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Asymmetric distance matching extension in 5-connected even planar triangulations

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Abstract

A matching M in a graph G is said to be extendable if there is a perfect matching in G which contains M. It has been known for some time that if the matching M has the property that the edges it contains are mutually far enough apart, it is more likely that M will extend. In previous studies, ensuring that edges were suitably far apart was achieved by fixing a distance d and requiring that *each* pair of edges in a set to be extended was at least distance d apart. In the present paper we study extending matchings in which the edges to be extended are pairwise at different distances one from the other. We call this an *asymmetric* matching extension. In particular, we focus on improving existing results

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by applying such an asymmetric distance restriction in which we single out one particular edge from M and require that it is at least a certain distance from the other edges in M while the remaining edges in the given matching are left unrestricted as to mutual distance. We shall confine ourselves to 5-connected even planar triangulations in order to guarantee that the graphs under study actually contain perfect matchings.

1 Introduction

In [11], Plummer introduced the notion of extendability in graphs. A graph G on at least $2m+2$ vertices is said to be *m-extendable* if for each matching M of size m, there is a perfect matching P_M of G such that $M \subset P_M$. In the original paper, it was shown that an m-extendable graph must be $m+1$-connected, and in the many investigations that followed, connectivity has proved to be an important factor in determining whether a class of graphs is m-extendable. Other fruitful considerations have involved regularity and genus. In [12], it was shown that no planar graph is 3-extendable. This remarkable result puts an upper bound on extendability for planar graphs well short of that imposed by connectivity considerations. A considerable number of papers have since appeared in the literature related to extendability and the reader is directed to surveys [13, 14, 15] as well as the book [18] for more information.

In 1996 Porteous and Aldred [17] asked about extending one matching to a perfect matching while completely avoiding a second disjoint matching. More precisely, a graph G with at least $2m + 2n + 2$ vertices which contains a perfect matching is said to satisfy property $E(m, n)$ (or simply "G is $E(m, n)$") if, for every pair of matchings M and N in G with $|M| = m$ and $|N| = n$ such that $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subseteq F$ and $N \cap F = \emptyset$. This property is a generalization of the widely studied concept of matching extension in that a graph is m-extendable if and only if it is $E(m, 0)$.

In [17], certain implications and non-implications were shown to exist among the $E(m, n)$ properties for different values of m and n. This generalized notion of extendability also inspired a number of papers some of which are surveyed in [16] and [18].

The present paper deals with $E(m, n)$ as it relates to certain planar graphs; more particularly, we shall focus on plane triangulations with an even number of vertices.

Let us briefly summarize what is known heretofore about the property $E(m, n)$ for plane triangulations. In general, no planar graph, triangulation or not, is $E(3, 0)$ ([12]), nor even $E(2, 1)$ ([1]). If a planar even triangulation is only 3-connected, it may not even contain a perfect matching as is evidenced, for example, by the graph (called a "Kleetope") shown in the following figure.

So we may proceed immediately to the case when the graph is (at least) 4-connected.

If G is a 4-connected planar even triangulation, it is $E(1, 1)$ ([1]), but not neces-

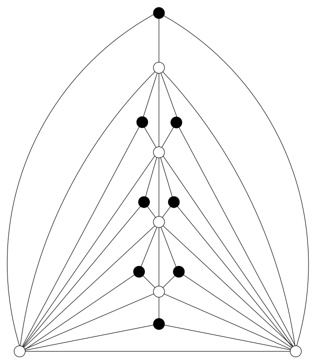

Figure 1: A 3-connected even planar triangulation with no perfect matching.

sarily $E(1,2)$ ([3]). It is also $E(0,3)$ ([1]), and hence by [17], also $E(0,2)$ and $E(0,1)$. But it is not necessarily $E(0,4)$ ([3]).

For 5-connected planar even triangulations the known results are combined in the next theorem.

Theorem 1.1. *Let G be a 5-connected planar even triangulation. Then:*

(i) *G is $E(2,0)$* ([9], [13]);

(ii) *G is $E(1,3)$* ([2] *Corollary* 3.3); *and*

(iii) *G is $E(0,7)$* ([2] *Theorem* 3.4).

Each of these results is known to be sharp. In particular, parts (ii) and (iii) have sharpness examples in which the matchings to be avoided consist of edges which are pairwise distance 2 apart.

In [2] the authors first showed that the distance between edges to be matched could affect whether or not they could be extended to a perfect matching. In particular, three independent edges in a planar 5-connected even triangulation do not necessarily extend to a perfect matching, but if each of the three edges lies at distance at least 2 from each of the other two, then the three do in fact so extend.

Later in [3] we defined the property $E_d(m,n)$ as follows. Let d be a positive integer and m and n, non-negative integers. A graph G is said to have the property $E_d(m,n)$ (or simply "G is $E_d(m,n)$") if given any two disjoint matchings M with $|M| = m$ and N with $|N| = n$ in G, where the distance between every two edges in M is at least d and the distance between every two edges in N is at least d, there is a perfect matching F in G such that $M \subseteq F$ and $N \cap F = \emptyset$. If we apply the distance d to the matching $L = M \cup N$, then the distance restriction is said to be *universal* and we say G is $E_{du}(m,n)$.

Applying distance restrictions in the manner described in the preceding paragraph has led to many results in matching extension theory (see, for example, [1, 2, 3, 4, 5, 6, 8]).

If we apply distance restrictions to our specified matchings we have the following improvements over the results in Theorem 1.1.

Theorem 1.2. *Let G be a 5-connected planar even triangulation. Then:*

(i) *G is $E_2(3,0)$* ([2] *Theorem* 2.1);

(ii) *G is $E_2(2,1)$* ([3] *Theorem* 4.1);

(iii) *G is $E_{2u}(1,5)$* ([3] *Theorem* 5.1);

(iv) *G is $E_3(1,n)$, for all non-negative integers n* ([3] *Theorem* 5.2); *and*

(v) *G is $E_3(0,n)$, for all non-negative integers n* ([3] *Corollary* 6.1).

Work in the present paper is motivated by the question: What is the effect of modifying the distance condition in an *asymmetric* way for the edges to be matched? This new approach means that we no longer specify a single distance, d, and require that all edges to be extended are pairwise at least distance d apart. Instead we allow that the specified distance may differ depending on which pair of edges we choose from the set to be extended. In particular, we are able to show that all of the results in Theorem 1.1 can be improved by applying such an *asymmetric* distance restriction in which we single out just one particular edge from $M \cup N$ and require that it is at least a certain distance from other edges in the given matching while the remaining edges in the given matching are left unrestricted as to mutual distance.

For general graph theoretic terminology, the reader is referred to [7] and in particular for more on matching theory, to [10]. In addition, however, we shall need the following concepts. Suppose a graph G contains two disjoint matchings E and F, such that G contains no perfect matching containing all edges in M while containing none of the edges in N. This is equivalent to saying that the graph $G' = G - V(E) - F$ does not contain a perfect matching. But then it follows by Tutte's classical result on matchings that G' must contain a set of vertices S (usually called a *Tutte set* or *barrier*) such that the number of odd components of $G' - S$ exceeds $|S|$.

We shall make use of the idea of the *bipartite distillation* G^* obtained from G via G' based upon E, S and F. This concept was first introduced in [1] and named 'bipartite distillation' in [2]. It is defined as follows: (1) Contract each odd component of $G' - S$ to a separate singleton and delete any multiple edges and loops thus formed, (2) delete all even components of $G' - S$, and (3) delete all edges in $G[V(E) \cup S]$ as well as those in F. Then let G^* be the bipartite graph thus obtained having $S \cup V(E)$ as the vertices of one partite set and the contracted components of $G' - S$ as the vertices in the other partite set. Clearly, G^* will be planar if the original graph G is planar.

2 Main Results

As mentioned above, a 5-connected planar even triangulation must have property $E_2(3,0)$, but not necessarily $E(3,0)$. Our first result below shows that in such a triangulation, requiring $d \geq 2$ among all pairs of the three edges in order to extend to a perfect matching is stronger than we need. In fact, suppose we have three independent edges such that one of them lies at distance at least 2 from each of the other two. Then the following theorem guarantees that the three given edges will always extend to a perfect matching.

Theorem 2.1. *Let G be a 5-connected even planar triangulation. Suppose $e, f, g \in E(G)$ are such that $d(e,f) \geq 2, d(e,g) \geq 2$ and $\{e,f,g\}$ is a matching in G. Then G contains a perfect matching which contains all of e, f and g.*

Proof. Let G, e, f and g be as in the statement of the theorem and suppose there is no matching containing all of e, f and g. That is, the graph $G' = G - V(e) - V(f) - V(g)$ has no perfect matching. By Tutte's theorem, we have a set of vertices $S = \{s_1, \ldots, s_k\} \subseteq V(G')$ such that $H = G' - S$ contains at least $|S|+2 = k+2$ odd components. (Note, the number of odd components in H must be the same parity as $|S| = k$, since $|V(G)|$ is even.) Moreover, since G is $E(2,0)$, H contains *exactly* $k+2$ odd components. To see this, suppose H has more than $|S|+2 = k+2$ odd components. By parity, this means there are at least $k+4$ odd components in H. Now consider $G'' = G - V(e) - V(f)$ and $S' = S \cup V(g)$. Then $H = G'' - S'$ has at least $k+4 = |S'|+2$ odd components and thus, G has no perfect matching that includes both the edges e and f, contradicting the fact that G is $E(2,0)$. In addition, by Theorem 1.2(i), if $d(f,g) \geq 2$, there is a perfect matching containing all three of e, f and g.

So let us suppose that $d(f,g) = 1$.

Now let G^* denote the bipartite distillation obtained from G via G' based on $E = \{e,f,g\}, S = \{s_1, \ldots, s_k\}$ and $F = \emptyset$. Then $|V(G^*)| = k+6+k+2 = 2k+8$. Moreover, by Euler's Formula for planar bipartite graphs, $2|V(G^*)| - 4 = 4k+12 \geq |E(G^*)| \geq 5(k+2)$ (since G is 5-connected), and hence $k \leq 2$.

Now in G contract edge e to a single vertex v_e. Then, since G is $E(2,0)$, v_e has neighbours in at least two of the odd components of H. Let $C_1, \ldots, C_k, C_{k+1}, C_{k+2}$ be the odd components of H and, without loss of generality, we may suppose v_e has neighbours in both C_1 and C_2.

In a fixed plane representation of graph G with edge e contracted to the single vertex v_e, let us scan about vertex v_e in a clockwise direction starting with a vertex $u_1 \in V(C_1)$ and eventually reaching a vertex u_j, the last neighbour of $v_e \in C_1$.

Then u_{j+1} must lie in $V(f) \cup V(g) \cup S$ (since it is not in C_1 and, being a neighbour of u_i, it cannot belong to another odd component of H). Continuing this clockwise scan, we must encounter neighbours of $v_e \in V(C_2)$. Let v_i be the last such neighbour so that $v_{i+1} \notin V(C_2)$. So $v_{i+1} \in V(f) \cup V(g) \cup S$. (See Figure 2.)

By our distance hypothesis, neither u_{j+1} nor v_{i+1} can lie in $V(f) \cup V(g)$. Thus u_{j+1} and v_{i+1} both lie in $S = \{s_1, \ldots s_k\}$.

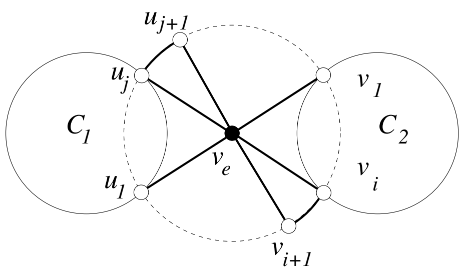

Figure 2: Scanning the neighbourhood of v_e.

Note that $u_{j+1} \neq v_{i+1}$, or else we would have a vertex cut of size 3 in G, formed by the two endvertices of e together with u_{j+1}. Consequently v_e is adjacent to two distinct vertices in S. Thus $|S| \geq 2$. From our earlier observation that $|S| = k \leq 2$, we must conclude that $k = 2$, $S = \{s_1, s_2\}$ and v_e is adjacent to both s_1 and s_2.

We can perform a similar exercise for each of the edges f and g. Since f and g are similar, we shall only detail the process for f. In G we contract f to a single vertex v_f, noting v_f has neighbours in at least two odd components of H. Scanning the neighbours of v_f clockwise reveals that v_f has at least two neighbours in $S \cup V(g)$. Since G is 5-connected, at most one of these neighbours of v_f can be in $V(g)$. Thus each of the vertices v_f and v_g has a neighbour in $S = \{s_1, s_2\}$.

Let $L = G[V(e) \cup V(f) \cup V(g) \cup \{s_1, s_2\}]$. Then L is connected and, with the embedding inherited from G, L is a plane graph with $N_L = |V(L)| =$ eight vertices, E_L edges and F_L faces.

Since G is a 5-connected planar triangulation and $H = G - V(L)$ has at least four odd components, L has at least four faces of size at least 5 in order to accommodate these odd components. By Euler's formula we have $N_L - E_L + F_L = 2$, which yields $E_L - F_L = 6$.

Now $2E_L = \sum_{i\geq 3} iF_{L,i}$ and $F_L = \sum_{i\geq 3} F_{L,i}$, where $F_{L,i}$ denotes the number of faces of size i in L. Thus $12 = 2E_L - 2F_L = \sum_{i\geq 3}(i-2)F_{L,i}$ from which we may conclude that $F_{L,i} = 0$ for all $i \neq 5$, $F_{L,5} = 4$ and thus $E_L = 10$ and $F_L = 4$. So each face in L is bounded by a 5-cycle and every edge of L belongs to two pentagonal faces. Hence L has no vertices of degree 1, no 3-cycles and no 4-cycles.

Let us denote $e = v_1v_2, f = w_1w_2$ and $g = x_1x_2$. We recall from earlier that $d(f, g) = 1$ so we have a path $w_1w_2x_1x_2$, without loss of generality. Also, each of v_1 and v_2 has degree in L at least 2, and since $d(e, f) \geq 2$ and $d(e, g) \geq 2$, and e has both s_1 and s_2 as neighbours, we have a path $s_1v_1v_2s_2$, again without loss of generality. The remaining four edges in L can only join $\{s_1, s_2\}$ to vertices in $\{w_1, w_2, x_1, x_2\}$. Since L has no 3-cycles or 4-cycles, neither s_1 nor s_2 can have more than two neighbours in $\{w_1, w_2, x_1, x_2\}$. Thus each of s_1 and s_2 has exactly two such neighbours and these must be w_1 and x_2. These edges are shown as dashed edges in

Figure 3. But then $s_1w_1s_2x_2s_1$ is a separating 4-cycle in G, a contradiction. □

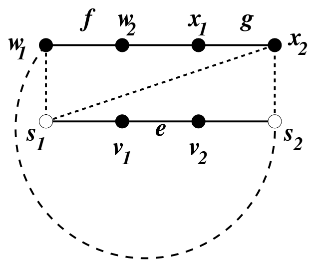

Figure 3: The subgraph L.

We note that Theorem 2.1 is sharp as the graph in Figure 2.2 of [2] is a 5-connected even planar triangulation which is not $E_2(4, 0)$.

Next let us recall that by Theorem 2.1 of [1], no planar graph (and therefore, no plane triangulation) satisfies property $E(2, 1)$. But by Theorem 4.1 of [3], every planar triangulation does satisfy $E_2(2, 1)$. The next theorem shows that the same asymmetric distance restriction assumed in Theorem 2.1 when applied to all three edges involved, again guarantees the desired extension.

Theorem 2.2. *Let G be a 5-connected even planar triangulation. Suppose $e, f, g \in E(G)$ are such that $d(e, f) \geq 2, d(e, g) \geq 2$ and $\{e, f, g\}$ is a matching in G. Then G contains a perfect matching which contains any pair of edges in $\{e, f, g\}$, while avoiding the third.*

Proof. Let G, e, f and g be as described in the statement of the theorem. Theorem 1.2(ii) guarantees that G contains a perfect matching which includes e and f, but not g, and a perfect matching including e and g, but not f. To establish the present theorem, we must now show that G also has a perfect matching including edges f and g, while avoiding e. Moreover, we may again assume by Theorem 1.2 (ii) that $d(f, g) = 1$, for otherwise we may conclude that the desired matching exists.

So, assuming that the desired matching does *not* exist, by Tutte's Theorem we have a set of vertices $S \subseteq V(G) - (V(f) \cup V(g))$ such that $H = G - V(f) - V(g) - S - e$ contains at least $|S| + 2$ odd components. Moreover, since G is $E(2, 0)$, we have precisely $|S| + 2$ odd components in H and the edge e joins vertices in two different odd components of H (see Figure 4).

Let G^* denote the bipartite distillation of G based on $E = \{f, g\}$, S and $F = \{e\}$. We now use the standard count on edges in G^* to obtain

$$4|S| + 8 \geq |E(G^*)| \geq 5|S| + 8$$

and hence $|S| = 0$.

But G is a triangulation, so the edge e lies in the boundary cycles of two triangular faces. Thus the endvertices of e must have two common neighbours neither of which

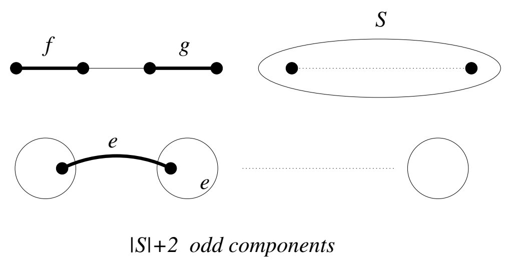

Figure 4:

can be in $V(f) \cup V(g)$ by our distance hypothesis, nor can they be in any of the components of H, since they must be adjacent to both endvertices of e and these endvertices lie in different odd components of H. This leaves only S available to contain the common neighbours of both endvertices of e. But we have already shown that $S = \emptyset$ and hence we have a contradiction. □

Theorem 2.2 is sharp. The sharpness is indicated by the graph in Figure 9 of [3], a 5-connected even planar triangulation which is not $E_{2u}(2, 2)$.

We know from [2] that if G is a 5-connected planar even triangulation, then G satisfies $E(1, 3)$, but not necessarily $E(1, 4)$. The sharpness example given there has the edge to be included at distance 1 from each of the four edges to be excluded, while the four edges to be avoided are pairwise distance 2 apart (i.e the graph is not $E_2(1, 4)$). (This sharpness example readily extends to an infinite family.)

In the next theorem we see that there are various asymmetric distance restrictions we can apply to the edges considered to give the desired matching.

Theorem 2.3. *Let G be a 5-connected planar even triangulation and let $\{e\}$ and $N = \{, f_1, f_2, f_3, f_4\} \subseteq E(G)$ be two disjoint matchings in G.*

If either

(i) $d(e, f_i) \geq 2$ *for* $i = 1, 2, 3, 4$,

(ii) $d(f_1, f_i) \geq 3$ *for* $i = 2, 3, 4$, *or*

(iii) $f_1 \in N$ *has* $d(f_1, e) \geq 2$, $d(f_1, f_i) \geq 2$, $i = 2, 3, 4$,

then G has a perfect matching that includes e, but none of the edges of N.

Proof. Suppose $G, e, f_1, \ldots, f_4$ are as in the statement of the theorem, but no perfect matching includes e while avoiding N. Then by Tutte's theorem there is a set $S \subseteq V(G) - V(e)$ such that $H = G - V(e) - N - S$ has at least $|S| + 2$ odd components. Moreover, since G is $E(1, 3)$, we have *exactly* $|S| + 2$ odd components and each $f_i \in N$ has its endvertices in two distinct odd components of H. Now, since G is a triangulation of the plane, we may conclude that each edge $f_i = x_i y_i \in N$ has two vertices in $S \cup V(e)$ adjacent to both x_i and y_i forming the two triangular faces including f_i.

Form G^*, the bipartite distillation of G based on e, N and S. By the 5-connectivity of G, there are at least $5(|S|+2)-(2\times 4) = 5|S|+2$ edges joining vertices in odd components of H to vertices in $S \cup V(e)$. This gives

$$|E(G^*)| \geq 5|S|+2.$$

On the other hand, G^* is a bipartite planar graph on $|S|+2+|S|+2$ vertices, so by Euler's formula,

$$|E(G^*)| \leq 2|V(G^*)| - 4 = 4|S|+4.$$

Consequently, $|S| \leq 2$.

To this point in our analysis we have not used the distance conditions applied to the edges in our matchings. In the following we consider each of the cases listed in the statement of the theorem in turn.

(i) Contract each edge f_i to a single vertex u_i for $i = 1, 2, 3, 4$. Let $U = \{u_1, u_2, u_3, u_4\}$. Since $d(e, f_i) \geq 2$ for $i = 1, 2, 3, 4$, each edge $f_i \in N$ lies in the boundaries of two triangular faces using two distinct vertices in S. Let G' be a graph with $V(G') = U \cup S$ and each vertex $u_i \in U$ is adjacent to those two vertices in S each of which forms a triangular face with f_i. Then $|V(G')| = 4+|S|$, $|E(G')| = 4\times 2 = 8$ and G' has $F_{G'}$ faces. By Euler's formula $(4+|S|)-8+F_{G'} = 2$ which implies $F_{G'} = 6-|S|$. If each cycle in G' has length at least 6, then $6(6-|S|) \leq 2|E(G')| = 2\times 8 = 16$. Thus $6-|S| \leq 2$ and so $|S| \geq 4$, a contradiction. Consequently G' has a 4-cycle and hence G has a separating 4-cycle which is a contradiction.

(ii) Now consider the two triangular faces containing f_1. Potentially both common neighbours of x_1 and y_1 are endvertices of e, but this would force the existence of a separating triangle. Thus at most one endvertex of e lies in a triangle containing f_1.

If neither endvertex of e forms a triangle with f_1, then x_1 and y_1 have two common neighbours in S. Consequently, each of f_2, f_3 and f_4 lies in two triangular faces including endvertices of e. But again, this forces the existence of a separating triangle in G.

Thus we may assume that each edge $f_i \in N$ lies in one triangle containing an endvertex of e and another triangle containing a vertex in S. Moreover, by our assumption that $d(f_1, f_i) \geq 3$ for $i = 2, 3, 4$, the triangulating vertices associated with f_1 are distinct from those associated with f_i, $i = 2, 3, 4$. Since $|S| \leq 2$, each of f_2, f_3 and f_4 belongs to two triangles, one including one endvertex of e (which is not a neighbour of x_1 or x_2) and the other containing one vertex in S (also not a neighbour of x_1 or x_2). But this forces the existence of a separating 4-cycle in G.

(iii) Finally, we note that, since $d(f_1, e) \geq 2$, the two triangular faces containing the edge f_1 must each be completed by a vertex in S adjacent to both endvertices of f_1. Consequently, $|S| = 2$. Moreover, since $d(f_1, f_i) \geq 2$ for each $i = 2, 3, 4$, neither of the two vertices in S can complete a triangular face containing the edge f_i. Thus, if $e = uv$, for each $f_i = x_iy_i$, $i = 2, 3, 4$ we have ux_i, uy_i, vx_i and $vy_i \in E(G)$. But this is impossible since G is 5-connected. This final contradiction establishes the result. □

Parts (ii) and (iii) of Theorem 2.3 can be seen to be sharp in the following sense. We cannot increase the size of N under the given distance restrictions and hope to guarantee the existence of a perfect matching that includes e and avoids N. Examples of this can be readily obtained using straightforward modifications of the 5-connected even planar triangulation which is not $E_2(1, 4)$ in Figure 3.1 of [2]. Part (i), however, can be strengthened as seen in the next theorem.

Theorem 2.4. *Let G be a 5-connected planar even triangulation and let $\{e, f_1, f_2, f_3, f_4, f_5\} \subseteq E(G)$. If $d(e, f_i) \geq 2$ for each $i = 1, 2, 3, 4, 5$, and $N = \{f_1, f_2, f_3, f_4, f_5\}$ is a matching, then G has a perfect matching that includes e and avoids N.*

Proof. Suppose $G, e, f_1, \ldots, f_5$ are as in the statement of the theorem, but no perfect matching of G includes e while avoiding N. Then by Tutte's theorem on perfect matchings there is a set $S \subseteq V(G) - V(e)$ such that $H = G - V(e) - N - S$ has at least $|S| + 2$ odd components. By Theorem 2.3, if we let N_i denote $N - f_i$, $i = 1, 2, \ldots, 5$, there is a perfect matching in G including e and avoiding N_i. Thus we must have exactly $|S| + 2$ odd components in H and each $f_i \in N$ has its endvertices in two distinct odd components of H.

Form G^*, the bipartite distillation of G based on $E = \{e\}, S$ and N. By the 5-connectivity of G, there are at least $5(|S| + 2) - (2 \times 5) = 5|S|$ edges joining vertices in odd components of H to vertices in $S \cup V(e)$. This gives

$$|E(G^*)| \geq 5|S|.$$

On the other hand, G^* is a bipartite planar graph on $|S| + 2 + |S| + 2$ vertices, so by Euler's formula,

$$5|S| \leq |E(G^*)| \leq 2|V(G^*)| - 4 = 4|S| + 4.$$

Consequently, $|S| \leq 4$.

Now, since G is a triangulation of the plane, we may conclude that each edge $f_i = x_iy_i \in N$ has two vertices in $S \cup V(e)$ adjacent to both x_i and y_i forming the two triangular faces which include f_i. By our distance restrictions (i.e. $d(e, f_i) \geq 2$, $i = 1, 2 \ldots, 5$) we see that these triangulating vertices are in S. Contract each edge f_i to a single vertex u_i for $i = 1, 2, 3, 4, 5$.

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and let G' be the bipartite graph with $V(G') = U \cup S$ and in which each vertex $u_i \in U$ is adjacent in G' to the two vertices in S which form triangles with the edge f_i in G. Note that G' inherits a plane embedding from G. Let $F_{G'}$ denote the number of faces in G'. Then $|V(G')| = 5 + |S|$, $|E(G')| = 5 \times 2 = 10$, G' has $F_{G'}$ faces and each vertex in U has degree 2 in G'. By Euler's formula,

$(5 + |S|) - 10 + F_{G'} = 2$ which implies $F_{G'} = 7 - |S|$. Since G is 5-connected, there cannot be any 4-cycles in G' as these would translate into separating 4-cycles in G. Since G' is bipartite, this means that all cycles in G' have length 6 or more. If each cycle in G' has length at least 6, then $6(7 - |S|) \leq 2|E(G')| = 2 \times 10 = 20$. Thus $7 - |S| \leq 3$ and so $|S| \geq 4$. Hence, $|S| = 4$.

From this we may deduce that $|V(G')| = 9$, $|E(G')| = 10$ and $F_{G'} = 3$. Now $20 = 2|E(G')| = \sum_{i \geq 6} iF_{G',i}$, where $F_{G',i}$ denotes the number of faces of size i in G'. Thus we have one 8-face and two 6-faces in G'. Suppose that $S = \{s_1, s_2, s_3, s_4\}$. Then G' is isomorphic to the graph in Figure 5.

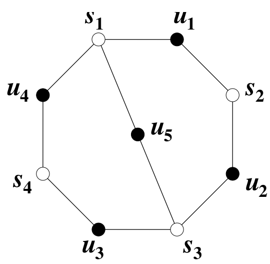

Figure 5: The graph G'.

Having established that $|S| = 4$, there are six odd components in H. Let $e = xy$. Then e lies in the interior of one of the three faces of G', and the only neighbours of e in G' are vertices in S. G' together with $e = xy$ and the edges joining x and y to their neighbours in $S \subset V(G')$ must produce a plane graph with at least six faces of size at least 5 to accommodate the six odd components in H. Moreover, there cannot be any 4-cycles in this graph as these would correspond to separating cycles in G which is 5-connected. Clearly, e and hence both x and y must lie in the same face of G' and e together with edges joining x and y to vertices in S on the boundary of this face must divide the face into at least four faces of size at least 5. Suppose that e is in one of the two 6-faces. Then it is straightforward to see that any permissible adjacencies for x and y can only result in dividing this 6-face into two faces of size at least 5.

So we may assume that e is in the 8-face. Clearly, if neither x nor y has more than one neighbour on the boundary of the 8-face, then we cannot divide this face sufficiently to produce the required number of faces of size at least 5. Consequently we may assume that x, say, has at least two such neighbours and is thus adjacent to antipodal vertices on the boundary of the 8-face. Since there are no 4-cycles allowed, x can have no other neighbours on the boundary of the 8-face. Now y lies in one of the two 6-faces resulting from the previous observation. But, while y may be adjacent to the two neighbours of x in S, if either of these adjacencies occurs, then y can have no further neighbours in S as such an additional adjacency would result in a separating 4-cycle in G. Thus the only other neighbour of y allowed to belong to the boundary of this face divides the face into two 5-faces giving just five faces of size at least 5 overall. This contradiction establishes the result. □

We note that Theorem 2.4 is sharp in that there are 5-connected even planar triangulations in which there are sets of seven edges, pairwise distance at least 2 apart and, having selected one of these edges, there is no perfect matching that contains this edge while avoiding all six remaining edges in the set. A family of such graphs was shown in Figure 36 of [3].

Theorem 3.4 of [2] is shown to be sharp by an example of a 5-connected even planar triangulation in which there is a set of eight edges, pairwise distance 2 apart, which cannot be completely avoided by any perfect matching of the graph. In the following theorem we show that by demanding that one edge in a matching of size 8 is distance at least 3 from all of the others, we can then find a perfect matching that avoids all eight edges.

Theorem 2.5. *Let G be a 5-connected planar even triangulation. Suppose $N = \{f_1, f_2, \ldots, f_8\}$ is a matching such that $d(f_1, f_i) \geq 3$ for each $i = 2, 3, \ldots, 8$. Then $G - N$ has a perfect matching.*

Proof. Let G and N be as in the statement of the theorem and suppose that $G - N$ has no perfect matching. Then by Tutte's 1-factor theorem, there is a set $S \subset V(G-N)$ such that $H = G - N - S$ has at least $|S| + 2$ odd components. Since G is $E(0, 7)$ (see Theorem 1.1(iii)), H has precisely $|S| + 2$ odd components. Moreover, each $f_i \in N$ has its endvertices in two distinct odd components of H. Now, since G is a triangulation of the plane, we may conclude that each edge $f_i = x_i y_i \in N$ has two vertices in S adjacent to both x_i and y_i forming the two triangular faces which include f_i.

Form G^*, the bipartite distillation of G based on N and S. By the 5-connectivity of G, there are at least $5(|S| + 2) - (2 \times 8)$ edges joining vertices in odd components of H to vertices in S. This gives $|E(G^*)| \geq 5|S| + 10 - 16 = 5|S| - 6$.

On the other hand, G^* is a bipartite planar graph on $|S| + 2 + |S|$ vertices, so by Euler's formula, $|E(G^*)| \leq 2|V(G^*)| - 4 = 4|S|$. Consequently, $|S| \leq 6$.

Now each edge $f_i \in N$ lies in the boundaries of two triangular faces of G. Each triangular face containing f_i must also contain a vertex from S. Suppose that the triangular faces containing f_1, say F_1 and F_2, contain s_1 and $s_2 \in S$, respectively. Since $d(f_1, f_i) \geq 3$ for each $i = 2, 3, \ldots, 8$, it follows that f_1 and f_i cannot have a common neighbour in S.

Contract each edge f_i to a single vertex u_i for $i = 2, 3, \ldots, 8$. Let $U = \{u_2, u_3, \ldots, u_8\}$ and $S' = S - \{s_1, s_2\}$.

Let $|S'| = s'$ and G' be a graph with $V(G') = U \cup S'$ with each vertex $u_i \in U$ adjacent to the two vertices in S' that form triangles with f_i, $i = 2, 3, 4$. Then $|V(G')| = 7 + s'$ and $|E(G')| = 7 \times 2 = 14$ and G' has $F_{G'}$ faces. By Euler's formula $(7 + s') - 14 + F_{G'} = 2$ which implies $F_{G'} = 9 - s'$. If each cycle in G' has length at least 6, then $6(9 - s') \leq 2|E(G')| = 14 \times 2 = 28$. Thus $9 - s' \leq 4$ and so $s' \geq 5$, a contradiction. Consequently, since G' is bipartite, it must contain a 4-cycle and hence G has a separating 4-cycle which is a contradiction. □

Theorem 2.5 is seen to be sharp in the sense that a straightforward modification of the example in Figure 3.4 of [2] gives a 5-connected even planar triangulation with a matching of size 9 in which one edge is as far away from the other eight as desired, but no perfect matching of the graph can avoid all nine of the edges in this matching.

3 Concluding remarks

We have seen that specifying one edge to be well distant from other edges in the matchings considered has been sufficient to strengthen the conclusions of Theorem 1.1. In applying more and varied distance restrictions to other edges in the matchings we may well be able to strengthen the conclusions further. We also believe that similar results may be obtained for classes of graphs embedded on other sufaces.

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