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Counting labeled threshold graphs with Eulerian numbers

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Abstract

A threshold graph is any graph which can be constructed from the empty graph by repeatedly adding a new vertex that is either adjacent to every vertex or to no vertices. The Eulerian number $\left\langle {n \atop k} \right\rangle$ counts the number of permutations of size n with exactly k ascents. Implicitly, Beissinger and Peled proved that the number of labeled threshold graphs on $n \geq 2$ vertices is

$$\sum_{k=1}^{n-1}(n-k)\left\langle {n-1 \atop k-1} \right\rangle 2^k.$$

Their proof used generating functions. We give a direct combinatorial proof of this result.

1 Introduction

This paper deals with threshold graphs, which can be defined recursively as follows. The empty graph is the unique threshold graph on 0 vertices. An n-vertex graph G is a threshold graph if and only if it can be obtained by taking a threshold graph G' on $n-1$ vertices and adding a new vertex which is either isolated or adjacent to every other vertex of G'.

Threshold graphs were first studied by Chvátal, and Hammer [3] in relation to linear programming, and since then they have been extensively studied. One such reason for this is that threshold graphs can be characterized in several different ways. For example, G is a threshold graph if and only if it contains no induced subgraph isomorphic to $2K_2$, P_4, or C_4 [6]. Variations such as random threshold graphs [4] and oriented threshold graphs [2] have been studied in recent years. We refer the reader

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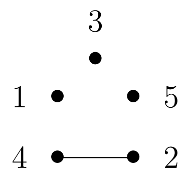

Figure 1: $\tilde{G} = T(24135, ++---)$

to the book "Threshold Graphs and Related Topics" [6] for more information and characterizations of threshold graphs.

It is easy to prove that the number of unlabeled threshold graphs on n vertices is exactly 2^{n-1}. Let t_n denote the number of labeled threshold graphs on n vertices. Beissinger and Peled found the exponential generating function of t_n to be $e^x(1-x)/(2-e^x)$ [1]. Using this they were able to derive an asymptotic formula for t_n, and implicitly they found an exact formula for t_n in terms of the Eulerian numbers $\left\langle {n \atop k} \right\rangle$, which we shall now define.

Let $\mathcal{S}_n$ denote the set of permutations of size n, where we treat our permutations as words written in one line notation. Given $\pi \in \mathcal{S}_n$, we say that position i with $1 \le i \le n-1$ is an ascent of π if $\pi_i < \pi_{i+1}$. Let $\mathrm{Asc}(\pi)$ denote the set of ascents of a permutation π and let $\mathrm{asc}(\pi) = |\mathrm{Asc}(\pi)|$. Define the Eulerian number $\left\langle {n \atop k} \right\rangle$ to be the number of permutations $\pi \in \mathcal{S}_n$ with $\mathrm{asc}(\pi) = k$. With this, a formula for t_n can be stated as follows.

Theorem 1 ([1]). *For $n \ge 2$, the number of labeled threshold graphs on n vertices is*

$$\sum_{k=1}^{n-1} (n-k) \left\langle {n-1 \atop k-1} \right\rangle 2^k.$$

This result can be derived from (16) of Beissinger and Peled [1] through some algebraic manipulation, though it is not immediately obvious that this is the case. Here we give a more direct and combinatorial proof of Theorem 1.

2 Proof of Theorem 1

We will say that a pair (π, w) is a threshold pair (of size n) if $\pi \in \mathcal{S}_n$ and if w is a word in $\{+1, -1\}^n$. Given a threshold pair (π, w), let $T(\pi, w)$ denote the labeled threshold graph obtained as follows. Let G_1 be the graph with a single vertex π_1. Given G_{i-1} with $2 \le i \le n$, define G_i by introducing a new vertex to G_{i-1} labeled π_i that is either connected to every vertex of G_{i-1} if $w_i = +1$, and otherwise π_i is an isolated vertex. We then let $T(\pi, w) = G_n$. As an example, Figure 1 shows $\tilde{G} := T(24135, ++---)$, where for ease of notation we have omitted the 1's in w. We will use $\tilde{G}$ as a working example throughout this paper.

There are several ways to write $\tilde{G}$, for example, $\tilde{G} = T(42351, -+---)$. We wish to standardize our choice of a threshold pair. To this end, we will say that a threshold pair (π, w) of size $n \geq 2$ is in standard form if $w_1 = w_2$ and if $w_i = w_{i+1}$ implies $\pi_i < \pi_{i+1}$ for all $1 \leq i < n$. For example, $(42351, -+---)$ is not in standard form but $(24135, ++---)$ is. Our first goal will be to prove the following.

Lemma 2. *Let G be a labeled threshold graph on $n \geq 2$ vertices. Then there exists a unique threshold pair (π, w) in standard form such that $G = T(\pi, w)$.*

To prove this, we require two more lemmas.

Lemma 3. *Let (π, w) and (σ, u) be threshold pairs of size $n \geq 2$ and let $G_1 := T(\pi, w)$ and $G_2 := T(\sigma, u)$. Then $G_1 = G_2$ as labeled graphs if and only if the following two conditions hold.*

(a) $w_k = u_k$ *for all* $k \geq 2$.

(b) *For every* $1 \leq i \leq n$, *if* $j = \pi_i^{-1}$ *and* $k = \sigma_i^{-1}$, *then either* $1 \in \{j, k\}$ *and* $w_\ell = w_{\max\{j,k\}}$ *for all* $1 < \ell \leq \max\{j, k\}$, *or for every* ℓ *with* $\min\{j, k\} \leq \ell \leq \max\{j, k\}$ *we have* $w_\ell = w_j = w_k$.

Proof. We first show that these conditions are necessary. We claim that condition (a) is necessary to have G_1 isomorphic to G_2, which certainly implies that (a) is necessary for G_1 and G_2 to be equal as labeled graphs. This claim is true when $n = 2$. Assume the claim has been proven up to some $n \geq 3$. If $w_n \neq u_n$, then exactly one of G_1 and G_2 will have an isolated vertex, so they cannot be isomorphic. Otherwise let G_1' be G_1 after deleting vertex π_n and G_2' be G_2 after deleting σ_n. Note that in both cases we either delete an isolated vertex or a vertex adjacent to every other vertex since $w_n = u_n$. Thus $G_1 \cong G_2$ if and only if $G_1' \cong G_2'$. The result follows by applying the inductive hypothesis to G_1' and G_2' since the words generating these graphs are the words w and u after deleting their last letters. Thus (a) is necessary.

We next show that (b) is necessary. Assume for contradiction that $G_1 = G_2$ and that (b) does not hold for some i. By the above claim, we can assume that (a) holds. Let $j = \pi_i^{-1}$ and $k = \sigma_i^{-1}$. If $j = k$ then (b) holds, a contradiction. Thus we can assume that $j \neq k$, and without loss of generality we can assume $j < k$. Let d_r be the degree of vertex i in G_r for $r = 1, 2$. First consider the case $j = 1$ and $w_k = +1$. In this case $d_1 = |\{\ell : w_\ell = +1,\ \ell > 1\}|$ and $d_2 = k - 1 + |\{\ell : w_\ell = +1,\ \ell > k\}$, where we used that $u_\ell = w_\ell$ for all $\ell > 1$ by (a). Thus $d_1 < d_2$ unless $w_\ell = +1$ for all $1 < \ell \leq k$. Because $G_1 = G_2$, this must be the case, so (b) holds for i, a contradiction. Essentially the same proof works if $j = 1$ and $w_k = -1$.

Now assume $j > 1$ and $w_j = +1$, so $d_1 = j - 1 + |\{\ell : w_\ell = +1,\ \ell > j\}|$. If $w_k = -1$ then $d_2 = |\{\ell : w_\ell = +1,\ \ell > k\}| < d_1$ since $j - 1 \geq 1$ by assumption. In this case we cannot have $G_1 = G_2$, so we can assume $w_k = +1$. This implies $d_2 = k - 1 + |\{\ell : w_\ell = +1,\ \ell > k\}|$. This will be strictly larger than d_1 unless $w_\ell = +1$ for all $j < \ell < k$. Thus (b) holds for i, a contradiction. Essentially the same proof works if $j > 1$ and $w_j = +1$. We conclude that (b) is necessary.

To show that these conditions are sufficient, let (π, w) and (σ, u) be threshold pairs satisfying (a) and (b). Fix some i and let $j = \pi_i^{-1}$ and $k = \sigma_i^{-1}$. We can assume without loss of generality that $j \leq k$. First consider the case $j = 1$ and $w_k = +1$. Then the neighborhood of i in G_1 is $\{\pi_\ell^{-1} : \ell > j,\ w_\ell = +1\}$, and the neighborhood of i in G_2 is $\{\pi_\ell^{-1} : \ell < k\} \cup \{\pi_\ell^{-1} : \ell > k,\ w_\ell = +1\}$, where again we used that $u_\ell = w_\ell$ for all $\ell > 1$. By (b), $w_\ell = +1$ for all $1 < \ell < k$, so these two sets are equal. The same result holds if $j = 1$ and $w_k = -1$.

Assume $j > 1$ and $w_j = +1$. We have $u_k = w_k = w_j = +1$ by (a) and (b). Thus the neighborhood of i in G_1 is $\{\pi_\ell^{-1} : \ell < j\} \cup \{\pi_\ell^{-1} : \ell > j,\ w_\ell = +1\}$, and the neighborhood of i in G_2 is $\{\pi_\ell^{-1} : \ell < k\} \cup \{\pi_\ell^{-1} : \ell > k,\ w_\ell = +1\}$. By (b) we have $w_\ell = +1$ for all $j < \ell < k$, so these sets are equal. The same result holds if $j > 1$ and $w_j = -1$. We conclude that the neighborhoods of every vertex is the same in both G_1 and G_2, and hence $G_1 = G_2$. □

Lemma 4. *If G is a threshold graph on $n \geq 2$ vertices, then there exists a threshold pair (π, w) such that $G = T(\pi, w)$.*

Proof. This certainly holds when $n = 2$, so assume it holds up to some $n \geq 3$. Because G is a threshold graph, there exists a labeled threshold graph H on $n - 1$ vertices such that G is isomorphic to H together with the additional vertex n which is either isolated or adjacent to every other vertex of G'. Denote this labeled graph that G is isomorphic to by K.

By our inductive hypothesis, $H = T(\pi', w')$ for some threshold pair (π', w'). Define π by $\pi_k = \pi'_k$ for $k < n$ and $\pi_n = n$. Define w by $w_k = w'_k$ for $k < n$ with $w_n = -1$ if K contains an isolated vertex and $w_n = +1$ otherwise. Then $K = T(\pi, w)$. By construction there exists a graph isomorphism $\sigma : V(K) \to V(G)$. Thus $T(\sigma \circ \pi, w)$ is isomorphic to G with the identity map serving as the graph isomorphism. In other words, $G = T(\sigma \circ \pi, w)$. □

Proof of Lemma 2. We first show that such a pair exists. Let (π', w') be a threshold pair with $G = T(\pi', w')$, which exists by Lemma 4. Define w by $w_k = w'_k$ for $k > 1$ and $w_1 = w_2$. Note that $T(\pi', w) = G$ by Lemma 3. Next define π by repeatedly flipping adjacent letters of π' that are out of order. More precisely, let $\pi^{(0)} = \pi'$. Inductively assume we have defined $\pi^{(j)}$. If $(\pi^{(j)}, w)$ is in standard form, take $\pi = \pi^{(j)}$. Otherwise there exists some index i such that $\pi_i^{(j)} > \pi_{i+1}^{(j)}$ and $w_i = w_{i+1}$. Define $\pi^{(j+1)}$ by $\pi_i^{(j+1)} = \pi_{i+1}^{(j)}$, $\pi_{i+1}^{(j+1)} = \pi_i^{(j)}$, and with $\pi_k^{(j+1)} = \pi_k^{(j)}$ for all other k. Note that this process eventually terminates (this can be seen, for example, by noting that the number of inversions decreases at each step), and that $T(\pi^{(j+1)}, w) = T(\pi^{(j)}, w)$ for all j by Lemma 3. As $T(\pi^{(0)}, w) = T(\pi', w) = G$, we conclude that $T(\pi, w) = G$, and hence such a pair exists.

To show that this pair is unique, assume that (σ, u) is also a threshold pair in standard form with $G = T(\sigma, u)$. By Lemma 3 we must have $u_k = w_k$ for all $k > 1$. Further, $u_1 = u_2 = w_2 = w_1$ since the pairs are in standard form. We next partition w into maximal segments that are all ± 1. To this end, let $p_0 = 1$. Inductively given p_{r-1}, define p_r to be the smallest integer p such that $w_p \neq w_{p_{r-1}}$, and let $p_r = n + 1$ if no such integer exists. Define $P_r = \{\pi_i : p_r \leq i < p_{r+1}\}$ and $S_r = \{\sigma_i : p_r \leq i < p_{r+1}\}$.

We claim that $P_r = S_r$ for all r. Indeed, assume that there exists some $i \in P_r$ and $i \in S_{r'}$ with, say, $r < r'$. Let $j = \pi_i^{-1}$ and $k = \sigma_i^{-1}$. By Lemma 3 we have $w_\ell = w_j$ for all $j \leq \ell \leq k$. In particular this holds for $\ell = p_{r+1}$ since $j < p_{r+1} \leq k$, which is a contradiction since $w_{p_r} = w_j$ by assumption of $\pi_j \in P_r$. We conclude that $P_r = S_r$ for all r. Because (π, w) is in standard form, we also must have $\pi_{p_r} < \pi_{p_r+1} < \cdots < \pi_{p_{r+1}-1}$ for all r, and the same inequalities hold with π replaced by σ. We conclude that $\pi_i = \sigma_i$ for all $p_r \leq i < p_{r+1}$ for all r, and hence $\pi = \sigma$, proving the result. $\square$

We now define our sets for the desired bijection. Let T_n denote the set of labeled threshold graphs on n vertices. Let $\mathcal{S}_n^+$ for $n \geq 2$ be the set of permutations of length n with $\pi_1 < \pi_2$. That is, these are the set of permutations which begin with an ascent. Define $P_n := \{(\pi, A) : \pi \in \mathcal{S}_n^+,\ A \subseteq \mathrm{Asc}(\pi)\}$.

Proposition 5. *There exists a bijection from T_n to P_n.*

Proof. Let G be a labeled threshold graph and (π^G, w^G) the unique threshold pair guaranteed by Lemma 2. Define $A'_G = \{i : w_i^G = w_{i+1}^G,\ 2 \leq i \leq n-1\}$. Let $A_G = A'_G \cup \{1\}$ if $w_1 = +1$ and let $A_G = A'_G$ if $w_1 = -1$. Define $\phi(G) = (\pi^G, A_G)$. For example, if $\tilde{G}$ is as in Figure 1, we have $(\pi^{\tilde{G}}, w^{\tilde{G}}) = (24135, ++---)$, and hence $\phi(\tilde{G}) = (24135, \{1,3,4\})$. We claim that the map ϕ gives the desired bijection.

We first show that ϕ is a map from T_n to P_n. Indeed, because (π^G, w^G) is in standard form, we have $w_1^G = w_2^G$ and hence $\pi_1^G < \pi_2^G$, so $\pi^G \in \mathcal{S}_n^+$. By similar reasoning we find that $A_G \subseteq \mathrm{Asc}(\pi^G)$, proving the claim.

Let (π, A) be an element of P_n. We define the word w as follows. Let $w_1 = w_2 = +1$ if $1 \in A$ and set $w_1 = w_2 = -1$ otherwise. Given w_k, let $w_{k+1} = w_k$ if $k \in A$ and otherwise let $w_{k+1} = -w_k$. We claim that $G = T(\pi, w)$ is the unique threshold graph with $\phi(G) = (\pi, A)$.

First observe that $A \subseteq \mathrm{Asc}(\pi)$ implies the pair (π, w) is in standard form. Thus $\phi(G) = (\pi, A_G)$, and it is not difficult to verify that $A_G = A$ by construction, so $\phi(G) = (\pi, A)$. Assume that H is also such that $\phi(H) = (\pi, A)$, so in particular $\pi^H = \pi$. We claim that $w_k^H = w_k$ for all k. Indeed, because $A_H = A$, we must have $w_1^H = w_1$, as this completely determines whether 1 is in A_H or not, and also $w_2^H = w_1^H = w_1 = w_2$ since both pairs are in standard form. Inductively assume that $w_k^H = w_k$ for some $2 \leq k \leq n-1$. If $k \in A$, then we must have $w_{k+1}^H = w_k^H = w_k = w_{k+1}$, and otherwise we have $w_{k+1}^H = -w_k^H = -w_k = w_{k+1}$. We conclude the result by induction. Thus $(\pi^H, w^H) = (\pi, w) = (\pi^G, w^G)$, and we conclude that $H = G$ by Lemma 2. Thus each element of P_n is mapped to by a unique element of T_n and the result follows. $\square$

All that remains is to enumerate P_n. To this end, we say that a permutation π has a descent in position i if $\pi_i > \pi_{i+1}$.

Lemma 6. *For all n and d with $n \geq 1$ and $0 \leq d \leq n-1$, let $\mathcal{S}_{n,d}^+$ be the set of permutations of size n which begin with an ascent and which have exactly d descents. If $P(n,d) := |\mathcal{S}_{n,d}^+|$, then $P(n,d) = (d+1)\left\langle {n-1 \atop d} \right\rangle$.*

We note that this result is proven in [7], but for completeness we include the full proof here. For this proof, we recall the following recurrence for the Eulerian numbers, which is valid for all $n \geq 1$ and $d \geq 0$ after adopting the convention $\left\langle {0 \atop d} \right\rangle = 0$ for $d > 0$, $\left\langle {0 \atop 0} \right\rangle = 1$, and $\left\langle {n \atop -1} \right\rangle = 0$ [5]:

$$\left\langle {n \atop d} \right\rangle = (d+1)\left\langle {n-1 \atop d} \right\rangle + (n-d)\left\langle {n-1 \atop d-1} \right\rangle. \tag{1}$$

Proof. The result is true for $d = 0$, so assume $d \geq 1$. For any fixed d the result is true for $n = 1$, so assume $n \geq 2$. To help us prove the result, we define $\mathcal{S}_{n,d}^-$ to be the set of permutations which begin with a descent and which have exactly d descents. Define $M(n,d) := |\mathcal{S}_{n,d}^-|$. By construction we have

$$P(n,d) + M(n,d) = \left\langle {n \atop d} \right\rangle. \tag{2}$$

Define the map $\phi : \mathcal{S}_{n,d}^+ \to \mathcal{S}_{n-1}$ by sending $\pi \in \mathcal{S}_{n,d}^+$ to the word obtained by removing the letter n from π. We wish to determine the image of ϕ. Let π be a permutation in $\mathcal{S}_{n,d}^+$, and let i denote the position of n in π. Note that $i \neq 1$ since π begins with an ascent. If $i = n$ or $\pi_{i-1} > \pi_{i+1}$ with $i > 2$, then $\phi(\pi)$ will continue to have d descents and begin with an ascent, so $\phi(\pi) \in \mathcal{S}_{n-1,d}^+$. If $i = 2$ and $\pi_1 > \pi_3$, then $\phi(\pi) \in \mathcal{S}_{n-1,d}^-$. If $\pi_{i-1} < \pi_{i+1}$, then $\phi(\pi) \in \mathcal{S}_{n-1,d-1}^+$.

It remains to show how many times each element of the image is mapped to by ϕ. If $\pi \in \mathcal{S}_{n-1,d}^+$, then n can be inserted into π in $d+1$ ways to obtain an element of $\mathcal{S}_{n,d}^+$ (it can be placed at the end of π or in between any $\pi_i > \pi_{i+1}$). If $\pi \in \mathcal{S}_{n-1,d-1}^+$, then n can be inserted in π in $n-d$ ways to obtain an element of $\mathcal{S}_{n,d}^+$ (it can be placed in between any $\pi_i < \pi_{i+1}$). If $\pi \in \mathcal{S}_{n-1,d}^-$, then n must be inserted in between $\pi_1 > \pi_2$ in order to have the word begin with an ascent. With this and the inductive hypothesis, we conclude that

$$\begin{aligned} P(n,d) &= (d+1)P(n-1,d) + (n-d)P(n-1,d-1) + M(n-1,d) \\ &= (d+1)^2\left\langle {n-2 \atop d} \right\rangle + (n-d)d\left\langle {n-2 \atop d-1} \right\rangle + M(n-1,d). \end{aligned} \tag{3}$$

By using (2), the inductive hypothesis, and (1); we find

$$\begin{aligned} M(n-1,d) &= \left\langle {n-1 \atop d} \right\rangle - P(n-1,d) \\ &= \left\langle {n-1 \atop d} \right\rangle - (d+1)\left\langle {n-2 \atop d} \right\rangle \\ &= (n-d)\left\langle {n-2 \atop d-1} \right\rangle. \end{aligned}$$

Substituting this into (3) and applying (1) again gives the result. □

Corollary 7. *Let $n \geq 2$ and $1 \leq k \leq n-1$. The number of permutations of $\mathcal{S}_n^+$ with exactly k ascents is $(n-k)\left\langle {n-1 \atop k-1} \right\rangle$.*

Note that there is no need to consider $k = 0$ as every permutation of $\mathcal{S}_n^+$ automatically has at least one ascent.

Proof. This quantity is exactly $(n-k)\left\langle {n-1 \atop n-1-k} \right\rangle$ by Lemma 6 after replacing d with $n-1-k$ (as any permutation of size n with k ascents has $n-1-k$ descents). It is well known and easy to prove that $\left\langle {m \atop x} \right\rangle = \left\langle {m \atop m-1-x} \right\rangle$ for $m > 0$ [5], from which the result follows. ◻

Proof of Theorem 1. By Proposition 5 it is enough to prove that P_n has this cardinality. Given $\pi \in \mathcal{S}_n^+$, the number of pairs $(\pi, A) \in P_n$ is exactly $2^{\mathrm{asc}(\pi)}$. By Corollary 7 we conclude that

$$|P_n| = \sum_{k=1}^{n-1} (n-k) \left\langle {n-1 \atop k-1} \right\rangle 2^k,$$

proving the result. ◻

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