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Hypercycle systems*

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We dedicate this paper to the good friend and colleague Lorenzo Milazzo who passed away in March 2019.

Abstract

The 3-uniform cycle of length 5 has five vertices a, b, c, d, e and five 3-element edges abc, bcd, cde, dea, eab. Similarly, an r-uniform k-cycle has k vertices arranged in a cyclic order, and k edges which are the r-element subsets formed by any r consecutive vertices. A hypercycle system $\mathcal{C}(r, k, v)$ of order v is a collection of r-uniform k-cycles on a v-element vertex set, such that each r-element subset is an edge in precisely one of those k-cycles. In this paper we study hypercycle systems with $r = 3$ and $k = 5$. The definition of 2-*split system* is introduced, and recursive constructions of hypercycle systems $\mathcal{C}(3, 5, v)$ are designed. We find, by a new difference method, hypercycle systems $\mathcal{C}(3, 5, v)$ of orders $v = 10$, 11, 16, 20 and 22. By recursion, they yield infinite families of hypercycle systems.

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1 Introduction

In this paper we study a type of combinatorial design which is a generalization of cycle decompositions of graphs.

A *hypergraph* H of order v is a pair (X, E), where X is the vertex set with $|X| = v$, and E is a family of subsets of X called *edges*. If all edges in E have size r, then the hypergraph is said to be r*-uniform*. The *complete* r-uniform hypergraph of order v, denoted by $K_v^{(r)}$, is the hypergraph in which E consists of all the r-element subsets of X. If $r = 3$, throughout this paper we shall write 3*-set* instead of 3-element set.

There are various ways to define cycles in hypergraphs. What we shall call a cycle in this paper is usually termed a 'tight cycle' in the literature. In our terminology, for any integers $k > r \geq 3$, an r*-uniform hypercycle of length* k, or simply a k*-cycle* when r is understood, consists of k vertices and k edges; namely, it is a cyclic sequence of k vertices of X in which any r consecutive vertices form an edge, and only those. It will be denoted by $C(r, k)$.

A *hypercycle system* $\mathcal{C}(r, k, v)$ of order v is a pair $(X, \mathcal{E})$ where X is the vertex set of cardinality $|X| = v$ and $\mathcal{E}$ is a family of k-cycles such that each edge of $K_v^{(r)}$ is contained in precisely one k-cycle of $\mathcal{E}$. In particular, if $k = v$, then $\mathcal{C}(r, v, v)$ is a *Hamiltonian hypercycle system.*

In this paper we propose to study $\mathcal{C}(r, k, v)$ for the case of k fixed, especially concentrating on $r = 3$ and $k = 5$. As a notational convention, we shall often use the integers $0, 1, \ldots, v-1$ for the vertices, so for instance the cyclic sequence $(0, 2, 4, 1, 3)$ represents a $C(3, 5)$ whose edges are the 3-sets $\{0, 2, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{0, 1, 3\}$, and $\{0, 2, 3\}$.

There are not many previous results about hypercycle systems $\mathcal{C}(r, k, v)$, and in fact initially only the Hamiltonian cases $\mathcal{C}(r, v, v)$ were studied. Namely, as a generalization of Hamiltonian cycles of graphs, in the first paper [4] on this subject, Katona and Kierstead defined a *Hamiltonian chain* for an r-uniform hypergraph, which means a hypercycle of length $|X|$ in our terminology. A decade later, Bailey and Stevens [1] developed a numerical algorithm to find $\mathcal{C}(r, v, v)$ systems for $r = 3$ and $v = 7$, 8; and also for $r = 4$ and $v = 9$. For $r = 3$ the authors showed that it is possible to connect any edge of $K_v^{(3)}$ to a triplet of differences and, by a difference method, they found *cyclic* $\mathcal{C}(3, v, v)$ systems for $v = 10$, 11, 16.

Meszka and Rosa [8] made further improvements, and with a computer program they determined all $\mathcal{C}(3, v, v)$ systems for $v \leq 32$. They also introduced the general definition of $\mathcal{C}(r, k, v)$. We find it important to recall the following composition theorem from that paper. The notation $S(3, s, v)$ stands for a *Steiner system* with parameters s and v, that is, a family B of s-element subsets (called *blocks*) of a v-element set X such that each 3-set in X is contained in precisely one member of B.

Theorem 1.1 ([8]) *If there exists a Steiner system $S(3, s, v)$ and a k-cycle system $\mathcal{C}(3, k, s)$, then there exists a $\mathcal{C}(3, k, v)$.*

The proof is obtained by replacing each block of the $S(3,s,v)$ with a copy of the $\mathcal{C}(3,k,s)$ whose vertex set is the block in question. Despite that the idea of substitution is simple, this theorem is important by the fact that the trivial $\mathcal{C}(3,5,5)$ exists (observe that $C(3,5)$ is self-complementary). Thus, if there exists a Steiner system $S(3,5,v)$, then a $\mathcal{C}(3,5,v)$ exists as well. Explicit consequences of this principle will be mentioned in the concluding section.

The results of the current paper were presented in September 2012 [3] by the third author. Shortly after the first write-up, Keevash [5] announced the milestone result that the Divisibility Conditions[1] (namely, that $\binom{v-i}{t-i}$ is a multiple of $\binom{k-i}{t-i}$ for every $i=0,1,\dots,t-1$) imply the existence of an $S(t,k,v)$ whenever v is sufficiently large with respect to k. The paper still seems to be under the refereeing process, but recently its simplified version [6] is also available. By this deep theorem, and by splitting each copy of $K_5^{(3)}$ (placed into the blocks of $S(3,5,v)$) into two edge-disjoint copies of $C(3,5)$, we immediately obtain the following existence result.

Theorem 1.2 *For every v sufficiently large, a cycle system $\mathcal{C}(3,5,v)$ exists whenever $v \equiv 2, 5, 17, 26, 41, 50 \pmod{60}$.*

In [8] it was also observed that the order of a $\mathcal{C}(3,5,v)$ must satisfy the congruence

$$v \equiv 1, 2, 5, 7, 10, 11 \pmod{15},$$

and all systems $\mathcal{C}(3,5,v)$ for $v \leq 17$ were found with the help of computer. For example, it was announced that there exist exactly two non-isomorphic systems $\mathcal{C}(3,5,7)$.

Simultaneously with [6], Keevash [7] announced a strong extension of his method, which in particular leads to necessary and sufficient arithmetic conditions for the decomposability of $K_v^{(r)}$ into copies of a fixed r-uniform hypergraph H assuming that v is sufficiently large. For the case of $H = \mathcal{C}(3,5,v)$ it implies the following result on the spectrum of $\mathcal{C}(3,5,v)$ systems, which was conjectured in an early version of the present paper.

Theorem 1.3 *For v sufficiently large, a $\mathcal{C}(3,5,v)$ exists if and only if $v \equiv 1$, 2, 5, 7, 10, 11 (mod 15).*

Currently a complete solution of the spectrum problem is out of reach, as the asymptotic existence result above cannot handle small cases. In this paper we make some steps towards the solution of this problem, by explicit constructions and recursive operations. We would like to emphasize that—contrary to previous work on small cycle systems—the results below have been found without any use of computers. At some points we shall indicate principles that helped us to construct cycle

[1] The Divisibility Conditions are sometimes referred to as Numerical Conditions. For $t=3$ and $k=5$, i.e. for Steiner systems $S(3,5,v)$, they mean the following three constraints: $10 \mid \frac{v(v-1)(v-2)}{6}$, $6 \mid \frac{(v-1)(v-2)}{2}$, $3 \mid (v-2)$. Proceeding in reverse order, they reduce to $3 \mid (v-2)$, $4 \mid (v-1)(v-2)$, $5 \mid v(v-1)(v-2)$. This yields the residue classes 2, 5, 17, 26, 41, 50 modulo 60.

systems with certain parameters by hand. Analogous ideas may turn out to be useful in larger cases as well.

We say that a $\mathcal{C}(3,5,w)$ hypercycle system $(X',\mathcal{E}')$ is a *subsystem* of a $\mathcal{C}(3,5,v)$ system $(X,\mathcal{E})$, if it holds that $X' \subset X$, and for each 3-set T inside X' the unique 5-cycle of $\mathcal{E}$ containing T is a 5-cycle of $\mathcal{E}'$, too (that is, $\mathcal{E}' \subset \mathcal{E}$).

Due to their highly useful role in constructions, we shall also study the existence of cycle systems of two restricted types. One of them is the class of *cyclic systems*, which admit a labeling $x_1, x_2, \ldots, x_v$ of the vertices in such a way that the rotation $x_i \mapsto x_{i+1}$ ($i = 1, \ldots, v$, where x_{v+1} is meant as x_1) is an automorphism; that is, if a cyclic sequence $(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5})$ of five vertices determines a $C(3,5)$ then, for every $j = 1, 2, \ldots, v-1$, also $(x_{i_1+j}, x_{i_2+j}, x_{i_3+j}, x_{i_4+j}, x_{i_5+j})$ determines a $C(3,5)$ in the same system (where subscript addition is taken modulo v, replacing 0 with v). The other type is what we call a 2-*split system*. Such systems always have even order, therefore we define them on $2v$ vertices rather than on v. More explicitly, a $\mathcal{C}(3,5,2v)$ is said to be a 2-*split system* if it contains two vertex-disjoint subsystems of type $\mathcal{C}(3,5,v)$. (Those two subsystems are not required to be isomorphic.) As a further definition, in a 2-split system, a *crossing* 5-*cycle* is a 5-cycle which does not contain edges from the two subsystems; in other words, the crossing 5-cycles establish the connection between the two subsystems of the $\mathcal{C}(3,5,2v)$. We use the term *crossing* 3-*set* analogously.

In Section 2 we determine necessary conditions for the existence of 2-split systems and of cyclic systems. Currently we cannot decide whether those conditions are also sufficient if v is large, although the relations obtained may characterize the corresponding spectra apart from finitely many exceptional cases. In the second part of the section we define the notion of difference triplets which will be applied later in designing particular hypercycle systems.

In Section 3 we present recursive constructions which can be applied to construct infinite classes of hypercycle systems $\mathcal{C}(3,5,v)$ from smaller systems. Some consequences, comments and conjectures are collected in the concluding section.

In Section 4 we describe some small cycle systems which are cyclic ($v = 11, 16$) or 2-split ($2v = 10, 20$) or both cyclic and 2-split ($2v = 22$). They can serve as building bricks of larger constructions. These small systems are created with the help of difference triplets.

We believe that our approach via difference triplets can turn out to be useful in constructing many further basic configurations, and hence also will lead to new infinite families of hypercycle systems. The 'difference method' has made the situation for small systems fairly transparent, helping to find them by hand without computer aid. For larger systems this may be difficult. However, while a standard computer search applies nearly brute force, the specific properties of differences required for our constructions may drastically reduce running time, and in this way systems of quite large orders may also become reachable.

2 Divisibility conditions and difference triplets

We begin this section with describing arithmetic conditions on the order v of hypercycle systems $\mathcal{C}(3,5,v)$, which are necessary, either in general, or for the existence of particular kinds of systems. After that we introduce the notion of difference triplets, and observe some of their properties.

2.1 Possible residue classes

The so-called Divisibility Conditions are well-known for Steiner systems, and they are traditionally used as a starting point in determining the spectrum of feasible orders for which a system with given parameters exists. Here we observe the analogous conditions for hypercycle systems $\mathcal{C}(3,5,v)$, and also mention their consequences for cyclic and 2-split systems. As we mentioned in the Introduction, the residue classes for unrestricted $\mathcal{C}(3,5,v)$ were also listed in [8].

Observation 2.1 (Divisibility Conditions) *Let $(X,\mathcal{E})$ be a hypercycle system $\mathcal{C}(3,\, 5,\, v)$ of order v.*

1. *Each copy of $C(3,5)$ in $\mathcal{E}$ contains five 3-sets, therefore $\frac{v(v-1)(v-2)}{6}$ is a multiple of 5, i.e.*
$$v \equiv 0,1,2 \pmod 5.$$
2. *Each vertex is incident with three 3-sets in each copy of $C(3,5)$, therefore $\frac{(v-1)(v-2)}{2}$ is a multiple of 3, i.e.*
$$v \equiv 1,2 \pmod 3.$$
3. *If $(X,\mathcal{E})$ is cyclic, then the rotational automorphisms cannot map any 3-set onto itself because v is not a multiple of 3. Thus, each orbit in $\mathcal{E}$ consists of v copies of $C(3,5)$ and contains precisely $5v$ 3-sets, therefore $\frac{v(v-1)(v-2)}{6}$ is a multiple of $5v$, i.e.*
$$v \equiv 1,2 \pmod 5.$$
4. *If there exists a 2-split system $\mathcal{C}(3,5,2v)$, then both v and $2v$ satisfy the first congruence above, i.e. more restrictively we have*
$$v \equiv 0,1 \pmod 5.$$

Certainly, if $\mathcal{C}(3,5,2v)$ is 2-split, then also the second property must be valid for both v and $2v$; but it is clear that once it holds for v, it also holds for $2v$, and vice versa.

From the four parts of Observation 2.1 one can easily deduce the following facts.

Corollary 2.1 (Feasible residue classes for the spectrum)

1. *If there exists a cyclic system* $\mathcal{C}(3,5,v)$*, then*
$$v \equiv 1, 2, 7, 11 \pmod{15}.$$

2. *If there exists a* 2*-split system* $\mathcal{C}(3,5,2v)$*, then*
$$v \equiv 1, 5, 10, 11 \pmod{15}.$$

3. *If there exists a cyclic* 2*-split system* $\mathcal{C}(3,5,2v)$*, then its order* $2v$ *satisfies*
$$2v \equiv 2, 22 \pmod{30}.$$

2.2 Difference triplets

Here we introduce a 'difference technique' distinct from the one presented in [1, 8]. The goal is to have a more transparent view on cyclic systems $\mathcal{C}(3,5,v)$; for this we define the notion of difference triplets. We assume that the vertices are $0, 1, \ldots, v-1$ and that they are arranged in a cyclic order in such a way that rotating each copy of $C(3,5)$ by 1 yields another copy of $C(3,5)$. The distance between vertices i and j is meant to be
$$\|i-j\| = \min(|i-j|,\, v-|i-j|).$$
In this way, we can assign a difference triplet
$$t_{i,j,k} = (\|i-j\|, \|j-k\|, \|k-i\|)$$
to any three vertices i, j, k with $0 \le i < j < k \le v-1$.

It has to be emphasized that the ordering condition $i < j < k$ is important in the definition, since $t_{j,i,k}$ would yield a cyclic triplet which usually is different from $t_{i,j,k}$ as the second and third components $\|j-k\|$ and $\|k-i\|$ are transposed. On the other hand, the formal definition above clearly yields $t_{i,j,k} = t_{j,k,i} = t_{k,i,j}$ for all choices of $\{i, j, k\}$. Moreover, difference triplets are rotation-invariant, i.e. $t_{i,j,k} = t_{i+1,j+1,k+1}$ holds for all i, j, k.

From the observations of Section 2.1 we see that v is not a multiple of 3, thus there can occur two kinds of difference triplets; we simplify their names by omitting the word 'triplet' in these particular terms:

- *symmetric differences*, of the form (a,a,b) where $2a = b$ or $2a + b = v$, and
- *reflected differences*, by which we mean a *pair* of the form $\{(a,b,c), (a,c,b)\}$ where $a + b = c$ or $a + b + c = v$, and $a \ne b \ne c \ne a$.

In accordance with the definition of $\|\cdot\|$, the case $a + b = c$ applies if $a + b \le v/2$, and we have $a + b + c = v$ if $a + b \ge v/2$. For instance, if $v = 7$ then the reflection of $(1,2,3)$ is $(1,3,2)$, although along the cycle the latter looks as $(1,4,2)$ (but '4' is not a valid distance for $v = 7$). That is, the meaning of b becomes clear only in the context of a and c together, since a triplet containing i and $i + a$ may have its third element $i + a + b$ or $i + a - b$ modulo v as well, depending on the value of c.

A *reflection* is a mapping $(a, b, c) \rightarrow (a, c, b)$ (which, by rotational symmetry, is equivalent to $(a, b, c) \rightarrow (b, a, c)$, as well as to $(a, b, c) \rightarrow (c, b, a)$). We extend the term also for 5-cycles. We say that two distinct 5-cycles are *reflected* if the five difference triplets of one of them are the reflected difference triplets of the other one, and vice versa.

It is clear that all the three difference triplets $(a,\ b,\ c)$, $(b,\ c,\ a)$ and $(c,\ a,\ b)$ generate always the same orbit under rotation $i \mapsto i + 1$. That is, a cyclic shift of the components of a difference triplet does not give different orbits. In particular, a symmetric difference (a, a, b) and its reflection (a, b, a) have the same orbit. The situation is opposite, however, if no value is repeated among a, b, c; this fact is expressed in the following proposition. It is simple but useful in determining the distribution of symmetric and reflected differences in the set of difference triplets of $K_v^{(3)}$.

Proposition 2.1 *In a $K_v^{(3)}$, with $v \geq 6$, two reflected differences generate two different orbits.*

Proof. Applying cyclic shift if necessary, we may assume that the two reflected difference triplets to be compared are $(a,\ b,\ c)$ and $(a,\ c,\ b)$. Identical orbits would require that the sets $\{i, i+a, i+a+b\}$ and $\{j, j+a, j-b\}$ coincide for some i and j. The difference a appears only between the first and second element in each set; this implies $i = j$. Hence, to obtain the same triplet we would need that $i+a+b$ coincides with $i - b$, but this is impossible because $b \neq c$. □

Lemma 2.1 *The number of difference triplets is $\frac{(v-1)(v-2)}{6}$. Moreover, depending on the parity of v we have:*

- (i) *if v is odd, then there are $\frac{v-1}{2}$ symmetric differences and $\frac{(v-1)(v-5)}{12}$ reflected differences;*
- (i) *if v is even, then there are $\frac{v-2}{2}$ symmetric differences and $\frac{(v-2)(v-4)}{12}$ reflected differences.*

Proof. Suppose that there are s symmetric and r reflected differences. Due to rotational symmetry, the $\frac{v(v-1)(v-2)}{6}$ vertex 3-sets determine equivalence classes of cardinality v with respect to difference triplets, and distinct classes have distinct $t_{i,j,k}$. This implies

$$s + 2r = \frac{(v-1)(v-2)}{6}.$$

Moreover, the parameter a in a symmetric difference (a, a, b) can take the values $a = 1, 2, \ldots, \lfloor \frac{v-1}{2} \rfloor$ and nothing else, because $b \geq 1$ and $2a + b \leq v$ must be valid. □

Lemma 2.2 *There exists a cyclic* $\mathcal{C}(3,5,v)$ *if and only if it is possible to find* $\frac{(v-1)(v-2)}{30}$ *5-cycles containing all the* $\frac{(v-1)(v-2)}{6}$ *difference triplets.*

Proof. Each $t_{i,j,k}$ must have a representative vertex 3-set in some copy of $C(3,5)$, and none of them can occur more than once because then rotation would also yield that the copies of $C(3,5)$ are not mutually disjoint. □

3 Hypercycle systems of small orders

In this section we obtain hypercycle systems $\mathcal{C}(3,5,v)$ for small orders v, namely v = 10, 11, 16, 20 and 22. The systems of orders v = 11, 16 are isomorphic to the ones found in [8] by computer search, but here we find base 5-cycles in particular positions with respect to their symmetric and reflected differences.

For the sake of completeness, at the end of the section we also put some remarks concerning $v = 17$, in which case the existence of $\mathcal{C}(3,5,v)$ follows immediately by that of $S(3,5,17)$.

3.1 Cyclic systems, $v = 11, 16$

Despite that $v = 7$ satisfies all the Divisibility Conditions, inspection shows that it does not allow a cyclic 5-cycle system. The next two feasible values are more favorable.

C(3,5,11)

A $\mathcal{C}(3,5,11)$ has 33 5-cycles and, by Lemma 2.1, there are five symmetric and five reflected differences. To obtain a cyclic $\mathcal{C}(3,5,11)$ we need three base 5-cycles. The idea here is to compose a 5-cycle which contains all the five symmetric differences, and to create a 5-cycle which takes precisely one difference triplet from each reflected difference. Then the reflection of the latter properly contains those triplets which have not been included so far.

This can be done with (0, 1, 6, 9, 2), (0, 1, 4, 2, 7), (0, 4, 9, 7, 10) (mod 11). The first one generates all the five symmetric differences, while the last two (viewed cyclically) are reflected, which fact can easily be verified if we compare the distances of their consecutive pairs: $1,3,2,5,4$ and $4,5,2,3,1$ respectively; they clearly are the inverses of each other.

C(3,5,16)

A $\mathcal{C}(3,5,16)$ has 112 5-cycles, and it contains 7 symmetric and 14 reflected differences. Hence a cyclic system requires $112/16 = 7$ base 5-cycles, the same as the number of symmetric differences.

The idea is to compose each base 5-cycle from one symmetric difference and two reflected differences. To do this, we constructed an edge-colored graph on 14 vertices; the vertices represented the reflected differences. Then we analyzed the possible combinations of symmetric and reflected differences which are possible to put together with a symmetric difference in one 5-cycle. An edge of color i was drawn between two vertices if those two reflected differences were included in a 5-cycle together with the symmetric difference $(i, i, \min\{2i, 16-2i\})$. In this graph we found a perfect matching such that each edge was taken from a distinct color class. This approach led to the following seven base 5-cycles:

$$(0,\ 8,\ 6,\ 14,\ 15),\ (0,\ 6,\ 5,\ 3,\ 1),\ (0,\ 7,\ 3,\ 10,\ 13),\ (0,\ 3,\ 5,\ 8,\ 4),$$
$$(0,\ 4,\ 6,\ 10,\ 5),\ (0,\ 7,\ 5,\ 12,\ 6),\ (0,\ 8,\ 3,\ 12,\ 5)\ \ (\text{mod } 16).$$

The list is put in increasing order of i. For instance, its first member (0, 8, 6, 14, 15) belongs to the symmetric difference $(1, 1, 2)$ established on the elements $0, 14, 15$. The structure of this base 5-cycle is shown in Figure 1.

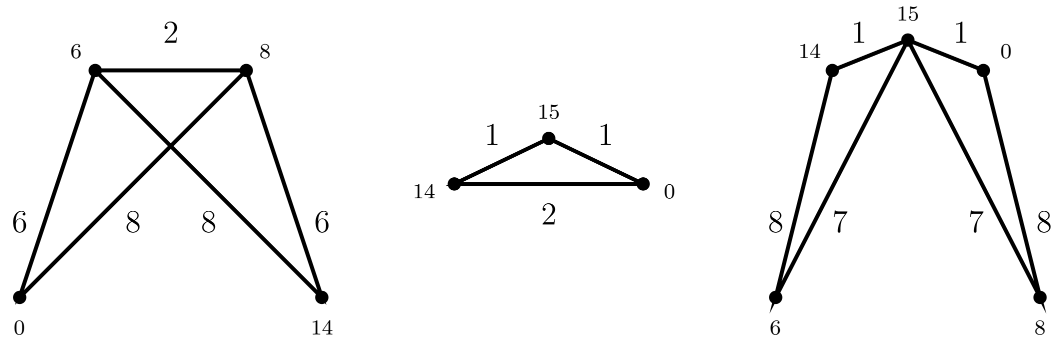

Figure 1: *Difference triplets of the* 5-*cycle* (0, 8, 6, 14, 15).

3.2 2-split systems, $2v = 10, 20$

One general principle, even when a *cyclic* system of order $2v$ does *not* exist, is to cover the crossing 3-sets in a cyclic manner. A transparent characteristic small example of this approach is the case of $2v = 10$.

C(3,5,10)

In order to emphasize the structure of the construction, instead of denoting the vertices by $0, 1, \ldots, 9$ in $\mathcal{C}(3, 5, 10)$, here we rather start from the embedded two subsystems isomorphic to $\mathcal{C}(3, 5, 5)$. We denote them by $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$, where $X_1 = \{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $X_2 = \{0_2, 1_2, 2_2, 3_2, 4_2\}$. Inside each of them, for $i = 1, 2$ we can take the trivial system with the two 5-cycles $(0_i, 1_i, 2_i, 3_i, 4_i)$ and $(0_i, 2_i, 4_i, 1_i, 3_i)$.

Then the $\mathcal{C}(3, 5, 10)$ system $(X, \mathcal{E})$ is defined on $X = X_1 \cup X_2$. Besides the four 5-cycles of $\mathcal{E}_1 \cup \mathcal{E}_2$ it contains the twenty 5-cycles which are the rotations of four base cycles as follows:

$$(0_2, 4_1, 0_1, 2_2, 3_1);\ (3_2, 0_1, 2_1, 2_2, 3_1);\ (3_1, 0_2, 1_2, 0_1, 4_2);\ (0_1, 0_2, 2_2, 4_1, 3_2)\,.$$

Rotation here is meant by keeping the subscripts unchanged, while increasing the main figures by 1, modulo 5, hence 0_i being the successor of 4_i, for $i = 1, 2$. For instance, the first base 5-cycle yields the following five 5-cycles including itself, too:

$$(0_2, 4_1, 0_1, 2_2, 3_1); (1_2, 0_1, 1_1, 3_2, 4_1); (2_2, 1_1, 2_1, 4_2, 0_1); (3_2, 2_1, 3_1, 0_2, 1_1); (4_2, 3_1, 4_1, 1_2, 2_1).$$

Each 5-cycle contains three vertices from one X_i, and two vertices from the other. Four of the five generated crossing 3-sets meet the majority class X_i in two vertices, and among them the distribution is $3 : 1$ or $1 : 3$ between the 3-sets with two consecutive vs. non-consecutive vertices. It is a matter of routine to check that each 3-set is generated; and then it follows by counting that each of them occurs precisely once.

C(3,5,20)

Let us construct a 2-split $\mathcal{C}(3,5,20)$ containing the two vertex-disjoint subsystems[2] $\mathcal{C}'(3,5,10)$ and $\mathcal{C}''(3,5,10)$, respectively, with vertex sets $X_1 = \{1,3,5,7,9,11, 13,15,17, 19\}$ and $X_2 = \{0,2,4,6,8, 10,12,14,16,18\}$. This partition of the vertex set permits that the difference triplets which contain three even integers generate all the 3-sets of vertices completely contained in X_1 or X_2; we call this kind of differences *even differences*.

By Lemma 2.1 a $\mathcal{C}(3,5,20)$ has 9 symmetric and 24 reflected differences. Because of the third Divisibility Condition these cannot be distributed in a completely cyclic way. But once the subsystems on X_1 and X_2 are inserted, they cover four symmetric and four reflected differences, hence there remain 5 symmetric and 20 reflected differences to be covered, which turns out to be possible in a completely cyclic way. For the sake of completeness we list all generators of the obtained $\mathcal{C}(3,5,20)$ in a systematic way, starting with the base cycles on X_1 and X_2.

The 5-cycles $(0,4,8,12,16)$, $(1,5,9,13,17)$, $(2,6,10,14,18)$, $(3,7,11,15,19)$ contain all the 3-sets which correspond to the even difference $(4,4,8)$, while the 5-cycles $(0,8,16,4,12)$, $(1,9,17, 5,13)$, $(2,10,18,6,14)$, $(3,11,19,7,15)$ contain all the 3-sets with even difference $(8,8,4)$.

The two base 5-cycles (0, 2, 8, 10, 18) and (0, 4, 10, 8, 14) (mod 20) contain one symmetric difference (namely $(2,2,4)$ and $(6,6,8)$, respectively) and two reflected differences each. Together with the previously given ones, these 5-cycles contain all the even differences of a $\mathcal{C}(3,5,20)$, so they generate the two disjoint subsystems $\mathcal{C}'(3,5,10)$ and $\mathcal{C}''(3,5,10)$.

Now let us consider the following two sets of crossing base 5-cycles. The first set contains five base 5-cycles: $(0,1,18,2,19)$, $(0,3,12,8,17)$, $(0,3,7,10,5)$, $(0,1,8,6,13)$, $(0,17,5,2,11)$ (mod 20). Each of them has one symmetric and two reflected differences.

[2]It would not be a proper approach to split 20 into four equal parts and take a 2-split system in the union of each pair. The reason is that the collection of 3-sets having their three vertices in three different parts does not admit a decomposition into 5-cycles; it does not even contain any 5-cycles. (A 5-cycle composed of such a kind of 3-sets would need at least five parts.)

The second set contains four base 5-cycles: $(0,1,6,14,5)$, $(1,0,15,7,16)$, $(0,9,16,6,7)$, $(16,7,0,10,9)$ (mod 20) where the first two and also the last two form a reflected pair.

The listed 5-cycles contain all the possible differences of a $\mathcal{C}(3,5,20)$, so a 2-split system is constructed.

3.3 Cyclic 2-split system, $2v = 22$

Since we already have a cyclic system of order 11, we will use its two copies on the alternating odd and even elements, respectively.

C(3,5,22)

A $\mathcal{C}(3,5,22)$ has ten symmetric and thirty reflected differences. Let us construct a 2-split $\mathcal{C}(3,5,22)$ with the two subsystems $\mathcal{C}'(3,5,11)$ and $\mathcal{C}''(3,5,11)$ having vertex sets $X_1=\{1,3,5,7,9,11,13,15,17,19,21\}$ and $X_2=\{0,2,4,6,8,10,12,14,16,18,20\}$ respectively. Applying the construction of $\mathcal{C}(3,5,11)$ from above, the 5-cycles of these two cyclic subsystems can be defined by the following base 5-cycles:

$$(0,\ 2,\ 12,\ 18,\ 4),\ (0,\ 2,\ 8,\ 4,\ 14),\ (0,\ 8,\ 18,\ 14,\ 20) \pmod{22}.$$

They contain all the even differences of a $\mathcal{C}(3,5,22)$, in particular the first base 5-cycle has five symmetric differences while the second and the third are mirror images of each other, hence they together generate reflected differences.

Now let us define all the crossing base 5-cycles. The five base 5-cycles $(0,1,6,16,21)$, $(0,3,8,14,19)$, $(0,2,19,10,5)$, $(0,9,16,8,15)$, $(0,19,6,4,13)$ (mod 22) contain one symmetric difference $(i,i,\min\{2i,22-2i\})$ for i odd (listed in the order of increasing i) and two reflected differences each.

The last three pairs of base 5-cycles $(0,1,14,13,3)$ and $(0,9,10,20,1)$; $(0,1,16,18,5)$ and $(0,7,5,18,1)$; $(0,3,7,4,15)$ and $(3,0,18,21,10)$ (mod 22) define three reflected base 5-cycles.

The fourteen listed base 5-cycles contain all the difference triplets of a $\mathcal{C}(3,5,22)$, and a cyclic 2-split system is obtained.

3.4 Cyclic system, $v = 17$

There exists a unique system $S(3,5,17)$. Similarly to Galois planes, this spherical geometry is cyclic, and it can be generated modulo 17 by the four basic blocks

$$(0,\ 1,\ 2,\ 8,\ 11),\quad (0,\ 1,\ 3,\ 5,\ 6),\quad (0,\ 2,\ 6,\ 10,\ 12),\quad (0,\ 3,\ 4,\ 7,\ 12).$$

Every base block admits a decomposition into two base 5-cycles, each of them generating one symmetric difference and two reflected differences. In this way we obtain a cyclic $\mathcal{C}(3,5,17)$ system whose eight base 5-cycles are

$$\begin{array}{llll}(0,\ 1,\ 2,\ 8,\ 11), & (0,\ 1,\ 3,\ 5,\ 6), & (0,\ 2,\ 6,\ 10,\ 12), & (0,\ 3,\ 4,\ 7,\ 12),\\ (0,\ 2,\ 11,\ 1,\ 8), & (0,\ 3,\ 6,\ 1,\ 5), & (0,\ 6,\ 12,\ 2,\ 10), & (0,\ 4,\ 12,\ 3,\ 7).\end{array}$$

The corresponding symmetric differences in this order are

$$(1,1,2),\ (2,2,4),\ (4,4,8),\ (5,5,7),\ (7,7,3),\ (3,3,6),\ (6,6,5),\ (8,8,1).$$

4 Recursive constructions of $\mathcal{C}(3,5,v)$

In this section we present three recursive constructions, which are useful to obtain infinite classes of hypercycle systems $\mathcal{C}(3,5,v)$. Some starting configurations will be constructed in Section 3, more specifically *2-split* systems of orders 10, 20 and 22, and further (cyclic) systems of orders 11 and 16.

Besides the already known cycle systems or their 2-split types, two auxiliary structures will be applied in performing the general steps. One of them is the class of *resolvable* systems $S(2,5,v)$; their divisibility conditions imply the necessity of $v \equiv 5 \pmod{20}$, and this is also known to be sufficient for all v but for $v = 45$, 345, 465, 645 (see [2, p. 127], Table 7.37).

The other tool is a structure called 3-*wise transversal design* of block size 5 and group size[3] w, commonly abbreviated as 3-TD$(5,w)$ or simply 3-TD when block size 5 and group size w are understood. It means a 3-tuple (X,G,B) where X is the vertex set of cardinality $5w$, G is a partition of X into five *groups* (subsets) of size w each, and B is a family of w^3 *blocks*, which are 5-element subsets of X intersecting each group of G. (Each block has precisely one vertex in each group.) In a 3-TD it is required that each 3-set which intersects three distinct groups of G is contained in precisely one block.

It is easy to define a particular 3-trasversal design with group size equal to $w \equiv 1, 5 \pmod 6$. Let the five groups be $G_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,w-1}\}$ for $1 \le i \le 5$, where we view the second subscripts as elemets of a group of order w, for instance $\mathbb{Z}_w$. Any block $b \in B$ is uniquely defined by $b = (x_{1,p}, x_{2,q}, x_{3,r}, x_{4,p+q+r}, x_{5,p+2q+3r})$. We have that $|B| = w^3$, and in particular any $(x_{i,p}, x_{j,q}, x_{k,r})$, for all $1 \le i < j < k \le 5$ and all $1 \le p,q,r \le w$, identifies one and only one block of B, because $(x_{i,p}, x_{j,q}, x_{k,r})$ defines a system of linear equations which admits a unique solution.

4.1 Main constructions

The basis for the first construction of an infinite class is the following result.

Theorem 4.1 *If there exist a $\mathcal{C}(3,5,v)$ and a resolvable Steiner system $S(2,5,v)$, then there exists a 2-split system $\mathcal{C}(3,5,2v)$.*

Proof. If X_1 and X_2 are two disjoint vertex sets of cardinality v, then let the two hypercycle systems $\mathcal{C}_1(3,5,v)$ and $\mathcal{C}_2(3,5,v)$ be $(X_1,\ \mathcal{E}_1)$ and $(X_2,\ \mathcal{E}_2)$; and let the two resolvable Steiner systems $S_1(2,5,v)$ and $S_2(2,5,v)$ be $(X_1,\ B_1)$ and $(X_2,\ B_2)$,

[3] In design theory, group size is sometimes called order; but we reserve ‘order’ for the number of vertices.

which define respectively the two parallel classes $F = (F_1, F_2, \ldots, F_m)$ and $F' = (F'_1, F'_2, \ldots, F'_m)$, where $m = \frac{(v-1)}{4}$.

Now let us construct a $\mathcal{C}(3,5,2v)$ system $(X, \mathcal{E})$ with $X = X_1 \cup X_2$, starting with the insertion of all the 5-cycles contained in $\mathcal{E}_1$ and $\mathcal{E}_2$ into $\mathcal{E}$.

If we fix a $b_{i,p} \in F_i$, then for any $b'_{i,q} \in F'_i$ we have that $b_{i,p}$ and $b'_{i,q}$ are vertex disjoint and $|b_{i,p} \cup b'_{i,q}| = 10$, so for every q with $1 \le q \le \frac{v}{5}$, all the crossing 5-cycles of the 2-split systems $\mathcal{C}(3,5,10)$ with vertex set $b_{i,p} \cup b'_{i,q}$ must be inserted in $\mathcal{E}$, as constructed in Section 3.2. Let us do this procedure with $b_{i,p}$ for all $1 \le i \le \frac{(v-1)}{4}$ and for all $1 \le p \le \frac{v}{5}$. The total number of 5-cycles in $\mathcal{E}$ is

$$\frac{v(v-1)(v-2)}{15} + 20\left(\frac{v}{5}\right)^2 \frac{v-1}{4} = \frac{v(v-1)(v-2)}{15} + \frac{v^2(v-1)}{5} = \frac{2v(2v-1)(2v-2)}{30}.$$

Any 3-set inside X_1 or X_2 is in a 5-cycle of $\mathcal{E}$, in fact it is contained in a 5-cycle of $\mathcal{E}_1$ or $\mathcal{E}_2$. Moreover a 3-set of X which contains only one or two vertices of X_1 is certainly in a crossing 5-cycle of a hypercycle $\mathcal{C}(3,5,10)$ having vertex set $b_{i,p} \cup b'_{i,q}$, where $b_{i,p}$ contains the one or two vertices of X_1 contained in the 3-set. □

Another recursion is based on the following result.

Theorem 4.2 *If there exist a Steiner system $S(3,5,v)$ and a 2-split system $\mathcal{C}(3,5,2w)$, with $w \equiv 1$ or $5 \pmod 6$, then there exists a $\mathcal{C}(3,5,vw)$.*

Proof. We want to construct a $\mathcal{C}(3,5,vw)$ hypercycle system $(X, \mathcal{E})$. Let $X' = \{x_1, \ldots, x_v\}$ be the vertex set of the $S(3,5,v)$. The vertex set X is obtained by replacing each $x_i \in X'$ with a set X_i of cardinality w, in such way that for any two distinct X_i and X_j we have $X_i \cap X_j = \emptyset$; so $X = \bigcup_{i=1}^{v} X_i$.

For any X_i it is possible to consider a $\mathcal{C}(3,5,w)$ hypercycle system $(X_i, \mathcal{E}_i)$. Let us insert into $\mathcal{E}$ all the 5-cycles of $\mathcal{E}_i$ for $1 \le i \le v$.

For any pair of two distinct i and j, with $1 \le i < j \le v$, we consider a 2-split system $\mathcal{C}(3,5,2w)$ which has vertex set $X_i \cup X_j$. Let $\mathcal{E}_{i,j}$ denote the family of its crossing 5-cycles; put it into $\mathcal{E}$ for all pairs i,j. Hence it remains to deal with the 3-sets meeting with exactly three of the sets X_i.

Let $b_k = (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5})$ be a block of the $S(3,5,v)$ Steiner system (X', B), for $1 \le k \le m = \frac{v(v-1)(v-2)}{60}$. Then it is possible to construct a 3-TD with group partition $G = \{X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}, X_{i_5}\}$ and family B_k of blocks. The vertices of any block of B_k can define a hypercycle system $\mathcal{C}(3,5,5)$, and from all those blocks together we obtain a family $\mathcal{E}'_k$ of 5-cycles. We insert also those into $\mathcal{E}$.

In this way we have obtained the following family $\mathcal{E}$ of 5-cycles:

$$\mathcal{E} \;=\; (\bigcup_{i=1}^{v}\mathcal{E}_i\;) \;\cup\; (\bigcup_{1\le i<j\le v}\mathcal{E}_{i,j}\;) \;\cup\; (\bigcup_{k=1}^{m}\mathcal{E}'_k\;).$$

In particular, for the sizes of the three parts we see

$$|\bigcup_{i=1}^{v}\mathcal{E}_i\;| = v\cdot\frac{w(w-1)(w-2)}{30} = \frac{vw^3-3vw^2+2vw}{30}\;;$$

$$\begin{aligned} |\bigcup_{1\le i<j\le v}\mathcal{E}_{i,j}\;| &= \frac{v(v-1)}{2}\left(\frac{2w(2w-1)(2w-2)}{30} - 2\cdot\frac{w(w-1)(w-2)}{30}\right)\\ &= \frac{3v^2w^3-3v^2w^2-3vw^3+3vw^2}{30}\;; \end{aligned}$$

$$|\bigcup_{k=1}^{m}\mathcal{E}'_k\;| \;=\; 2w^3\cdot\frac{v(v-1)(v-2)}{60} \;=\; \frac{v^3w^3-3v^2w^3+2vw^3}{30}.$$

It is simple to verify that the cardinality of $\mathcal{E}$ is

$$|\mathcal{E}| = \frac{vw(vw-1)(vw-2)}{30}.$$

It remains to prove that each 3-set of X is in one 5-cycle of $\mathcal{E}$. In fact any 3-set inside an X_i, for $1\le i\le v$, is contained in one 5-cycle of $\mathcal{E}_i$; a 3-set which contains a vertex in X_i and two vertices in X_j, for $1\le i,j\le v$, is in a 5-cycle of $\mathcal{E}_{i,j}$; and a 3-set which contains vertices from three distinct X_i, X_j, X_p is in a 5-cycle of an $\mathcal{E}'_k$. The theorem is proved. □

In fact a strengthening of the above theorem is possible by replacing Steiner systems with cycle systems, as follows.

Theorem 4.3 *If there exists a hypercycle system $\mathcal{C}(3,5,v)$ and a 2-split system $\mathcal{C}(3,5,2w)$ with $w\equiv 1$ or $5 \pmod 6$, then there exists a $\mathcal{C}(3,5,vw)$.*

Proof.

The construction is analogous to the one in the preceding theorem. We replace each vertex x_i $(1\le i\le v)$ of a hypercycle system $\mathcal{C}(3,5,v)$ with a set X_i of cardinality w, substitute a $\mathcal{C}(3,5,w)$ into each X_i, and insert the crossing 5-cycles of a 2-split $\mathcal{C}(3,5,2w)$ between each pair X_i, X_j for all $1\le i<j\le v$. What remains is to find a decomposition into 5-cycles for the collection of those 3-sets which meet three distinct $X_{i_1}, X_{i_2}, X_{i_3}$. This will be carried out with the help of 3-TD$(5,w)$ systems, using the structure of $\mathcal{C}(3,5,v)$.

Any three vertices $x_{i_1}, x_{i_2}, x_{i_3}$ in $\mathcal{C}(3,5,v)$ belong to precisely one 5-cycle, say $C = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}$. We now consider a 3-TD on $X_{i_1} \cup \cdots \cup X_{i_5}$, where the sets X_{i_j} are the groups of size w. Each block B of the 3-TD covers ten 3-sets; we put those five of them into the $\mathcal{C}(3,5,vw)$ system which correspond to the members of the 5-cycle C. In a cyclical listing these are $\{B\cap X_{i_1}, B\cap X_{i_2}, B\cap X_{i_3}\}$, $\{B\cap X_{i_2}, B\cap X_{i_3}, B\cap X_{i_4}\}$, $\ldots$, $\{B \cap X_{i_5}, B \cap X_{i_1}, B \cap X_{i_2}\}$. In this way a 5-cycle is specified for each 3-set which meets three distinct parts X_i of the construction. Observe that any such 3-set T uniqely determines a 5-cycle $C(T)$ of $\mathcal{C}(3,5,v)$; then $C(T)$ uniqely determines a 5-tuple of parts $X_{i_j}(T)$ $(j = 1,\ldots,5)$ such that $T \subset X_{i_1}(T) \cup \cdots \cup X_{i_5}(T)$; inside those five parts T uniqely determines the block $B(T)$ such that $T \subset B(T)$; and then precisely one 5-cycle is specified inside $B(T)$. It follows that the obtained 5-cycles — together with the copies of $\mathcal{C}(3,5,w)$ inside each X_i, and with the crossing cycles of the copies of the 2-split $\mathcal{C}(3,5,2w)$ system between any two X_i, X_j — form a $\mathcal{C}(3,5,vw)$. □

5 Conclusion

We have designed two recursive ways to construct hypercycle systems $\mathcal{C}(3,5,v)$ from which some infinite classes can be obtained. Here we briefly summarize their scheme, together with the earlier one mentioned in the introduction:

Rule 1. $\mathcal{C}(3,5,s) + S(3,s,v) \longrightarrow \mathcal{C}(3,5,v)$ (Meszka and Rosa [8]);

Rule 2. $\mathcal{C}(3,5,v)$ + resolvable $S(2,5,v) \longrightarrow$ 2-split $\mathcal{C}(3,5,2v)$ (Theorem 4.1);

Rule 3. 2-split $\mathcal{C}(3,5,2v)$ with $v \equiv 1$ or $5 \pmod 6$ + $S(3,5,w) \longrightarrow \mathcal{C}(3,5,vw)$ (Theorem 4.2);

Rule 4. 2-split $\mathcal{C}(3,5,2v)$ with $v \equiv 1$ or $5 \pmod 6$ + $\mathcal{C}(3,5,w) \longrightarrow \mathcal{C}(3,5,vw)$ (Theorem 4.3).

These rules can be applied iteratively, starting from the hypercycle systems mentioned above, and applying the following known facts from design theory:

- The necessary condition $v \equiv 5 \pmod{20}$ for the existence of a resolvable system $\mathrm{S}(2,5,v)$ is known to be sufficient for all v but for $v = 45$, 345, 465, 645 ([2, p. 127], Table 7.37). This means that almost all $v \equiv 5 \pmod{20}$ are applicable in Rule 2.

- Spherical geometries $\mathrm{S}(3,q+1,q^n+1)$ exist for every prime power q and integer $n \geq 2$ ([2, p. 103], Theorem 5.11.2). This yields the sequence of $\mathrm{S}(3,5,4^n+1)$ for Rule 3, and more generally some pairs (q,n) for Rule 1 with suitable values of $s = q+1$. Also, a system $\mathrm{S}(3,5,101)$ is known to exist.

- If an $\mathrm{S}(3,q+1,v+1)$ and an $\mathrm{S}(3,q+1,w+1)$ both exist, then there exists an $\mathrm{S}(3,q+1,vw+1)$ ([2, p. 103], Theorem 5.12).

A partial list of parameters, from which one can start to build larger systems, is shown in Table 1. Then in Table 2 we exhibit a possible sequence of operations that can be performed on the basis of those initial configurations. Table 3 gives a more transparent summary of known constructions.

system type	initial values of v
5-cycle system general $\mathcal{C}(3,5,v)$	5, 7, 10, 11, 16, 20, 22
2-split 5-cycle system $\mathcal{C}(3,5,2v)$	5, 10, 11 ($2v = 10$, 20, 22)
resolvable $S(2,5,v)$	25, 65, 85, 105, 125, etc.
$S(3,5,v)$	5, 17, 26, 65, 101, 257
$S(3,s,v)$ with $s > 5$	82, 730 ($s = 10$); 362 ($s = 20$)

Table 1: Some starting configurations

Next we mention some consequences.

Corollary 5.1 *Some of the infinite classes of hypercycle systems that can be obtained are as follows.*

1. *There exists a* $\mathcal{C}(3,5,v)$ *for every order* $v = 4^n + 1$, $v = 9^n + 1$, $v = 19^n + 1$, $v = 25^n + 1$, $v = 49^n + 1$, *and* $v = 169^n + 1$, *for any* $n \geq 1$.
2. *There exists a* 2*-split* $\mathcal{C}(3,5,2v)$ *for every order* $2v = 2 \cdot (4^{2m+1} + 1)$, $m \geq 0$.
3. *There exists a* $\mathcal{C}(3,5,v)$ *for every order* $v = (4^n + 1) \cdot (4^{2m+1} + 1)$, $n \geq 1$, $m \geq 0$.
4. *There exists a* $\mathcal{C}(3,5,v)$ *for every order* $v = w \cdot (4^n + 1)$ *where* $w = 11$, 25, 85, 125, 325, *for any* $n \geq 1$.

Proof.

1: by Rule 1, from the existence of $S(3,s,v)$ systems (spherical geometries, in particular) where $s-1$ is a prime power, and from the existence of $\mathcal{C}(3,5,s)$ systems with $s = 5$, 10, 20, 26, 50, and 170.

2: by Rule 2, using *1.* and the fact that $4^{2m+1} + 1 \equiv 5 \pmod{20}$ for all m, hence a resolvable $S(2,5,4^{2m+1}+1)$ exists. Note that the possible four exceptions v from the orders of resolvable systems are not of the form $4^{2m+1} + 1$. (The even powers of 4 do not work here.)

3: by Rule 3, the 2-split system taken from *2.*, applying the spherical geometries of block size 5.

rule	systems used	$\mathcal{C}(3,5,v)$ obtained with $v =$
1	$\mathcal{C}(3,5,5)$ + $S(3,5,17\,\vert\,26\,\vert\,65\,\vert\,101\,\vert\,257)$	17, 26, 65, 101, 257
3	2-split $\mathcal{C}(3,5,10)$ + $S(3,5,5\,\vert\,17\,\vert\,26\,\vert\,65\,\vert\,101)$	25, 85, 130, 325, 505
3	2-split $\mathcal{C}(3,5,22)$ + $S(3,5,5\,\vert\,17\,\vert\,26\,\vert\,65\,\vert\,101)$	55, 187, 286, 715, 1111
2	$\mathcal{C}(3,5,v)$ + resolvable $S(2,5,v)$ (v = 25, 65, 85, 325, 505)	**2-split** $2v$ = 50, 130, 170, 650 650, 1010
3	2-split $\mathcal{C}(3,5,50)$ + $S(3,5,5\,\vert\,17)$	125, 425
2	$\mathcal{C}(3,5,125\,\vert\,425)$ + resolvable $S(2,5,125\,\vert\,425)$	**2-split** $2v$ = 250, 850
3	2-split $\mathcal{C}(3,5,250)$ + $S(3,5,5)$	625
3	2-split $\mathcal{C}(3,5,130)$ + $S(3,5,17)$	1105
1	$\mathcal{C}(3,5,10)$ + $S(3,10,9^2+1\,\vert\,9^3+1)$	82, 730
1	$\mathcal{C}(3,5,20)$ + $S(3,20,19^2+1)$	362
1	$\mathcal{C}(3,5,26)$ + $S(3,26,25^2+1)$	626
4	2-split $\mathcal{C}(3,5,10\,\vert\,22\,\vert\,50\,\vert\,130\,\vert\,170\,\vert\,250)$ + $\mathcal{C}(3,5,7)$	35, 77, 175, 455, 595, 875
2	$\mathcal{C}(3,5,385)$ + resolvable $S(2,5,385)$	**2-split** $2v$ = 770
4	2-split $\mathcal{C}(3,5,10\,\vert\,22\,\vert\,50\,\vert\,130)$ + $\mathcal{C}(3,5,16)$	80, 176, 400, 1040
4	2-split $\mathcal{C}(3,5,10\,\vert\,22\,\vert\,50)$ + $\mathcal{C}(3,5,20)$	100, 220, 500
4	2-split $\mathcal{C}(3,5,10)$ + $\mathcal{C}(3,5,22\,\vert\,55\,\vert\,77\,\vert\,82\,\vert\,110\,\vert\,187)$	110, 275, 385, 410, 550, 935
4	2-split $\mathcal{C}(3,5,22)$ + $\mathcal{C}(3,5,11\,\vert\,22\,\vert\,55\,\vert\,80\,\vert\,82\,\vert\,100)$	121, 242, 605, 880, 902, 1100

Table 2: Some iterative constructions with $v \leq 1111$

5	7	**10**	11	16	17	**20**	**22**	25
26	35	**50**	55	65	77	80	82	85
100	101	110	121	125	**130**	**170**	175	176
187	220	242	**250**	257	275	286	325	362
385	400	410	425	455	500	505	550	595
605	625	626	**650**	715	730	**770**	**850**	875
880	902	935	**1010**	1040	1100	1105	1111	

Table 3: Orders $\leq$ 1111 of known constructions: 5-cycle systems and **2-split** systems

4: by Rule 3, combining spherical geometries of order $4^n + 1$ with those w which have been obtained from the 2-split systems of Table 2 and are not included in *3.* It should be noted that not all of the known 2-split systems are applicable here; for instance $20/2 \equiv 4 \pmod 6$, hence $\mathcal{C}(3, 5, 20)$ is not a suitable choice. □

Remark 5.1 *Table 5.17 of [2, pp. 103–104] lists all existing, non-existing, and unsettled cases of $S(t, k, t+n)$ up to $n = 200$. Concentrating on the possible applicability of Rule 1, the latter (unsettled) relevant ones are $v = 41$, 50, 62, 77, 86, 110, 122, 125, 137, 146, 161, 170, 182, 185, 197 for $S(3, 5, v)$; $v = 77$, 92, 107, 112, 127, 142, 182, 197 for $S(3, 7, v)$; and $v = 146$ for $S(3, 10, v)$. There are several ones among them — namely 50, 77, 110, 125, 170 — which have been solved above for 5-cycle systems. We note further that some infinite classes of the constructed $\mathcal{C}(3, 5, v)$ systems have $v \equiv 1 \pmod 3$, hence they surely cannot be obtained from an $S(3, 5, v)$ that would require $v \equiv 2 \pmod 3$. This is the case with all the systems in parts 2. and 3. of the corollary, and also with the ones belonging to $w = 11$ and $w = 125$ in part 4.*

Conjecture 5.1 *Suppose that v is sufficiently large.*

1. *If $v \equiv 1, 2, 7, 11 \pmod{15}$, then there exists a cyclic system $\mathcal{C}(3, 5, v)$.*
2. *If $v \equiv 1, 5, 10, 11 \pmod{15}$, then there exists a 2-split system $\mathcal{C}(3, 5, 2v)$.*
3. *If $2v \equiv 2, 22 \pmod{30}$, then there exists a cyclic 2-split system $\mathcal{C}(3, 5, 2v)$.*
4. *If $v \equiv 1, 11 \pmod{15}$, then there exists a 2-split system $\mathcal{C}(3, 5, 2v)$ on the vertex set $X = X_1 \cup X_2$, with $X_i = \{x_{i,j} \mid 1 \le j \le v\}$ for $i = 1, 2$, such that both the involution $x_{1,j} \leftrightarrow x_{2,j}$ $(1 \le j \le v)$ and the 2-orbit permutation which is cyclic on both X_1 and X_2 (i.e., $x_{i,j} \to x_{i,j+1}$ for $i = 1, 2$ and for all $1 \le j \le v$, where $x_{i,v+1} = x_{i,1}$) are automorphisms.*

We close this paper with some recursive conjectures, which are perhaps more reachable than the ones above.

Conjecture 5.2 *For every v and w we have:*

1. *If there exist hypercycle systems $\mathcal{C}(3, 5, v)$ and $\mathcal{C}(3, 5, w)$, then there exists a system $\mathcal{C}(3, 5, (v - 1)(w - 1) + 1)$.*
2. *If there exist cyclic systems $\mathcal{C}(3, 5, v)$ and $\mathcal{C}(3, 5, w)$, then there exists a cyclic system $\mathcal{C}(3, 5, (v - 1)(w - 1) + 1)$.*
3. *If there exist 2-split systems $\mathcal{C}(3, 5, 2v)$ and $\mathcal{C}(3, 5, 2w)$, then there exists a 2-split system $\mathcal{C}(3, 5, (2v - 1)(2w - 1) + 1)$.*
4. *If there exist cyclic 2-split systems $\mathcal{C}(3, 5, 2v)$ and $\mathcal{C}(3, 5, 2w)$, then there exists a cyclic 2-split $\mathcal{C}(3, 5, (2v - 1)(2w - 1) + 1)$.*

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