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Total Roman domination edge-supercritical and edge-removal-supercritical graphs

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Abstract

A total Roman dominating function on a graph G is a function $f: V(G) \to \{0,1,2\}$ such that every vertex v with $f(v) = 0$ is adjacent to i some vertex u with $f(u) = 2$, and the subgraph of G induced by the set of all vertices w such that $f(w) > 0$ has no isolated vertices. The weight of f is $\Sigma_{v \in V(G)} f(v)$. The total Roman domination number $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function on G. A graph G is k-γ_{tR}-edge-critical if $\gamma_{tR}(G+e) < \gamma_{tR}(G) = k$ for every edge $e \in E(\overline{G}) \neq \emptyset$, and k-γ_{tR}-edge-supercritical if it is k-γ_{tR}-edge-critical and $\gamma_{tR}(G+e) = \gamma_{tR}(G) - 2$ for every edge $e \in E(\overline{G}) \neq \emptyset$. A graph G is k-γ_{tR}-edge-stable if $\gamma_{tR}(G+e) = \gamma_{tR}(G) = k$ for every edge $e \in E(\overline{G})$ or $E(\overline{G}) = \emptyset$. For an edge $e \in E(G)$ incident with a degree 1 vertex, we define $\gamma_{tR}(G-e) = \infty$. A graph G is k-γ_{tR}-edge-removal-critical if $\gamma_{tR}(G-e) > \gamma_{tR}(G) = k$ for every edge $e \in E(G)$, and k-γ_{tR}-edge-removal-supercritical if it is k-γ_{tR}-edge-removal-critical and $\gamma_{tR}(G-e) \geq \gamma_{tR}(G) + 2$ for every edge $e \in E(G)$. A graph G is k-γ_{tR}-edge-removal-stable if $\gamma_{tR}(G-e) = \gamma_{tR}(G) = k$ for every edge $e \in E(G)$. We investigate connected γ_{tR}-edge-supercritical graphs and exhibit infinite classes of such graphs. In addition, we characterize γ_{tR}-edge-removal-critical and γ_{tR}-edge-removal-supercritical graphs. Furthermore, we present a connection between k-γ_{tR}-edge-removal-supercritical and k-γ_{tR}-edge-stable graphs, and similarly between k-γ_{tR}-edge-supercritical and k-γ_{tR}-edge-removal-stable graphs.

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1 Introduction

We consider the behaviour of the total Roman domination number of a graph G upon the addition or removal of edges to and from G. A *dominating set* S in a graph G is a set of vertices such that every vertex in $V(G)-S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set in G. A *total dominating set* S (abbreviated by *TD-set*) in a graph G with no isolated vertices is a set of vertices such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* $\gamma_t(G)$ (abbreviated by *TD-number*) is the cardinality of a minimum total dominating set in G. For $S \subseteq V(G)$ and a function $f : S \to \mathbb{R}$, define $f(S) = \Sigma_{s\in S} f(s)$. A *Roman dominating function* (abbreviated by *RD-function*) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to some vertex u with $f(u) = 2$. The *weight* of f, denoted by $\omega(f)$, is defined as $f(V(G))$. The *Roman domination number* $\gamma_R(G)$ (abbreviated by *RD-number*) is defined as $\min\{\omega(f) : f$ is an RD-function on $G\}$. For an RD-function f, let $V_f^i = \{v \in V(G) : f(v) = i\}$ and $V_f^+ = V_f^1 \cup V_f^2$. Thus, we can uniquely express an RD-function f as $f = (V_f^0, V_f^1, V_f^2)$.

As defined by Chang and Liu [6], a *total Roman dominating function* (abbreviated by *TRD-function*) on a graph G with no isolated vertices is a Roman dominating function with the additional condition that $G[V_f^+]$ has no isolated vertices. The *total Roman domination number* $\gamma_{tR}(G)$ (abbreviated by *TRD-number*) is the minimum weight of a TRD-function on G; that is, $\gamma_{tR}(G) = \min\{\omega(f) : f$ is a TRD-function on $G\}$. A TRD-function f such that $\omega(f) = \gamma_{tR}(G)$ is called a *$\gamma_{tR}(G)$-function*, or a *γ_{tR}-function* if the graph G is clear from the context; *γ_R-functions* are defined analogously. Total Roman domination was also studied by Ahangar, Henning, Samodivkin and Yero [1].

The addition of an edge to a graph has the potential to change its total domination or total Roman domination number. Van der Merwe, Mynhardt and Haynes [12] studied *γ_t-edge-critical graphs*, that is, graphs G for which $\gamma_t(G+e) < \gamma_t(G)$ for each $e \in E(\overline{G})$ and $E(\overline{G}) \neq \emptyset$. Similarly, Lampman, Mynhardt and Ogden [11] defined an edge $e \in E(\overline{G})$ to be *critical* with respect to total Roman domination (abbreviated *TRD-critical*) if $\gamma_{tR}(G + e) < \gamma_{tR}(G)$. An edge $e \in E(\overline{G})$ is *supercritical* with respect to total Roman domination (abbreviated *TRD-supercritical*) if $\gamma_{tR}(G+e) \leq \gamma_{tR}(G) - 2$. A graph G with no isolated vertices is *total Roman domination edge-critical*, or simply *γ_{tR}-edge-critical*, if every edge $e \in E(\overline{G}) \neq \emptyset$ is TRD-critical. We say that G is *k-γ_{tR}-edge-critical* if $\gamma_{tR}(G) = k$ and G is γ_{tR}-edge-critical. Similarly, if every edge $e \in E(\overline{G}) \neq \emptyset$ is TRD-supercritical, then G is *γ_{tR}-edge-supercritical*; *γ_t-edge-supercritical* graphs are defined analogously. An edge $e \in E(\overline{G})$ is *stable* with respect to total Roman domination (abbreviated *TRD-stable*) if $\gamma_{tR}(G+e) = \gamma_{tR}(G)$. If every edge $e \in E(\overline{G})$ is TRD-stable, or if $E(\overline{G}) = \emptyset$, we say that G is *γ_{tR}-edge-stable.*

The removal of an edge from a graph G also has the potential to change its total domination or total Roman domination number. Desormeaux, Haynes and Henning [8] studied *γ_t-edge-removal-critical graphs*, that is, graphs G for which

$\gamma_t(G-e) > \gamma_t(G)$ for each $e \in E(G)$. We consider the same concept for total Roman domination. An edge $e \in E(G)$ is *removal-critical* with respect to total Roman domination (abbreviated *TRD-ER-critical*) if $\gamma_{tR}(G) < \gamma_{tR}(G-e)$. We say that an edge $e \in E(G)$ is *removal-supercritical* with respect to total Roman domination (abbreviated *TRD-ER-supercritical*) if $\gamma_{tR}(G) + 2 \leq \gamma_{tR}(G-e)$. Note that the removal of an edge $e \in E(G)$ incident with a degree 1 vertex would result in $G-e$ containing an isolated vertex. For such an edge $e \in E(G)$, Desormeaux et al. [8] defined $\gamma_t(G-e) = \infty$. Likewise, we define $\gamma_{tR}(G-e) = \infty$ when $e \in E(G)$ is an edge incident with a degree 1 vertex. Furthermore, we define $E_P(G) \subseteq E(G)$ to be the set of edges in G which are not incident with a degree 1 vertex; that is, the set of edges e such that $\gamma_{tR}(G-e) < \infty$. Hence every edge $e \in E(G) - E_P(G)$ is TRD-ER-supercritical. A graph G with no isolated vertices is *total Roman domination edge-removal-critical*, or simply *γ_{tR}-ER-critical*, if every edge $e \in E(G)$ is TRD-ER-critical. We say that G is *k-γ_{tR}-ER-critical* if $\gamma_{tR}(G) = k$ and G is γ_{tR}-ER-critical. Similarly, if every edge $e \in E(G)$ is TRD-ER-supercritical, then G is *γ_{tR}-ER-supercritical*; *γ_t-ER-supercritical* graphs are defined analogously. An edge $e \in E(G)$ is *removal-stable* with respect to total Roman domination (abbreviated *TRD-ER-stable*) if $\gamma_{tR}(G) = \gamma_{tR}(G-e)$. If every edge $e \in E(G)$ is TRD-ER-stable, we say that G is *γ_{tR}-edge-removal-stable*, or simply *γ_{tR}-ER-stable*.

Pushpam and Padmapriea [13] established bounds on the total Roman domination number of a graph in terms of its order and girth. Total Roman domination in trees was studied by Amjadi, Nazari-Moghaddam, Sheikholeslami and Volkmann [2], as well as by Amjadi, Sheikholeslami and Soroudi [3]. The authors of [3] also studied Nordhaus-Gaddum bounds for total Roman domination in [4]. Campanelli and Kuziak [5] considered total Roman domination in the lexicographic product of graphs. We refer the reader to the well-known books [7] and [9] for graph theory concepts not defined here. Frequently used or lesser known concepts are defined where needed.

We begin with some previous results on the total domination and total Roman domination numbers of a graph in Section 2, and γ_{tR}-edge-critical graphs in Section 3. In Section 4, we investigate the existence of connected γ_{tR}-edge-supercritical graphs and demonstrate that each such graph contains a cycle. After characterizing 5-γ_{tR}-edge-critical graphs in Section 5, we investigate 6-γ_{tR}-edge-supercritical graphs in Section 6. In Section 7, we characterize γ_{tR}-ER-critical graphs. A similar characterization of γ_{tR}-ER-supercritical graphs is presented in Section 8, where we also note that every γ_{tR}-ER-supercritical graph is γ_{tR}-edge-stable. The analogous result for γ_{tR}-edge-supercritical and γ_{tR}-ER-stable graphs is given in Section 9. We conclude in Section 10 with ideas for future research.

2 Preliminaries

Before investigating γ_{tR}-edge-critical and γ_{tR}-ER-critical graphs, we present some basic results relating the domination, total domination, and total Roman domination numbers of a graph. Our first result is a direct corollary to Observation 6.42 and Theorem 6.47 in [9], and provides bounds on the total domination number of a graph G in terms of its domination number.

Proposition 2.1. [9] *For a graph G with no isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

As noted in Section 1, total Roman domination was studied by Ahangar et al. [1]. There, they provided two results which bound the total Roman domination number of a graph in terms of its domination number and total domination number, respectively. Note the similarities between the bounds in Propositions 2.1 and 2.3.

Proposition 2.2. [1] *For a graph G with no isolated vertices, $2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G)$.*

Proposition 2.3. [1] *If G is a graph with no isolated vertices, then $\gamma_t(G) \leq \gamma_{tR}(G) \leq 2\gamma_t(G)$. Furthermore, $\gamma_{tR}(G) = \gamma_t(G)$ if and only if G is the disjoint union of copies of K_2.*

Note that Proposition 2.3 characterizes the graphs G for which $\gamma_{tR}(G) = \gamma_t(G)$. The problem of determining whether $\gamma_{tR}(G) = 2\gamma(G)$, $\gamma_{tR}(G) = 2\gamma_t(G)$ or $\gamma_{tR}(G) = 3\gamma(G)$ was shown to be NP-hard by Poureidi and Jafari Rad [14]. Ahangar et al. [1] also characterized the graphs which nearly attain the lower bound in Proposition 2.3; that is, the graphs G for which $\gamma_{tR}(G) = \gamma_t(G) + 1$.

Proposition 2.4. [1] *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{tR}(G) = \gamma_t(G) + 1$ if and only if $\Delta(G) = n - 1$, that is, G has a universal vertex.*

We now consider the graphs with the smallest possible TRD-number, namely 3, which were characterized by Lampman et al. [11].

Proposition 2.5. [11] *For a graph G of order $n \geq 3$ with no isolated vertices, $\gamma_{tR}(G) = 3$ if and only if $\Delta(G) = n - 1$, that is, G has a universal vertex.*

When combined with Proposition 2.4, Proposition 2.5 implies that, for a connected graph G of order $n \geq 3$, $\gamma_{tR}(G) = \gamma_t(G) + 1$ if and only if $\gamma_{tR}(G) = 3$. This result provides a tighter lower bound on the TRD-number of a connected graph with no universal vertex with respect to its TD-number.

Observation 2.6. *If G is a connected graph of order $n \geq 3$ such that $\Delta(G) \leq n-2$, then $\gamma_t(G) + 2 \leq \gamma_{tR}(G) \leq 2\gamma_t(G)$.*

Lampman et al. [11] also provided an alternate characterization of the graphs G with total Roman domination number 3, as well as a characterization of the graphs G with total Roman domination number 4, in terms of the domination and total domination numbers of the graph.

Proposition 2.7. [11] *If G is a connected graph of order $n \geq 3$, then $\gamma_{tR}(G) \in \{3, 4\}$ if and only if $\gamma_t(G) = 2$. Moreover, $\gamma(G) = 1$ when $\gamma_{tR}(G) = 3$, and $\gamma(G) = 2$ when $\gamma_{tR}(G) = 4$.*

3 γ_{tR}-Edge-critical graphs

As noted in Section 1, the addition of an edge to a graph has the potential to change its total domination or total Roman domination number. Van der Merwe et al. [12] studied this effect with respect to the total domination number, providing bounds on the total domination number of the graph $G + e$, where $e \in E(\overline{G})$, in terms of the total domination number of G.

Proposition 3.1. [12] *For a graph G with no isolated vertices, if $uv \in E(\overline{G})$, then $\gamma_t(G) - 2 \leq \gamma_t(G + uv) \leq \gamma_t(G)$.*

These bounds also hold with respect to the total Roman domination number of the graph $G + e$ obtained by adding an edge $e \in E(\overline{G})$ to G, as shown by Lampman et al. [11].

Proposition 3.2. [11] *Given a graph G with no isolated vertices, if $uv \in E(\overline{G})$, then $\gamma_{tR}(G) - 2 \leq \gamma_{tR}(G + uv) \leq \gamma_{tR}(G)$.*

For any edge $uv \in E(G)$, there are $3^2 = 9$ ways for a TRD-function f to assign the values in $\{0, 1, 2\}$ to u and v. However, the following observation restricts the possible values assigned to a degree 1 vertex and its unique neighbour when f is a $\gamma_{tR}(G)$-function. Note that, for a graph G and a vertex $v \in V(G)$, the *open neighbourhood* of v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, and the *closed neighbourhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$.

Observation 3.3. *For a graph G with no isolated vertices, if $deg(u) = 1$ and $N_G(u) = \{v\}$, then, for any $\gamma_{tR}(G)$-function f, either $f(u) = f(v) = 1$, or $f(v) = 2$ and $f(u) \in \{0, 1\}$. Furthermore, there exists a $\gamma_{tR}(G)$-function g such that $\{g(u), g(v)\} \neq \{1, 2\}$.*

Similarly, Lampman et al. [11] provided a result restricting the possible values assigned to the vertices of a TRD-critical edge uv by a γ_{tR}-function f on $G + uv$. We mildly abuse set-theoretic notation by denoting the case where $f(u) = f(v) = i$ for $i \in \{0, 1, 2\}$ by $\{f(u), f(v)\} = \{i, i\}$.

Proposition 3.4. [11] *Given a graph G with no isolated vertices, if $uv \in E(\overline{G})$ is a TRD-critical edge and f is a $\gamma_{tR}(G+uv)$-function, then $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$. If, in addition, $deg(u) = deg(v) = 1$, then there exists a $\gamma_{tR}(G + uv)$-function f such that $f(u) = f(v) = 1$.*

We now consider γ_{tR}-edge-critical graphs. Recall that a graph G with no isolated vertices is γ_{tR}-edge-critical if $\gamma_{tR}(G+e) < \gamma_{tR}(G)$ for every edge $e \in E(\overline{G}) \neq \emptyset$. For a graph $G \neq K_2$, the unique neighbour of an end-vertex of G is called its *support vertex*. In this case, the end-vertex is referred to as a *pendant vertex*, and the edge incident with it a *pendant edge*. An *endpath* in a graph G is a path from a vertex v, where $\deg(v) \geq 3$, to a pendant vertex, such that all of the internal vertices of the path have degree 2. We begin with some results from [11] which provide necessary conditions for a graph G to be γ_{tR}-edge-critical.

Proposition 3.5. [11] *For a graph G with no isolated vertices, if G has a pendant vertex w with support vertex x such that $G[N(x) - \{w\}]$ is not complete, then G is not γ_{tR}-edge-critical.*

Proposition 3.6. [11] *For a graph G with no isolated vertices, if G has two endpaths $v_0, v_1, \ldots, v_k$ and $u_0, u_1, \ldots, u_m$, where $k, m \geq 3$ and v_k and u_m are pendant vertices, then G is not γ_{tR}-edge-critical.*

We conclude this section by considering the graphs G which have the largest TRD-number, namely $|V(G)|$. A *subdivided star* is a tree obtained from a star on at least three vertices by subdividing each edge exactly once. A *double star* is a tree obtained from two disjoint non-trivial stars by joining the two central vertices (choosing either central vertex in the case of K_2). The *corona* $\text{cor}(G)$ (sometimes denoted by $G \circ K_1$) of G is obtained by joining each vertex of G to a new end-vertex.

Connected graphs G for which $\gamma_{tR}(G) = |V(G)|$ were characterized in [1]. There, Ahangar et al. defined $\mathcal{G}$ as the family of connected graphs obtained from a 4-cycle v_1, v_2, v_3, v_4, v_1 by adding $k_1 + k_2 \geq 1$ vertex-disjoint paths P_2, and joining v_i to an end-vertex of k_i such paths, for $i \in \{1, 2\}$. Note that possibly $k_1 = 0$ or $k_2 = 0$. Furthermore, they defined $\mathcal{H}$ to be the family of graphs obtained from a double star by subdividing each pendant edge once and the non-pendant edge $r \geq 0$ times.

Proposition 3.7. [1] *If G is a connected graph of order $n \geq 2$, then $\gamma_{tR}(G) = n$ if and only if one of the following holds.*
- (i) *G is a path or a cycle;*
- (ii) *G is the corona of a graph;*
- (iii) *G is a subdivided star;*
- (iv) *$G \in \mathcal{G} \cup \mathcal{H}$.*

Lampman et al. [11] used this result to characterize the connected graphs of order $n \geq 4$ which are n-γ_{tR}-edge-critical. For $r \geq 0$, they defined $\mathcal{H}_r \subseteq \mathcal{H}$ as the family of graphs in $\mathcal{H}$ where the non-pendant edge was subdivided r times.

Proposition 3.8. [11] *A connected graph G of order $n \geq 4$ is n-γ_{tR}-edge-critical if and only if G is one of the following graphs:*
- (i) *C_n, $n \geq 4$;*
- (ii) *$cor(K_r)$, $r \geq 3$;*
- (iii) *a subdivided star of order $n \geq 7$;*

(iv) $G \in \mathcal{G}$;
(v) $G \in \mathcal{H} - \mathcal{H}_0 - \mathcal{H}_2$.

4 γ_{tR}-Edge-supercritical graphs

We now consider γ_t-edge-supercritical and γ_{tR}-edge-supercritical graphs. Note that, by Proposition 3.1, a graph G with no isolated vertices is γ_t-edge-supercritical when $\gamma_t(G+e) = \gamma_t(G) - 2$ for every $e \in E(\overline{G}) \neq \emptyset$. Similarly, by Proposition 3.2, a graph G with no isolated vertices is γ_{tR}-edge-supercritical when $\gamma_{tR}(G+e) = \gamma_{tR}(G) - 2$ for every $e \in E(\overline{G}) \neq \emptyset$. We begin with a result by Haynes, Mynhardt and Van der Merwe [10] characterizing γ_t-edge-supercritical graphs, as well as the lemma required to prove this result.

Lemma 4.1. [10] *If G is a graph with no isolated vertices and $u, v \in V(G)$ such that $d(u,v) = 2$, then $\gamma_t(G) - 1 \leq \gamma_t(G+uv)$.*

Proposition 4.2. [10] *A graph G is γ_t-edge-supercritical if and only if G is the union of two or more non-trivial complete graphs.*

Lampman et al. [11] considered whether an analogous result holds for γ_{tR}-edge-supercritical graphs. They determined that a result analogous to Lemma 4.1 does not hold with respect to total Roman domination, and thus, even if a result similar to Proposition 4.2 holds, it cannot be proved via the technique employed by Haynes et al. in [10]. However, they did establish that an analogous sufficient condition does hold for γ_{tR}-edge-supercritical graphs, which we now present.

Proposition 4.3. [11]

(i) *There are no 5-γ_{tR}-edge-supercritical graphs.*
(ii) *If G is the disjoint union of $k \geq 2$ complete graphs, each of order at least 3, then G is $3k$-γ_{tR}-edge-supercritical.*

Lampman et al. [11] left the existence of connected γ_{tR}-edge-supercritical graphs as an open problem, which we investigate here. We begin by demonstrating the existence of connected $2n$-γ_{tR}-edge-supercritical graphs for $n \geq 4$.

Proposition 4.4. *If $G = cor(K_n)$ for $n \geq 4$, then G is γ_{tR}-edge-supercritical.*

Proof. By Proposition 3.7, $\gamma_{tR}(G) = 2n$. Label the vertices of G such that $u_1, u_2, \ldots, u_n$ are the pendant vertices with support vertices $w_1, w_2, \ldots, w_n$, respectively. Consider $uv \in E(\overline{G})$. Then at least one of u and v has degree 1 in G; say $\deg_G(u) = 1$. Note that we may assume $u = u_1$, without loss of generality. We consider two cases:

Case 1: Suppose $v = u_2$ (without loss of generality). Consider $f : V(G) \to \{0,1,2\}$ defined by $f(u_1) = f(u_2) = 1$, $f(w_3) = f(w_4) = \cdots = f(w_n) = 2$, and $f(z) = 0$ for all other $z \in V(G)$.

Case 2: Suppose $v = w_2$ (without loss of generality). Consider $f : V(G) \to \{0, 1, 2\}$ defined by $f(w_2) = f(w_3) == f(w_n) = 2$, and $f(z) = 0$ for all other $z \in V(G)$.

In either case, f is a TRD-function on $G + uv$ with $\omega(f) = 2n - 2$. Hence G is γ_{tR}-edge-supercritical. □

Having proved the existence of connected γ_{tR}-edge-supercritical graphs, we now present the following necessary condition for a graph G to be γ_{tR}-edge-supercritical.

Proposition 4.5. *If G is a γ_{tR}-edge-supercritical graph, then G contains no adjacent endpaths.*

Proof. Suppose for a contradiction that G contains two adjacent endpaths $w, v_1, \ldots, v_n$ and $w, u_1, \ldots, u_m$. Since G is γ_{tR}-edge-supercritical, Proposition 3.5 implies that $n, m \geq 2$. Moreover, by Proposition 3.6, at least one of n and m is equal to 2; say $n = 2$. Consider $u_1v_1 \in E(\overline{G})$ and a γ_{tR}-function f on $G + u_1v_1$. Since $n = 2$, Observation 3.3 implies that $f(v_1) > 0$. If $f(u_1) > 0$, define $f' : V(G) \to \{0, 1, 2\}$ by $f'(w) = \min\{2, f(w) + 1\}$ and $f'(x) = f(x)$ for all other $x \in V(G)$. Otherwise, if $f(u_1) = 0$, then by Proposition 3.4, $f(v_1) = 2$. Thus, by Observation 3.3, we may assume without loss of generality that $f(v_2) = 0$. Hence $f(w) > 0$. Therefore, define $f' : V(G) \to \{0, 1, 2\}$ by $f'(u_1) = 1$ and $f'(x) = f(x)$ for all other $x \in V(G)$. In either case, f' is a TRD-function on G with $\omega(f') \leq \omega(f) + 1$, contradicting G being γ_{tR}-edge-supercritical. Therefore G contains no adjacent endpaths. □

As a result of Proposition 4.5, every γ_{tR}-edge-supercritical graph contains a cycle, as we now show.

Corollary 4.6. *There are no γ_{tR}-edge-supercritical trees.*

Proof. Suppose for a contradiction that T is a γ_{tR}-edge-supercritical tree. By Propositions 3.7 and 3.8, T cannot be a path. Therefore T contains at least one branch vertex (that is, a vertex of degree 3 or more), and hence two adjacent endpaths, contradicting Proposition 4.5. Therefore, there are no γ_{tR}-edge-supercritical trees. □

5 5-γ_{tR}-Edge-critical graphs

As seen in Section 2, Lampman et al. characterized connected 4-γ_{tR}-edge-critical graphs in [11]. There, they also provided necessary conditions for a graph G to be 5-γ_{tR}-edge-critical (see Proposition 5.1). In this section, we develop a characterization of 5-γ_{tR}-edge-critical graphs from these necessary conditions.

Proposition 5.1. [11] *For any graph G, if G is 5-γ_{tR}-edge-critical, then G is either 3-γ_t-edge-critical or $G = K_2 \cup K_n$ for $n \geq 3$, in which case G is 4-γ_t-edge-supercritical.*

Before characterizing 5-γ_{tR}-edge-critical graphs, we characterize the connected graphs with total Roman domination number 5, as follows.

Theorem 5.2. *For a connected graph G, $\gamma_{tR}(G) = 5$ if and only if $\gamma_t(G) = 3$ and there exist a $\gamma(G)$-set S and a $\gamma_t(G)$-set T such that $S \subset T$.*

Proof. Suppose $\gamma_{tR}(G) = 5$. By Proposition 2.2, $\gamma(G) \leq 2$. Furthermore, by Proposition 2.5, G has no universal vertex. Therefore $\gamma(G) > 1$, and thus $\gamma(G) = 2$. Moreover, Observation 2.6 implies that $\gamma_t(G) \leq 3$. By Proposition 2.7, $\gamma_t(G) \neq 2$, and thus $\gamma_t(G) = 3$. Now, consider a $\gamma_{tR}(G)$-function f such that $G[V_f^+]$ contains the minimum number of components. If $|V_f^2| = 0$, then by Proposition 3.7, $G \cong P_5$ or $G \cong C_5$. In either case, there exist a $\gamma(G)$-set S and a $\gamma_t(G)$-set T such that $S \subset T$. If $|V_f^2| = 2$, then V_f^2 is a $\gamma(G)$-set and V_f^+ is a $\gamma_t(G)$-set, where $V_f^2 \subset V_f^+$ as required. Otherwise, assume $|V_f^2| = 1$; say $f(u) = 2$. Since f was chosen such that $G[V_f^+]$ contains the minimum number of components, it is easy to see that $G[V_f^+]$ is connected. Therefore, $G[V_f^1] \cong P_3 = v, w, x$ such that $uv \in E(G)$ but $uw, ux \notin E(G)$. Taking $S = \{u, w\}$ and $T = \{u, v, w\}$ gives the required result.

Conversely, suppose $\gamma_t(G) = 3$ and there exist a $\gamma(G)$-set S and a $\gamma_t(G)$-set T such that $S \subset T$. Then $\gamma(G) < 3$, and thus $\gamma(G) = 2$, as G clearly has no universal vertex. Therefore, by Proposition 2.2, $4 \leq \gamma_{tR}(G) \leq 6$. Furthermore, Proposition 2.7 implies that $\gamma_{tR}(G) \neq 4$. Hence $\gamma_{tR}(G) \in \{5, 6\}$. Suppose for a contradiction that $\gamma_{tR}(G) = 6$. Since $\gamma_t(G) = 3$, $G[T] \cong K_3$ or $G[T] \cong P_3$. Clearly $G[T] \ncong K_3$, otherwise $G[S]$ would be connected, contradicting $\gamma_t(G) = 3$. Thus $G[T] \cong P_3$; say $G[T]$ is the path u, v, w. Since $S \subset T$, clearly $S = \{u, w\}$. However, the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(u) = f(w) = 2$, $f(v) = 1$, and $f(y) = 0$ for all other $y \in V(G)$ is then a TRD-function on G with $\omega(f) = 5$, contradicting $\gamma_{tR}(G) = 6$. Therefore $\gamma_{tR}(G) = 5$. □

The characterization of 5-γ_{tR}-edge-critical graphs follows.

Proposition 5.3. *A graph G is 5-γ_{tR}-edge-critical if and only if either G is 3-γ_t-edge-critical and there exist a $\gamma(G)$-set S and a $\gamma_t(G)$-set T such that $S \subset T$, or $G = K_2 \cup K_n$ for $n \geq 3$, in which case G is 4-γ_t-edge-supercritical.*

Proof. If G is 5-γ_{tR}-edge-critical, then the result follows directly from Proposition 5.1 and Theorem 5.2. Conversely, suppose G is 3-γ_t-edge-critical and there exists a $\gamma(G)$-set S and a $\gamma_t(G)$-set T such that $S \subset T$. Then $\gamma_t(G + e) = 2$ for every $e \in E(\overline{G})$. Therefore Proposition 2.7 implies that $\gamma_{tR}(G + e) \in \{3, 4\}$ for every $e \in E(\overline{G})$. Since $\gamma_t(G) = 3$ and there exist a $\gamma(G)$-set S and a $\gamma_t(G)$-set T such that $S \subset T$, Theorem 5.2 implies that $\gamma_{tR}(G) = 5$, and thus G is 5-γ_{tR}-edge-critical. Otherwise, if $G = K_2 \cup K_n$ for $n \geq 3$, then G is clearly 5-γ_{tR}-edge-critical. □

6 6-γ_{tR}-Edge-supercritical graphs

We now consider γ_{tR}-edge-supercritical graphs with total Roman domination number 6, which, by Proposition 4.3, is the smallest TRD-number possible for a γ_{tR}-edge-supercritical graph. We begin by characterizing the disconnected 6-γ_{tR}-edge-supercritical graphs.

Proposition 6.1. *A disconnected graph G is 6-γ_{tR}-edge-supercritical if and only if $G \cong K_n \cup K_m$, where $n, m \geq 3$.*

Proof. First, suppose G is 6-γ_{tR}-edge-supercritical. Since $\gamma_{tR}(H) \geq 2$ for any graph H without isolated vertices, with equality if and only if $H = K_2$, G has two or three components. If G has three components, then $G = K_2 \cup K_2 \cup K_2$ and $\gamma_{tR}(G+e) = 6$ for any $e \in E(\overline{G})$, contradicting G being 6-γ_{tR}-edge-supercritical. Thus G has two components; say H_1 and H_2. Now, either (say) $H_1 = K_2$ and $\gamma_{tR}(H_2) = 4$, or $\gamma_{tR}(H_1) = \gamma_{tR}(H_2) = 3$. In the former case, Proposition 2.5 implies that H_2 is not complete. Thus $\gamma_{tR}(H_2 + e) \geq 3$ for any edge $e \in E(\overline{H_2}) \neq \emptyset$, contradicting our assumption that G is 6-γ_{tR}-edge-supercritical. In the latter case, H_i has a universal vertex for $i = 1, 2$. If H_i is not complete, then $\gamma_{tR}(H_i+e) = 3$, and thus $\gamma_{tR}(G+e) = 6$ for each edge $e \in E(\overline{H_i}) \neq \emptyset$, contradicting G being 6-γ_{tR}-edge-supercritical. We conclude that H_1 and H_2 are complete graphs of order at least 3, as required. The converse follows directly from Proposition 4.3. □

We now consider connected 6-γ_{tR}-edge-supercritical graphs, beginning with a result bounding the diameter of such a graph.

Proposition 6.2. *If G is a connected 6-γ_{tR}-edge-supercritical graph, then*

$$2 \leq \operatorname{diam}(G) \leq 3.$$

Proof. Clearly $2 \leq \operatorname{diam}(G)$, otherwise $E(\overline{G}) = \emptyset$ and hence G is not γ_{tR}-edge-critical. Now, suppose for a contradiction that $\operatorname{diam}(G) \geq 4$. Let u and v be vertices such that $d(u, v) = 4$; say u, x, y, z, v is a $u - v$ path. Since G is 6-γ_{tR}-edge-supercritical, $\gamma_{tR}(G + uv) = 4$. Consider a γ_{tR}-function f on $G + uv$. By Proposition 3.4, $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$. If $f(u) = f(v) = 1$, then, in order to totally Roman dominate $\{x, y, z\}$, there exists some vertex $w \in N_G(u)$ (without loss of generality) such that $w \in N_G(x) \cap N_G(y) \cap N_G(z)$. But then u, w, z, v is a shorter $u-v$ path, a contradiction. Otherwise, if $f(u) = 2$ (without loss of generality), then, in order to totally Roman dominate $\{y, z\}$, there exists some vertex $w \in N_G(u)$ such that $w \in N_G(y) \cap N_G(z)$. Again, u, w, z, v is a shorter $u - v$ path, a contradiction. Therefore $\operatorname{diam}(G) \leq 3$. □

In Section 4, we demonstrated the existence of connected $2n$-γ_{tR}-edge-supercritical graphs for each $n \geq 4$. We now demonstrate the existence of an infinite class of 6-γ_{tR}-edge-supercritical graphs. We define the graph G_r below, and show that G_r is such a graph for each $r \geq 2$. Note that $\operatorname{diam}(G_r) = 3$.

Let G_r be the graph constructed from K_{2r} as follows: Label the vertices of K_{2r} as $u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_r$, and remove from K_{2r} the perfect matching u_iw_i where $1 \leq i \leq r$. Add a vertex disjoint K_3 component to K_{2r}, and label the added vertices x, y, z. Let z be adjacent to both u_i and w_i, and y be adjacent to u_i, for $1 \leq i \leq r$. Finally, add two more vertices u_0 and w_0, such that $u_0x, u_0u_i, w_0y, w_0w_i \in E(G_r)$ for $1 \leq i \leq r$. See Figure 1.

Theorem 6.3. *If $r \geq 2$, then G_r is 6-γ_{tR}-edge-supercritical.*

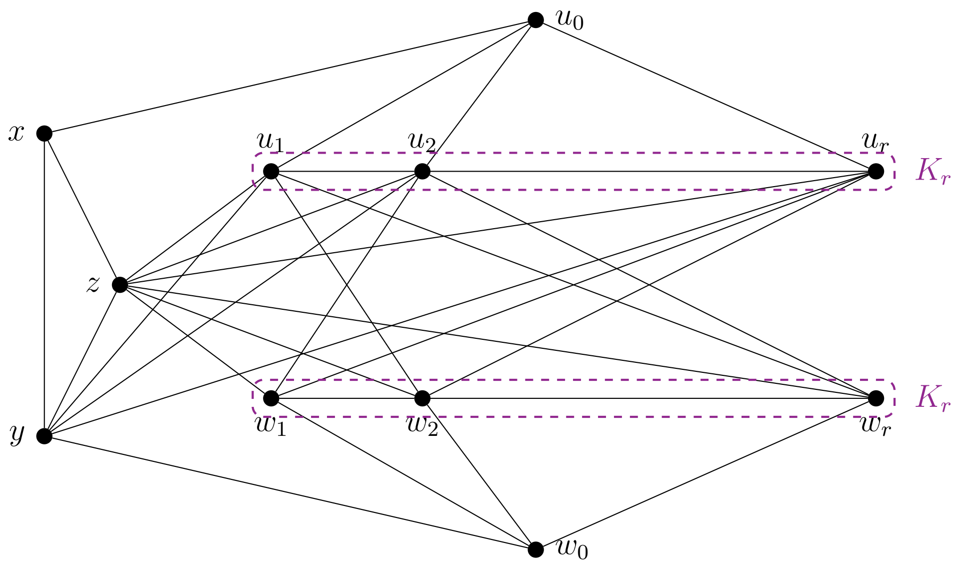

Figure 1: The graph G_r, where $r \geq 2$.

Proof. We first show that $\gamma_{tR}(G_r) = 6$ for $r \geq 2$. By inspection, $\gamma(G) > 2$. Therefore, since $\{x, y, z\}$ dominates G_r, $\gamma(G) = 3$. Furthermore, this is a TD-set on G, and thus $\gamma_t(G) = 3$. By Proposition 2.2, $\gamma_{tR}(G) \leq 6$. Moreover, Proposition 2.7 and Theorem 5.2 imply that $\gamma_{tR}(G) > 5$, and hence $\gamma_{tR}(G) = 6$.

Now, consider any edge $vv' \in E(\overline{G_r})$. Consider the following cases:

Case 1: Let $v = u_0$. Then, without loss of generality, $v' \in \{y, z, w_0, w_1\}$. If $v' \in \{y, w_1\}$, consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(v') = f(z) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$. Otherwise, if $v' \in \{z, w_0\}$, consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(v') = f(y) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$.

Case 2: Let $v = z$. Then $v' = w_0$. Consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(u_1) = f(z) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$.

Case 3: Let $v = y$. Then, without loss of generality, $v' = w_1$. Consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(y) = f(u_1) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$.

Case 4: Let $v = w_0$. Then, without loss of generality, $v' \in \{x, u_1\}$. Consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(v') = f(z) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$.

Case 5: Let $v = x$. Then, without loss of generality, $v' \in \{u_1, w_2\}$. Consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(u_1) = f(w_2) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$.

Case 6: Let $v = u_1$ (without loss of generality). Then $v' = w_1$. Consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(y) = f(u_1) = 2$ and $f(b) = 0$ for all other

$b \in V(G_r)$.

In each case, f is a TRD-function on G_r+vv' with $\omega(f) = 4$. Therefore, by Proposition 3.2, $\gamma_{tR}(G_r + e) = 4$ for any $e \in E(\overline{G_r})$. Thus G_r is 6-γ_{tR}-edge-supercritical. □

Corollary 6.4. *For $r \geq 2$, there exists a connected 6-γ_{tR}-edge-supercritical graph on $5 + 2r$ vertices.*

7 γ_{tR}-Edge-removal-critical graphs

We now consider the effect that the removal of an edge has on the total Roman domination number of a graph. The following observations follow directly from Propositions 3.2 and 3.4, and Observation 3.3.

Observation 7.1. *Given a graph G with no isolated vertices, if $uv \in E_P(G)$, then $\gamma_{tR}(G) \leq \gamma_{tR}(G - uv) \leq \gamma_{tR}(G) + 2$.*

Observation 7.2. *For a graph G with no isolated vertices, if $uv \in E(G)$ is a TRD-ER-critical edge, then, for any $\gamma_{tR}(G)$-function f, $\{f(u), f(v)\} \in \{\{0, 2\}, \{1, 2\}, \{2, 2\}, \{1, 1\}\}$.*

As with TRD-ER-critical edges, we now present a result restricting the possible values assigned to the vertices of a TRD-ER-supercritical edge $e \in E(G)$ by a γ_{tR}-function f on G.

Proposition 7.3. *For a graph G with no isolated vertices, if $uv \in E(G)$ is a TRD-ER-supercritical edge, then there exists a $\gamma_{tR}(G)$-function f such that $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 0\}, \{1, 1\}\}$.*

Proof. Let $G' = G - uv$. By Observation 7.2, $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 1\}, \{2, 0\}, \{1, 1\}\}$ for any $\gamma_{tR}(G)$-function f. Suppose for a contradiction that $\{f(u), f(v)\} = \{1, 2\}$ for every $\gamma_{tR}(G)$-function f, and consider one such function. Say $f(u) = 2$ and $f(v) = 1$. Then by Observation 3.3, $\deg_G(u) > 1$ and $\deg_G(v) > 1$. Now, f is a RD-function on G', with u and v being the only possible isolated vertices in $G'[V_f^+]$. Note that at least one of u and v must be isolated in $G'[V_f^+]$, otherwise f is also a TRD-function on G', contradicting uv being TRD-ER-critical.

Suppose for a contradiction that v is isolated in $G'[V_f^+]$. That is, $f(x) = 0$ for all $x \in N_G(v) - \{u\}$. Since $\deg_G(u) > 1$, there exists some $w \in N_G(u) - \{v\}$. But $f(w) = 0$ for each such w, otherwise $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(v) = 0$ and $f'(z) = f(z)$ for all other $z \in V(G)$ would be a TRD-function on G, contradicting the minimality of f. That is, u is also isolated in $G'[V_f^+]$. But then $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 0$, $g(w) = 1$ for some $w \in N(u) - \{v\}$, and $g(z) = f(z)$ for all other $z \in V(G)$ is a $\gamma_{tR}(G)$-function with $g(u) = 2$ and $g(v) = 0$, contradicting our assumption.

Therefore u is the only isolated vertex in $G'[V_f^+]$. But then $g : V(G) \to \{0, 1, 2\}$ defined by $g(w) = 1$ for some $w \in N_G(u) - \{v\}$ and $g(z) = f(z)$ for all other $z \in V(G)$ is a TRD-function on G with $\omega(g) = \omega(f) + 1$, contradicting uv being a TRD-ER-supercritical edge. □

Corollary 7.4. *For a graph G with no isolated vertices, if $uv \in E(\overline{G})$ is a TRD-supercritical edge, then there exists a $\gamma_{tR}(G+uv)$-function f such that $\{f(u), f(v)\} \in \{\{2, 2\}, \{2, 0\}, \{1, 1\}\}$.*

We now consider γ_t-ER-critical and γ_{tR}-ER-critical graphs. Recall that a graph G with no isolated vertices is γ_t-ER-critical if $\gamma_t(G - e) > \gamma_t(G)$ for every $e \in E(G)$, and similarly γ_{tR}-ER-critical if $\gamma_{tR}(G - e) > \gamma_{tR}(G)$ for every $e \in E(G)$. Connected γ_t-ER-critical graphs G were characterized in [8]. There, Desormeaux et al. defined $\mathcal{T}$ to be the family of trees T such that T is either a nontrivial star, or a double star, or can be obtained from a subdivided star by adding zero or more pendant edges to the non-leaf vertices.

Proposition 7.5. [8] *A connected graph G is γ_t-ER-critical if and only if $G \in \mathcal{T}$.*

Note that a disconnected graph G is γ_t-ER-critical if and only if each component of G is itself γ_t-ER-critical. As a result, Proposition 7.5 provides the following characterization of all γ_t-ER-critical graphs.

Observation 7.6. *A graph G is γ_t-ER-critical if and only if G is the union of $k \geq 1$ graphs $G_i \in \mathcal{T}$, for $1 \leq i \leq k$.*

We investigate whether a similar characterization holds for γ_{tR}-ER-critical graphs. Note that as with γ_t-ER-critical graphs, a disconnected graph G is γ_{tR}-ER-critical if and only if each component of G is itself γ_{tR}-ER-critical. Similarly, a disconnected graph G is γ_{tR}-ER-supercritical if and only if each component of G is itself γ_{tR}-ER-supercritical. As a result, we focus specifically on connected γ_{tR}-ER-critical and γ_{tR}-ER-supercritical graphs. We begin with a result restricting the values that a $\gamma_{tR}(G)$-function f can assign to the vertices of a γ_{tR}-ER-critical graph based on their degree.

Proposition 7.7. *Let G be a connected γ_{tR}-ER-critical graph. For any γ_{tR}-function f on G, if $f(u) = 0$, then* $\deg(u) = 1$. *Moreover, $\delta(G) = 1$.*

Proof. Let G be a connected k-γ_{tR}-ER-critical graph of order n, and f any $\gamma_{tR}(G)$-function. Suppose for a contradiction that there exists $u \in V(G)$ such that $f(u) = 0$ and $\deg(u) \geq 2$. Then there exist $v, w \in N_G(u)$. By Observation 7.2, $f(v) = f(w) = 2$. But then f is also a TRD-function on $G - uv$, contradicting uv being TRD-ER-critical. Hence $\deg(u) = 1$. Now, if $\delta(G) \geq 2$, then $V_f^1 = V(G)$; that is, $k = n$. But then Observation 7.1 implies that $\gamma_{tR}(G-e) = n = k$ for all $e \in E(G)$, contradicting our assumption that G is γ_{tR}-ER-critical. Hence $\delta(G) = 1$. □

Note that Proposition 7.7 implies that every component of a γ_{tR}-ER-critical graph contains at least one degree 1 vertex. We now present a result demonstrating that a connected γ_{tR}-ER-critical graph G cannot contain any cycles.

Proposition 7.8. *If G is a connected γ_{tR}-ER-critical graph, then G is a tree.*

Proof. Suppose for a contradiction that G is a connected γ_{tR}-ER-critical graph which contains a cycle; say $v_1, v_2, \ldots, v_k, v_1$, for $k \geq 3$. Consider a γ_{tR}-function f on G. By Proposition 7.7, $f(v_i) > 0$ for $1 \leq i \leq k$. But then f is also a TRD-function on $G - v_1v_2$, contradicting G being γ_{tR}-ER-critical. Hence G cannot contain a cycle, and thus, since G is connected, G is a tree. □

Our next result restricts the distance between any two vertices of a γ_{tR}-ER-critical graph G which are in V_f^+ for some $\gamma_{tR}(G)$-function f.

Proposition 7.9. *Let G be a connected γ_{tR}-ER-critical graph. If $u, v \in V(G)$ and f is a γ_{tR}-function on G such that $f(u) > 0$ and $f(v) > 0$, then $d(u, v) \leq 2$.*

Proof. Let G be a connected γ_{tR}-ER-critical graph. Then, by Proposition 7.8, G is a tree. Let $f(u) > 0$ and $f(v) > 0$, and suppose for a contradiction that $u, w_1, \ldots, w_k, v$ is the unique path from u to v, where $k \geq 2$. Consider a γ_{tR}-function f on G. Then Proposition 7.7 implies that $f(w_i) > 0$ for all $1 \leq i \leq k$. But then f is a TRD-function on $G - w_1w_2$, contradicting G being γ_{tR}-ER-critical. □

Corollary 7.10. *Let G be a connected γ_{tR}-ER-critical graph. If $u, v \in V(G)$ such that* $\deg(u) > 1$ *and* $\deg(v) > 1$*, then $d(u, v) \leq 2$. Moreover,* $\operatorname{diam}(G) \leq 4$.

We now present a characterization of the graphs G which are γ_{tR}-edge-removal-critical. Consider for a moment a star graph S_n, which is defined to be the complete bipartite graph $K_{1,n}$, with $n \geq 1$. Let $\mathcal{F}_n$ be the family of graphs constructed from S_n by appending $k_1, k_2, \ldots, k_n$ (where $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$) pendant vertices to each pendant vertex of S_n. In what follows, we label the vertices of a graph $G \in \mathcal{F}_n$ as follows: Let c be the central vertex (choosing either central vertex in the case of S_1), and u_i $(1 \leq i \leq n)$ the pendant vertices, in the original star S_n. For each such vertex u_i, let $v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}$ be the pendant vertices added to u_i. See Figure 2.

Theorem 7.11. *A connected graph G with no isolated vertices is γ_{tR}-ER-critical if and only if G is a member of $\mathcal{F}_n$, for some $n \geq 1$, such that $k_1, k_2, \ldots, k_n \neq 1$.*

Proof. Let G be a connected γ_{tR}-ER-critical graph. We begin by showing that $G \in \mathcal{F}_n$ for $n \geq 1$. By Proposition 7.8, G is a tree. Let $S = \{v \in V(G) : \deg_G(v) > 1\}$. If $G \cong S_n$ for $n \geq 1$, then $G \in \mathcal{F}_n$ as required. So assume $|S| \geq 2$. We claim that $G[S] \cong S_n$ for $n \geq 1$. Suppose for a contradiction that $E_P(G[S]) \neq \emptyset$. Then there exist $u, v \in S$ such that $d(u, v) \geq 3$. But then, by definition of S, $\operatorname{diam}(G) > 4$, contradicting Corollary 7.10. Hence $G[S] \cong S_n$ for $n \geq 1$, and thus $G \in \mathcal{F}_n$.

Now, consider a graph $G \in \mathcal{F}_n$ for some $n \geq 1$. In what follows, let the vertices of G be labelled as described in the definition of $\mathcal{F}_n$.

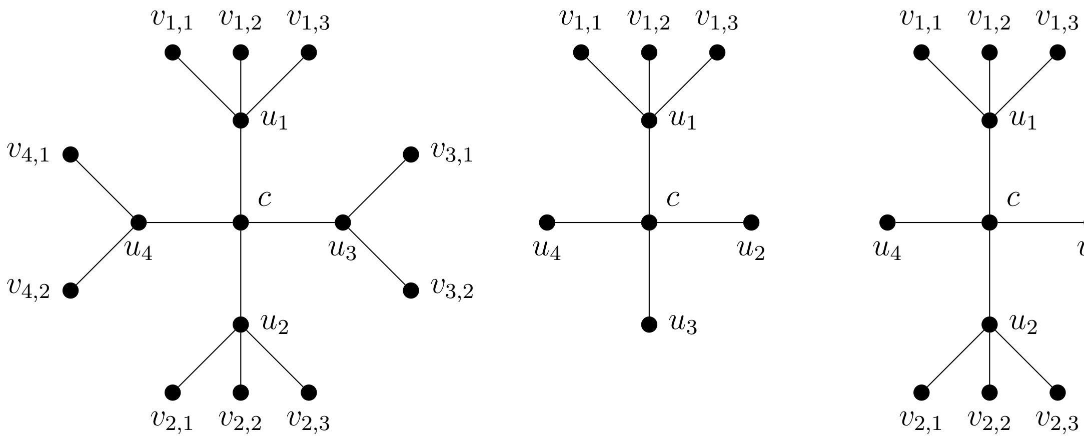

Figure 2: Examples of graphs in $\mathcal{F}_4$

Case 1: Suppose $G \in \mathcal{F}_n$ for some $n \geq 1$ such that $k_1, k_2, \ldots, k_n \neq 1$. If G is a star or a double star, then G is clearly γ_{tR}-ER-critical. Therefore, assume $n \geq 2$ and $k_1 \geq k_2 \geq 2$. Let $2 \leq l \leq n$ be such that $k_i = 0$ if and only if $i > l$. Note that $E_P(G) = \{cu_i : 1 \leq i \leq l\}$. We consider two cases.

Case A: Suppose $l = n$. Then it can be easily seen that $f : V(G) \to \{0, 1, 2\}$ defined by $f(c) = 1$, $f(u_i) = 2$ for all $1 \leq i \leq n$, and $f(b) = 0$ for all other $b \in V(G)$ is a $\gamma_{tR}(G)$-function. If $n \geq 3$, then $G - cu_i$ $(1 \leq i \leq n)$ is the disjoint union of a star on at least 3 vertices with a graph $H \in \mathcal{F}_{n-1}$, where $n - 1 \geq 2$. Otherwise, if $n = 2$, $G - cu_i$ is the disjoint union of two stars, each on at least 3 vertices. In either case, it can be easily seen that $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(v_{i,1}) = 1$ and $f'(z) = f(z)$ for all other $z \in V(G)$ is a $\gamma_{tR}(G - cu_i)$-function with $\omega(f') = \omega(f) + 1$, for each $1 \leq i \leq n$.

Case B: Suppose $l < n$. Then it can be easily seen that $f : V(G) \to \{0, 1, 2\}$ defined by $f(c) = 2$, $f(u_i) = 2$ for all $1 \leq i \leq l$, and $f(b) = 0$ for all other $b \in V(G)$ is a $\gamma_{tR}(G)$-function. Since $2 \leq l < n$, we have $n \geq 3$. Hence $G - cu_i$ $(1 \leq i \leq l)$ is the disjoint union of a star on at least 3 vertices with a graph $H \in \mathcal{F}_{n-1}$, where $n - 1 \geq 2$. Thus, it can be easily seen that $f' : V(G) \to \{0, 1, 2\}$ defined by $f'(v_{i,1}) = 1$ and $f'(z) = f(z)$ for all other $z \in V(G)$ is a $\gamma_{tR}(G - cu_i)$-function with $\omega(f') = \omega(f) + 1$, for each $1 \leq i \leq l$.

Therefore, in each case, G is γ_{tR}-ER-critical, as required.

Case 2: Otherwise, suppose $G \notin \mathcal{F}_n$ for any $n \geq 1$ such that $k_1, k_2, \ldots, k_n \neq 1$. Thus $G \in \mathcal{F}_n$ for $n \geq 1$ where $k_i = 1$ for some $1 \leq i \leq n$. If $n = 1$, then G is also a member of $\mathcal{F}_2$. Therefore, it suffices to consider $n \geq 2$. Consider a $\gamma_{tR}(G)$-function f such that $|V_f^2|$ is a minimum. Then $f(u_i) = f(v_{i,1}) = 1$. Moreover, Proposition 7.7 implies that $f(c) > 0$. Suppose first that $n = 2$, and let $j \neq i$.

If $k_j = 0$, then $f(u_j) = f(c) = 1$ by our choice of f. If $k_j \geq 1$, then $f(u_j) > 0$ by Proposition 7.7. Otherwise, suppose $n \geq 3$. If $k_j = 0$ for all $j \neq i$, then G is also a member of $\mathcal{F}_2$ with $k_1 = n - 1 \geq 2$ and $k_2 = 0$, contradicting our assumption. Hence there exists $j \neq i$ such that $k_j \geq 1$, and thus by Proposition 7.7, $f(u_j) > 0$. Note that, in each case, there exists $j \neq i$ such that $f(u_j) > 0$. But $u_j, c, u_i, v_{i,1}$ is a path in G, contradicting Proposition 7.9. Hence G is not γ_{tR}-ER-critical. □

Corollary 7.12. *A graph G with no isolated vertices is γ_{tR}-ER-critical if and only if G is the disjoint union of $m \geq 1$ graphs $G_i \in \mathcal{F}_{n_i}$, for some $n_i \geq 1$ such that $k_1, k_2, \ldots, k_{n_i} \neq 1$, for $1 \leq i \leq m$.*

8 γ_{tR}-Edge-removal-supercritical graphs

Having classified γ_{tR}-ER-critical graphs, we now classify the graphs G which are γ_{tR}-ER-supercritical.

Theorem 8.1. *A connected graph G with no isolated vertices is γ_{tR}-ER-supercritical if and only if G is either a non-trivial star, or a double star where each non-pendant vertex has degree at least* 3.

Proof. Suppose G is γ_{tR}-ER-supercritical. If $E_P(G) = \emptyset$, then $G = S_n$ for $n \geq 1$. Otherwise, assume $E_P(G) \neq \emptyset$. We claim that $|E_P(G)| = 1$. Suppose for a contradiction that $|E_P(G)| \geq 2$, and consider a path u, v, w, x, y in G. Let f be a $\gamma_{tR}(G)$-function. Then, by Proposition 7.7, $v, w, x \in V_f^+$. Moreover, since Proposition 7.8 implies that G is a tree, by Corollary 7.10, $\deg(u) = \deg(y) = 1$. Thus Observation 3.3 implies that $f(u) \leq 1$ and $f(y) \leq 1$. But then $g : V(G) \to \{0, 1, 2\}$ defined by $g(u) = 1$ and $g(z) = f(z)$ for all other $z \in V(G)$ is a $\gamma_{tR}(G - vw)$-function with $\omega(g) \leq \omega(f) + 1$, contradicting vw being TRD-ER-supercritical. Hence $|E_P(G)| = 1$, and thus G is a double star.

Conversely, $G = S_n$ for $n \geq 1$ is, by definition, γ_{tR}-ER-supercritical. Otherwise, suppose G is a double star. Then $\gamma_{tR}(G) = 4$. Moreover, $E_P(G) = \{uv\}$ where u and v are the two non-pendant vertices. If each non-pendant vertex has degree at least 3, then by Proposition 2.5, $\gamma_{tR}(G - uv) = 6$, since the removal of the non-pendant edge disconnects the graph into two stars each on at least 3 vertices. Therefore G is γ_{tR}-ER-supercritical. Otherwise, if G has a non-pendant vertex of degree 2, then $\gamma_{tR}(G - uv) \leq 5$, since since the removal of the non-pendant edge disconnects the graph into two stars, at least one of which is on only two vertices. Therefore G is not γ_{tR}-ER-supercritical. □

Corollary 8.2. *A graph G with no isolated vertices is γ_{tR}-ER-supercritical if and only if G is the disjoint union of $m \geq 1$ graphs G_i such that, for each $1 \leq i \leq m$, G_i is either a non-trivial star, or a double star where each non-pendant vertex has degree at least* 3.

We conclude this section by observing a link between connected γ_{tR}-ER-supercritical and γ_{tR}-edge-stable graphs, which follows directly from the previous theorem.

Corollary 8.3. *If G is a connected k-γ_{tR}-edge-removal-supercritical graph, then G is k-γ_{tR}-edge-stable.*

9 γ_{tR}-Edge-removal-stable graphs

We now consider graphs G which are γ_{tR}-edge-removal-stable. Recall that a graph G is γ_{tR}-ER-stable when $\gamma_{tR}(G-e) = \gamma_{tR}(G)$ for all $e \in E(G)$. We begin with two observations that follow directly from the definitions of $\gamma_{tR}(G-e)$, where e is a pendant edge of G, and a TRD-ER-stable edge, respectively.

Observation 9.1. *If G is a γ_{tR}-ER-stable graph, then $\delta(G) > 1$.*

Observation 9.2. *If G is a γ_{tR}-ER-stable graph, then for any $e \in E(G)$ there exists a γ_{tR}-function f on G such that f is also a $\gamma_{tR}(G-e)$-function.*

Consider again the graph G_r defined in Section 6. There, we showed that, for $r \geq 2$, G_r is 6-γ_{tR}-edge-supercritical. In addition, it can be shown that G_r is γ_{tR}-ER-stable. Furthermore, note that the union of $k \geq 2$ complete graphs each of order at least 3 is both $3k$-γ_{tR}-edge-supercritical (by Proposition 4.3) and $3k$-γ_{tR}-ER-stable (by Proposition 3.7). Similarly, $\mathrm{cor}(K_n)$ for $n \geq 4$ is $2n$-γ_{tR}-edge-supercritical (by Proposition 3.8) and every non-pendant edge $e \in E(G)$ is TRD-ER-stable (by Proposition 3.7). In light of these results, we present the following theorem.

Theorem 9.3. *If G is a γ_{tR}-edge-supercritical graph, then every non-pendant edge $e \in E(G)$ is TRD-ER-stable.*

Proof. Let G be a γ_{tR}-edge-supercritical graph. Then G contains no K_2 components. Suppose for a contradiction that there exists a non-pendant edge $uw \in E(G)$ that is TRD-ER-critical. Then $\deg(u) \geq 2$ and $\deg(w) \geq 2$. Let $v \in N_G(w) - \{u\}$.

Claim: $N_G[u] \neq N_G[w]$.

Proof of Claim: Suppose for a contradiction that $N_G[u] = N_G[w]$. Let $S = N_G[u] - \{u, w\}$. Then $v \in S$. Consider a $\gamma_{tR}(G)$-function f. By Observation 7.2, $\{f(u), f(w)\} \in \{\{2,2\},\{2,1\},\{2,0\},\{1,1\}\}$. We claim that $G[S]$ has no universal vertex. Suppose for a contradiction that v is a universal vertex of $G[S]$. Note that possibly $S = \{v\}$. If $f(u) = f(w) = 1$ and $f(v) = 0$, consider $f' : V(G) \to \{0,1,2\}$ defined by $f'(u) = f'(w) = 0$, $f'(v) = 2$ and $f'(b) = f(b)$ for all other $b \in V(G)$. Otherwise, if $f(u) = f(w) = 1$ and $f(v) > 0$, consider $f' : V(G) \to \{0,1,2\}$ defined by $f'(w) = 0$, $f'(v) = 2$ and $f'(b) = f(b)$ for all other $b \in V(G)$. Finally, if $f(u) = 2$ (without loss of generality), consider $f' : V(G) \to \{0,1,2\}$ defined by $f'(u) = f(v)$, $f'(v) = f(u)$ and $f'(b) = f(b)$ for all other $b \in V(G)$. In any case, f' is a $\gamma_{tR}(G)$-function. Moreover, f' is also a TRD-function on $G - uw$, contradicting uw being TRD-ER-critical. Therefore $G[S]$ has no universal vertex, and thus there exists some vertex $x \in S - \{v\}$ such that $vx \in E(\overline{G})$.

Now, consider a γ_{tR}-function g on $G + vx$. By Proposition 3.4, $\{g(x), g(v)\} \in \{\{2,2\},\{2,1\},\{2,0\},\{1,1\}\}$. If $g(x) > 0$ and $g(v) > 0$, then $g' : V(G) \to \{0,1,2\}$ defined by $g'(u) = \min\{2, g(u)+1\}$ and $g'(b) = g(b)$ for all other $b \in V(G)$ is a TRD-function on G with $\omega(g') \le \omega(g)+1$, contradicting vx being TRD-supercritical. Hence $\{g(x), g(v)\} = \{2,0\}$; say $g(v) = 2$ and $g(x) = 0$ (without loss of generality). Then $g(u) = g(w) = 0$, otherwise $g' : V(G) \to \{0,1,2\}$ defined by $g'(x) = 1$ and $g'(b) = g(b)$ for all other $b \in V(G)$ would be a TRD-function on G with $\omega(g') = \omega(g)+1$, contradicting vx being TRD-supercritical. Hence $h : V(G) \to \{0,1,2\}$ defined by $h(u) = h(x) = 1$ and $h(b) = g(b)$ for all other $b \in V(G)$ is a $\gamma_{tR}(G)$-function, which, since uw is TRD-ER-critical, contradicts Observation 7.2. Therefore, $N_G[u] \neq N_G[w]$. (□)

As a result of the above claim, we can choose $v \in N_G(w) - \{u\}$ such that $uv \in E(\overline{G})$. Now, consider a γ_{tR}-function f on $G+uv$. By Proposition 3.4, $\{f(u), f(v)\} \in \{\{2,2\},\{2,1\},\{2,0\},\{1,1\}\}$. If $f(u) > 0$ and $f(v) > 0$, then $f' : V(G) \to \{0,1,2\}$ defined by $f'(w) = 1$ and $f'(b) = f(b)$ for all other $b \in V(G)$ is a TRD-function on G with $\omega(f') \le \omega(f) + 1$, contradicting uv being TRD-supercritical. Hence $\{f(u), f(v)\} = \{2,0\}$. We show that $f(u) = 0$ and $f(v) = 2$.

Suppose for a contradiction that $f(u) = 2$ and $f(v) = 0$. Clearly $f(w) = 0$, otherwise $f' : V(G) \to \{0,1,2\}$ defined by $f'(v) = 1$ and $f'(b) = f(b)$ for all other $b \in V(G)$ would be a TRD-function on G with $\omega(f') = \omega(f) + 1$, contradicting uv being TRD-supercritical. Hence $g : V(G) \to \{0,1,2\}$ defined by $g(w) = g(v) = 1$ and $g(b) = f(b)$ for all other $b \in V(G)$ is a $\gamma_{tR}(G)$-function. However, since u is not isolated in $G[V_f^+]$, g is also a TRD-function on $G - uw$, contradicting uw being TRD-ER-critical. Hence $f(u) = 0$ and $f(v) = 2$.

Now, $f(N_G(u)) = 0$, otherwise $f' : V(G) \to \{0,1,2\}$ defined by $f'(u) = 1$ and $f'(b) = f(b)$ for all other $b \in V(G)$ would be a TRD-function on G with $\omega(f') = \omega(f)+1$, contradicting uv being TRD-supercritical. Furthermore, since $\deg_G(u) \ge 2$, there exists some vertex $y \in N_G(u) - \{w\}$. Note that $f(y) = 0$. Hence $f' : V(G) \to \{0,1,2\}$ defined by $f'(y) = f'(u) = 1$ and $f'(b) = f(b)$ for all other $b \in V(G)$ is a $\gamma_{tR}(G)$-function. However, f' is also a TRD-function on $G - uw$, contradicting uw being TRD-ER-critical. Therefore $\deg_G(u) = 1$; that is, uw is a pendant edge. □

Corollary 9.4. *If G is a γ_{tR}-edge-supercritical graph with $\delta(G) \ge 2$, then G is γ_{tR}-ER-stable.*

10 Future Work

Consider for a moment connected 6-γ_{tR}-edge-supercritical graphs. We showed in Section 6 that, for any connected 6-γ_{tR}-edge-supercritical graph G, $2 \le \mathrm{diam}(G) \le 3$. Furthermore, note that each graph G_r, with $r \ge 2$, introduced in Section 6 has diameter 3. We now consider the 6-γ_{tR}-edge-supercritical graphs G for which

$\text{diam}(G) = 2$. We begin with the following lemma, which provides a lower bound for the minimum degree of a connected graph G with diameter 2, based on its TRD-number.

Lemma 10.1. *If G is a connected graph with $\text{diam}(G) = 2$ and $\gamma_{tR}(G) = k$, then $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$.*

Proof. Suppose for a contradiction that there is a vertex $v \in V(G)$ with $\deg(v) < \lfloor \frac{k}{2} \rfloor$. Since $\text{diam}(G) = 2$, $N_G(v)$ is a dominating set of G. Thus the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(v) = 1$, $f(x) = 2$ for all $x \in N_G(v)$, and $f(z) = 0$ for all other $z \in V(G)$ is a TRD-function on G with $\omega(f) \leq 2(\lfloor \frac{k}{2} \rfloor - 1) + 1$. That is, $\omega(f) \leq 2\lfloor \frac{k}{2} \rfloor - 1 < k$, contradicting $\gamma_{tR}(G) = k$. □

Corollary 10.2. *If G is a connected γ_{tR}-edge-supercritical graph with $\text{diam}(G) = 2$, then $\delta(G) \geq 3$.*

The previous corollary follows directly from Proposition 4.3. In light of this result, we present the following proposition which provides necessary conditions for a connected graph G with $\text{diam}(G) = 2$ to be 6-γ_{tR}-edge-supercritical. Characterizing connected 6-γ_{tR}-edge-supercritical graphs with diameter 2, and indeed with diameter 3, remain open problems.

Lemma 10.3. *If G is a connected 6-γ_{tR}-edge-supercritical graph with $\text{diam}(G) = 2$, then G is 3-γ_t-edge-critical and 3-γ-edge-critical.*

Proof. Let G be a connected 6-γ_{tR}-edge-supercritical graph with $\text{diam}(G) = 2$. Then, for any edge $e \in E(\overline{G})$, $\gamma_{tR}(G + e) = 4$. Thus, by Proposition 2.7, $\gamma_t(G + e) = \gamma(G + e) = 2$. Now, Proposition 2.7 also implies that $\gamma_t(G) > 2$. Furthermore, by Proposition 3.1, $\gamma_t(G) \leq 4$. If $\gamma_t(G) = 4$, then G is 4-γ_t-edge-supercritical, which, since G is connected, contradicts Proposition 4.2. Hence $\gamma_t(G) = 3$. Now, by Proposition 2.1, $2 \leq \gamma(G) \leq 3$. Suppose for a contradiction that $\gamma(G) = 2$, and consider a $\gamma(G)$-set $S = \{u, v\}$. Note that, since $\gamma_t(G) = 3$, $uv \in E(\overline{G})$. However, since $\text{diam}(G) = 2$, there exists some $w \in N_G(u) \cap N_G(v)$. Hence $T = \{u, v, w\}$ is a $\gamma_t(G)$-set. But then $S \subset T$, contradicting Theorem 5.2. Hence $\gamma(G) = 3$, and thus G is 3-γ_t-edge-critical and 3-γ-edge-critical. □

Question 1. *Do there exist connected 6-γ_{tR}-edge-supercritical graphs with diameter 2?*

Having demonstrated the existence of connected 6-γ_{tR}-edge-supercritical graphs with diameter 3 in Section 6, we now consider the γ_{tR}-functions on these graphs G_r, where $r \geq 2$.

Proposition 10.4. *For $r \geq 2$, if $v \in V(G_r)$, then there exists a $\gamma_{tR}(G)$-function f such that $v \in V_f^+$.*

Proof. Let $v \in V(G_r)$, where $r \geq 2$. Then, without loss of generality, $v \in \{x, y, z, u_0, u_1, w_0, w_1\}$. If $v \in \{x, y, z\}$ consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(x) = f(y) = f(z) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$. Otherwise, if $v \in \{u_1, w_1\}$, consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(u_1) = f(w_1) = f(z) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$. Otherwise, if $v = u_0$, consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(u_0) = f(u_1) = f(w_2) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$. Otherwise, if $v = w_0$, consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(x) = f(y) = f(w_0) = 2$ and $f(b) = 0$ for all other $b \in V(G_r)$. In any case, we have a $\gamma_{tR}(G)$-function f such that $v \in V_f^+$, as required. □

Corollary 10.5. *For $r \geq 2$ and $n \geq 3$, $G_r \cup K_n$ is 9-γ_{tR}-edge-critical.*

Proof. Consider the graph $H \cong G_r \cup K_n$ where $r \geq 2$ and $n \geq 3$. Clearly $\gamma_{tR}(H) = 9$. Consider an edge $uv \in E(\overline{H})$. If $uv \in E(\overline{G_r})$, Theorem 6.3 implies that uv is supercritical, and thus critical, with respect to total Roman domination. Otherwise, suppose that $u \in V(G_r)$ and $v \in V(K_n)$. By Proposition 10.4, there exists a $\gamma_{tR}(G_r)$-function g such that $u \in V_g^+$. Consider the function $f : V(G_r) \to \{0, 1, 2\}$ defined by $f(w) = g(w)$ for all $w \in V(G_r)$, $f(v) = 2$, and $f(x) = 0$ for all other $x \in V(K_n)$. Then f is a TRD-function on $H + uv$ with $\omega(f) = 8$, and hence H is 9-γ_{tR}-edge-critical. □

By Propositions 4.3 and 6.1, the disjoint union of a disconnected 6-γ_{tR}-edge-supercritical graph G and K_n for $n \geq 3$ is γ_{tR}-edge-supercritical, and thus γ_{tR}-edge-critical. Moreover, it can be easily seen that the union of $\mathrm{cor}(K_m)$ and K_n, with $m \geq 4$ and $n \geq 3$, is also γ_{tR}-edge-critical. In light of our previous result, we pose the following conjectures. Note that the second conjecture would be a direct result of the first.

Conjecture 1. *If G is a γ_{tR}-edge-supercritical graph and $v \in V(G)$, then there exists a $\gamma_{tR}(G)$-function f such that $v \in V_f^+$.*

Conjecture 2. *If G is a k-γ_{tR}-edge-supercritical graph, then $G \cup K_n$ is $(k+3)$-γ_{tR}-edge-critical, for $n \geq 3$.*

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