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The star avoidance game

Adrian Beker

Trinity College
Cambridge CB2 1TQ
U.K.
ab2454@cam.ac.uk

Abstract

Let n, k be positive integers. The $(k+1)$-star avoidance game on K_n is played as follows. Two players take it in turn to claim a (previously unclaimed) edge of the complete graph on n vertices. The first player to claim all edges of a subgraph isomorphic to a $(k+1)$-star loses. Equivalently, each player must keep all degrees in the subgraph formed by his/her edges at most k. If all edges have been chosen and neither player has lost, the game is declared a draw. We prove that, for each fixed value of k, the game is a win for the second player for all n sufficiently large.

1 Introduction

Many natural combinatorial games occur as follows. We have a finite set (called the *board*), some subsets of which are designated as *lines*. Two players take it in turn to claim a (previously unclaimed) element of the board. The first player to complete a line loses (and the other player is declared the winner). If all elements have been chosen and neither player has lost, the game is declared a draw. Due to the winning criterion, games of the described kind are called *misère* games. Games with the usual winning criterion of making the desired object are called 'achievement games'—see Beck [1] for a discussion of both kinds of game and Slany [4] for background on misère games. For a related more general overview of results and methods in combinatorial game theory, see Beck [2].

A subfamily of misère games of particular interest is the class of *sim-like* games. These games have board the edge set of the complete graph K_n. The lines are subsets which form a subgraph isomorphic to a graph from some fixed family $\mathcal{F}$. The first example of such a game we are aware of is the game of *Sim*, which is played on K_6 and has $\mathcal{F} = \{K_3\}$ (see Simmons [3]). For a survey of sim-like games from a more computational perspective, see Slany [4].

It is an easy consequence of Ramsey's theorem that any sim-like game is not a draw for all sufficiently large boards. Indeed, take $G \in \mathcal{F}$ and consider $n \geq R(k, k)$,

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where k is the order of G. Then if all edges of K_n have been chosen, at least one player has claimed all edges of a k-clique and hence of a subgraph isomorphic to G.

In this paper we will consider the sim-like game given by $\mathcal{F} = \{S_{k+1}\}$, where k is a fixed positive integer (and S_{k+1} denotes the graph $K_{1,k+1}$), also called the $(k+1)$-*star avoidance game*. This is one of the simplest and most natural misère games on a graph. Our main result is the following:

Theorem 2.1. *The second player wins the $(k+1)$-star avoidance game on K_n whenever $n \geq 200k$.*

In Section 2, we give a proof of Theorem 2.1. In Section 3, we conclude the paper with some remarks and a discussion of related open problems. Throughout the paper, all graphs in consideration are simple (i.e. do not contain multiple edges or self-loops) and we assume standard notation from graph theory. In particular, recall that in a graph G, $d(G)$, $\Delta(G)$ and $\delta(G)$ denote the average, maximum, and minimum degrees of a vertex respectively. Furthermore, given a vertex v of G, we use $d_G(v)$ to denote its degree in G. We also abbreviate the first and second player to PI and PII respectively. A *round* comprises a move of PI followed by a move of PII.

2 Proof of Theorem 2.1

The aim of this section is to provide a proof of the following theorem:

Theorem 2.1. *The second player wins the $(k+1)$-star avoidance game on K_n whenever $n \geq 200k$.*

We first give a brief overview of our approach. In the context of the $(k+1)$-star avoidance game we define a *valid subgraph* to be a subgraph of K_n of maximum degree at most k. We note that a straightforward way for PII to win would be to build a valid subgraph of size $\mathrm{ex}(n, S_{k+1}) = \left\lfloor \frac{nk}{2} \right\rfloor$. However, in the case when nk is even, this approach would require careful adjustments to the opponent's actions in the final stage of the game, and it turns out that in fact slightly less is needed. Instead, we contend that PII can build a subgraph of size $\left\lfloor \frac{nk-1}{2} \right\rfloor$ with the property that, in the case when nk is even, there exists an unclaimed edge that extends it to a valid subgraph. We then use the fact that PI's last move is uniquely determined to argue that PII can force a win.

In order to prove the main claim, we define an auxiliary game, called the *pair clipping game*, which is strictly speaking not a positional game. However, the strategy required for PII to win can be viewed as very similar to a fast winning strategy in the Maker-Breaker perfect matching game. The difference is in the winning criterion, which is slightly modified according to the needs of our problem. Furthermore, we require this game to be played on a general graph instead of K_n. We make use of a winning strategy in the pair clipping game by building layers of almost perfect matchings until we reach PII's goal.

Finally, the main part of the proof deals with finding a strategy for the pair clipping game, which is provided by Theorem 2.2. By the way the game is defined, one can think of the graph as representing obstacles for PII, i.e. previously claimed edges in the context of the star avoidance game. PII essentially employs a greedy strategy which entails inductively controlling both the average and the maximum degree of the graph.

We start by defining the auxiliary game. Let G be a graph and let $n = |G|$. The *pair clipping game on* G, PCG(G), is defined as follows. It is played by two players and consists of rounds of the following form:

> PI adds at most one edge which is not already present in G and then PII removes two non-adjacent vertices from G.

PII wins the game if, after $\lfloor \frac{n-1}{2} \rfloor$ rounds, G becomes empty. Note that the meaning of 'G' herein is twofold — PCG(G) denotes an instance of the pair clipping game played with G as the starting graph, whereas in the description of the rules of the game, 'G' refers to the current state of the graph. To avoid confusion, we will denote by G_j the state of the graph G after j moves have been made. Thus, G_0 is the initial graph and $|G_{2j}| = |G_{2j+1}| = |G_0| - 2j$ holds for all $j \geq 0$. Likewise, we will denote by $\{u_j, v_j\}$ the pair of vertices chosen in the j-th turn (or an arbitrary element of $V(G_{j-1})^{(2)}$ if PI does nothing in the j-th turn).

We define the notion of a nice pair of vertices, which is central to the greedy strategy. Let G be a graph and let $u, v \in V(G)$ be distinct. We say that the pair $\{u, v\}$ is *nice* if $uv \notin E(G)$ and $d_G(u) + d_G(v) \geq 2d(G)$, i.e. uv is not an edge and the average of the degrees of u and v is at least the average degree of the whole graph. The following lemma guarantees the existence of such a pair in graphs with not too large maximum degree:

Lemma 2.1. *Let G be a graph with $|G| \geq 2$ and $\Delta(G) \leq \frac{1}{2}|G| - 1$. Then G has a nice pair.*

Proof. If $|G| = 2$, then we are done since G is empty, so suppose $|G| \geq 3$. Let $H = \overline{G}$ be the complement of G and let $n = |G| = |H|$. Then note that $\delta(H) = n - 1 - \Delta(G) \geq \frac{1}{2}n$, so by Dirac's theorem, H has a Hamiltonian cycle $v_1 v_2 \dots v_n$. Averaging over this cycle, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} \frac{d_H(v_j) + d_H(v_{j+1})}{2} = \frac{1}{n}\sum_{j=1}^{n} d_H(v_j) = d(H).$$

Hence, there exists $j \in [n]$ such that $\frac{d_H(v_j)+d_H(v_{j+1})}{2} \leq d(H)$. But note that $v_j v_{j+1} \notin E(G)$ and

$$d_G(v_j) + d_G(v_{j+1}) = (n-1-d_H(v_j)) + (n-1-d_H(v_{j+1})) \geq 2(n-1) - 2d(H) = 2d(G),$$

so $\{v_j, v_{j+1}\}$ is a nice pair in G. □

We now introduce a certain notion of sparseness of graphs which will be used in Theorem 2.2. Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be functions. Given a graph G, we say G is *g-sparse* if $d(G) \leq g(|G|)$. If G additionally satisfies $\Delta(G) \leq f(|G|)$, we say G is

(f,g)*-sparse.* The following lemma is mostly technical and describes the effect of removing a nice pair on the sparseness of a graph:

Lemma 2.2. *Let $\alpha > 0$ and let $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be given by $g(n) = \alpha n + 1$. Let G be a g-sparse graph with $|G| \geq 4$ and suppose that G' is obtained from G by adding at most one edge. Let $G'' = G' - u - v$, where $uv \notin E(G')$. If $d_{G'}(u) + d_{G'}(v) \geq 2d(G)$, then G'' is g-sparse. In particular, if $\{u, v\}$ is nice in G', then G'' is g-sparse.*

Proof. Observe that

$$\begin{aligned} d(G'') = \frac{2e(G'')}{|G''|} &= \frac{2e(G') - 2(d_{G'}(u) + d_{G'}(v))}{|G'| - 2} \leq \frac{|G| \cdot d(G) + 2 - 2 \cdot 2d(G)}{|G| - 2} \\ &= \frac{(|G| - 4)d(G) + 2}{|G| - 2} \leq \frac{(|G| - 4)(\alpha|G| + 1) + 2}{|G| - 2} = \frac{\alpha|G|(|G| - 4)}{|G| - 2} + 1 \\ &< \alpha(|G| - 2) + 1 = \alpha|G''| + 1, \end{aligned}$$

so G'' is g-sparse. If $\{u, v\}$ is nice in G', the hypotheses of the lemma hold since $d(G') \geq d(G)$. □

The next lemma is straightforward and serves mainly for the purposes of the base cases in the proof of Theorem 2.2:

Lemma 2.3. *Let G be a graph. Then PII wins PCG(G) if either*

(i) $|G| \geq 3$ *and G is* $(1,1)$*-sparse or*

(ii) $|G| \geq 5$ *and G is* 1*-sparse.*

Proof. To prove part (i), we use induction on the order of G. If $|G| \in \{3, 4\}$, then $\Delta(G) \leq 1$, so it is easy to see that PII wins. If $|G| \geq 5$, then PII removes u_1 and any non-adjacent vertex—this can be done since $d_{G_1}(u_1) \leq 2 \leq |G_1| - 2$. In this way, G_2 is $(1,1)$-sparse, so we are done by the induction hypothesis.

For part (ii), we also use induction on $|G|$. Note that we may assume that $\Delta(G) \geq 2$ since otherwise G is $(1,1)$-sparse and we are done by part (i). If $|G| \in \{5, 6\}$, then it is easy to check that PII can ensure that G_2 is $(1,1)$-sparse, so part (i) again applies. If $|G| \geq 7$, then PII removes a vertex of degree at least 2 in G_1 and any non-adjacent vertex – this can be done since $\Delta(G_1) \leq e(G_1) \leq \frac{|G_1|}{2} + 1 \leq |G_1| - 2$. In this way, we have $e(G_2) \leq e(G_1) - 2 \leq \frac{|G_1|}{2} - 1 = \frac{|G_2|}{2}$, so G_2 is 1-sparse, as desired. □

Theorem 2.2. *Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be given by $f(n) = \frac{n-1}{2}$, $g(n) = \frac{n}{100} + 1$. Then PII wins PCG(G) for any (f,g)-sparse graph G.*

Proof. We proceed by induction on the order of G. Suppose that G is a graph with $n = |G|$ and $\Delta(G) \leq f(n)$, $d(G) \leq g(n)$. If $n \in \{1, 2\}$, then G is empty, so PII immediately wins. If $n \in \{3, 4\}$, then G is $(1,1)$-sparse, so we are done by part (i) of

Lemma 2.3. Hence, we may assume that $n \geq 5$. Consider first the case when $n < 10$. Then we have

$$|G| \cdot d(G) \leq ng(n) \leq n\left(\frac{n}{100} + 1\right) = n + \frac{n^2}{100} < n + 1.$$

Since $|G| \cdot d(G) = 2e(G)$ is an integer, we in fact have $|G| \cdot d(G) \leq n$, whence $d(G) \leq 1$. Therefore, we are done by part *(ii)* of Lemma 2.3. From now on, we assume that $n \geq 10$. We consider two cases:

Case 1. $\Delta(G) \leq f(n-2) - 1$

We have $\Delta(G_1) \leq \Delta(G_0) + 1 \leq f(n-2) < \frac{1}{2}n - 1$. By Lemma 2.1, G_1 has a nice pair $\{u_2, v_2\}$, so PII removes $\{u_2, v_2\}$. By Lemma 2.2, G_2 is g-sparse, and hence (f, g)-sparse because $\Delta(G_2) \leq \Delta(G_1)$. Thus, we are done by the induction hypothesis.

Case 2. $f(n-2) - 1 < \Delta(G) \leq f(n)$

In the j-th round, PII acts as follows:

- Remove a vertex of maximum degree in G_{2j-1} and any non-adjacent vertex.

We first make the following easy observations:

Claim A. *For all $j \geq 0$ such that PII can make a move in each of the first $j+1$ rounds, the following hold:*

(i) $\Delta(G_{2j}) \leq \Delta(G_{2j+1}) \leq \Delta(G_{2j}) + 1$;

(ii) $e(G_{2j+2}) \leq e(G_{2j+1}) - \Delta(G_{2j+1}) \leq e(G_{2j}) - \Delta(G_{2j}) + 1$;

(iii) $\Delta(G_{2j+2}) \leq \Delta(G_{2j})$.

Proof. We note that *(i)* and *(ii)* are clear. To see that *(iii)* holds, note that this is clear if $\Delta(G_{2j+1}) = \Delta(G_{2j})$ because $\Delta(G_{2j+2}) \leq \Delta(G_{2j+1})$. On the other hand, if $\Delta(G_{2j+1}) = \Delta(G_{2j}) + 1$, then any vertex of maximum degree in G_{2j+1} must be incident to the edge $u_{2j+1}v_{2j+1}$, so the claim follows. □

Let $r(n) = \left\lfloor \frac{n-1}{4} \right\rfloor$. For all $1 \leq j \leq r(n)$, note that we have

$$|G_{2j-1}| - 2 = n - 2j \geq n - 2r(n) \geq f(n) + 1 \geq \Delta(G_0) + 1 \geq \Delta(G_{2j-2}) + 1 \geq \Delta(G_{2j-1}),$$

so PII can make a move in the j-th round. Therefore, by the induction hypothesis, it suffices to show that there exists $j \in [r(n)]$ such that G_{2j} is (f, g)-sparse. So suppose for contradiction that this is not the case.

Claim B. *For all $1 \leq j \leq r(n)$, the following hold:*

(i) $\Delta(G_{2j}) > f(n-2j)$,

(ii) $d(G_{2j}) \leq g(n-2j)$, *i.e.* G_{2j} *is g-sparse.*

Proof. We proceed by induction on j. Note that (i) follows from (ii) combined with the assumption that G_{2j} is not (f,g)-sparse, so it suffices to prove (ii). Letting $s = d_{G_{2j-1}}(u_{2j}) + d_{G_{2j-1}}(v_{2j})$, we know that $s \geq \Delta(G_{2j-1}) \geq \Delta(G_{2j-2})$. By Lemma 2.2, it suffices to show that $s \geq 2d(G_{2j-2})$. To this end, we note that if $j = 1$, then

$$s \geq \Delta(G_0) \geq f(n-2) - \frac{1}{2} \overset{(\dagger)}{\geq} 2g(n) \geq 2d(G_0),$$

as desired. Note that the inequality $(\dagger)$ is equivalent to $n \geq \frac{25}{3}$, which indeed holds by assumption. Similarly, if $j > 1$, then we have

$$s \geq \Delta(G_{2j-2}) \overset{(i)}{\geq} f(n-2j+2) + \frac{1}{2} \overset{(*)}{\geq} 2g(n-2j+2) \overset{(ii)}{\geq} 2d(G_{2j-2}),$$

as desired. Note that we used the induction hypothesis in the inequalities (i) and (ii). Moreover, the inequality $(*)$ is equivalent to $n - 2j + 2 \geq \frac{25}{6}$, which holds as $n - 2j + 2 \geq n - 2 \cdot \frac{n-1}{4} + 2 = \frac{n+5}{2} \geq \frac{15}{2}$. □

Using part (ii) of Claim A and part (i) of Claim B, we obtain

$$\begin{aligned} e(G_0) &\geq e(G_0) - e(G_{2r(n)}) + \Delta(G_{2r(n)}) = \Delta(G_{2r(n)}) + \sum_{j=1}^{r(n)} (e(G_{2j-2}) - e(G_{2j})) \\ &\geq \Delta(G_{2r(n)}) + \sum_{j=1}^{r(n)} (\Delta(G_{2j-2}) - 1) \\ &\geq f(n - 2r(n)) + f(n-2) - 1 + \sum_{j=2}^{r(n)} \left(f(n-2j+2) - \frac{1}{2} \right) \\ &\geq \sum_{j=1}^{r(n)+1} \left(\frac{n}{2} - j \right) - 1 = \frac{(r(n)+1)(n - r(n) - 2)}{2} - 1 \\ &\geq \frac{\frac{n}{4}\left(n - \frac{n-1}{4} - 2\right)}{2} - 1 = \frac{n(3n-7)}{32} - 1. \end{aligned}$$

On the other hand, the assumption on the g-sparseness of G_0 implies that

$$e(G_0) = \frac{|G_0| \cdot d(G_0)}{2} \leq \frac{ng(n)}{2} = \frac{n\left(\frac{n}{100} + 1\right)}{2}.$$

Since $\frac{n(3n-7)}{32} - 1 > \frac{n\left(\frac{n}{100}+1\right)}{2}$, we obtain the desired contradiction. □

The following notation will be found useful in the proof of Theorem 2.1, and in fact applies to any sim-like game. For $j \in \{1, 2\}$, let $H_{j,t}$ be the graph $(V(K_n), E_{j,t})$, where $E_{j,t}$ is the set of edges taken by the j-th player up to his/her t-th turn. We also let Γ_t be the graph $(V(K_n), E_{1,t} \cup E_{2,t})$. We will usually abuse notation by omitting t when the turn is understood.

Using this notation, we have that $\Delta(H_j) \leq k$ for $j \in \{1, 2\}$ and hence $\Delta(\Gamma) \leq 2k$ holds before any player loses. In particular, the game cannot be a draw for $n \geq 2k+2$.

Proof of Theorem 2.1. Let n, k be positive integers such that $n \geq 200k$ and consider the $(k+1)$-star avoidance game on K_n. Suppose for contradiction that PII doesn't win the game. The following claim is key to the proof:

Claim. *PII can ensure that eventually one or two vertices in H_2 have degree $k-1$ while the rest have degree k, and additionally the vertices of degree $k-1$ span no edges in Γ.*

We first show that the Claim implies the desired result. Note that PI certainly loses on his/her $\left(\lfloor \frac{nk}{2} \rfloor + 1\right)$-st move. Let Φ denote the strategy provided by the Claim. If nk is odd, PII follows Φ and hence wins. So suppose nk is even. Then PII follows Φ for the first $\frac{nk}{2} - 2$ rounds. After PI's $\left(\frac{nk}{2} - 1\right)$-st move, let e_1, e_2 denote PII's next move according to Φ and the pair of vertices that would have degree $k-1$ in H_2 if PII claimed e_1, respectively. Note that PI's next move is fixed at this moment, so if it is among $\{e_1, e_2\}$, PII simply wins by claiming it. Otherwise, PII claims e_1 and e_2 in that order and hence wins.

Proof of Claim. PII's strategy is divided into k stages. For all j, at the beginning of the j-th stage, at most 2 vertices in H_2 will have degree j and the rest will have degree $j-1$. We will show by induction on j that PII will be able to maintain this property. So fix some $j \in [k]$ and throughout the j-th stage, let $S = \{v \in V(H_2) \mid d_{H_2}(v) = j-1\}$. PII follows the winning strategy for PCG(G), where $G = \Gamma[S]$. By Theorem 2.2, PII is able to do so because $|G| \geq n-2 \geq 200k-2$ and $\Delta(G) \leq k+j-1 \leq 2k-1$ hold at the beginning. Consequently, the conclusion is immediate in the case when $j = k$. Otherwise, if $j < k$, then as long as S is non-empty, PII chooses a vertex in S and a vertex of degree j in H_2. PII can do so since $\Delta(\Gamma) \leq 2k$ and there are at least $n-2 \geq 200k-2$ vertices of degree j in H_2. This results in at most 2 vertices of H_2 having degree $j+1$ and the rest having degree j, as desired. □

3 Concluding remarks and open problems

As a consequence of the main result, there exists a function $h : \mathbb{N} \to \mathbb{N}$ with the property that $h(k)$ is the least positive integer n_0 such that the $(k+1)$-star avoidance game on K_n is a PII win for all $n \geq n_0$. Hence, Theorem 2.1 can be rephrased as the assertion that $h(k) \leq 200k$ holds for all positive integers k. It is clear that this bound is not optimal, in particular it is possible to verify with the aid of a computer that $h(1) = 3$ and $h(2) = 5$. We are also aware that certain modifications of the presented approach may lead to replacing 200 by a smaller constant. As has already been remarked in Section 2, the game is not a draw for $n \geq 2k+2$. Hence, we find it natural to ask the following question:

Question 3.1. *Is it true that $h(k) \leq 2k+2$ for all positive integers k?*

We have little intuition as to the correct answer to this question. In particular, we doubt that our approach can be modified so as to settle this problem. The reason for this is that Question 1 having an affirmative answer would probably have to do with the game having a lot of symmetries rather than it being very sparse, in the sense that $\text{ex}(n, S_{k+1})$ is much smaller than $\binom{n}{2}$ as n grows large. It would also be interesting to find a complete characterisation of the outcomes of the game:

Question 3.2. *For each pair of positive integers (k, n), is the $(k+1)$-star avoidance game on K_n a PI win, a PII win or a draw?*

A related problem is to explore sim-like games with other graphs in place of S_{k+1}. A good starting point would be to pursue similar results for certain classes of sparse graphs, for example trees, in particular paths. Finally, no example of a sim-like game that is a PI win is known. The following question of Johnson, Leader and Walters (see [5]) remains open:

Question 3.3. *Does there exist a sim-like game that is a PI win?*

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