3}, \ldots, x_{i,4}, x_{i,1}, x_{i,m})$ for i odd and $1 \leq i \leq t$;
- $(x_{i,3}, x_{i,2}, x_{i,4}, x_{i,6}, \ldots, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, \ldots, x_{i,5}, x_{i,1}, x_{i,m})$ for i even and $1 \leq i \leq t$;
- $(x_{i,3}, x_{i,1}, x_{i,4}, x_{i,6}, \ldots, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, \ldots, x_{i,5}, x_{i,2}, x_{i,m})$ for i odd and $t+1 \leq i \leq 2t$; and
- $(x_{i,3}, x_{i,1}, x_{i,5}, x_{i,7}, \ldots, x_{i,m-2}, x_{i,m-1}, x_{i,m-3}, \ldots, x_{i,4}, x_{i,2}, x_{i,m})$ for i even and $t+1 \leq i \leq 2t$.

□

Using the 10×9 array X given in (4) and constructed from the proof of Lemma 4.6, the required set of difference 9-tuples found by reordering the entries in the rows of X as prescribed in the proof of Theorem 5.2 are:

$$
\begin{array}{rrrrrrrrr}
\{(-4, & -12, & 18, & -28, & -37, & 27, & -17, & 15, & 38), \\
(-2, & -13, & 17, & -27, & 37, & 28, & -18, & 16, & -38), \\
(-6, & -8, & 20, & -30, & -39, & 29, & -19, & 13, & 40), \\
(-3, & -10, & 19, & -29, & 39, & 30, & -20, & 14, & -40), \\
(-5, & -7, & 22, & -32, & -41, & 31, & -21, & 11, & 42), \\
(5, & 12, & -22, & 32, & 41, & -31, & 21, & -16, & -42), \\
(4, & 9, & -23, & 33, & -43, & -34, & 24, & -14, & 44), \\
(6, & 10, & -24, & 34, & 43, & -33, & 23, & -15, & -44), \\
(3, & 7, & -25, & 35, & -45, & -36, & 26, & -11, & 46), \\
(2, & 8, & -26, & 36, & 45, & -35, & 25, & -9, & -46)\}.
\end{array}
$$

Hence this set of 10 difference 9-tuples gives a signed Langford 9-cycle difference set of order $t = 5$ and defect $d = 2$.

Theorem 5.2 has the following immediate corollaries.

Corollary 5.3 *Let $m \geq 3$ be an integer.*

- *For all $t \geq 1$ and $n \geq 2mt+1$, there exists a cyclic m-cycle system of $\overrightarrow{X}(n; \pm[1, mt])$.*
- *For all $t \geq 3$ and $n \geq 2mt+3$, there exists a cyclic m-cycle system of $\overrightarrow{X}(n; \pm[2, mt+1])$.*
- *For all $t \geq 5$ and $n \geq 2mt+5$, there exists a cyclic m-cycle system of $\overrightarrow{X}(n; \pm[3, mt+2])$.*

Corollary 5.4 *For all integers $m \geq 3$ and $t \geq 1$, there exists a cyclic m-cycle system of K^*_{2mt+1}.*

A goal for future work is to find signed Langford sequences of order t and defect d for all $t \geq 2d-1$. One might also consider generalizing signed Langford sequences in many of the ways in which Langford sequences have been generalized (see [11] for a survey).

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