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On a spanning tree with specified leaves in a bipartite graph

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Abstract

Let G be a graph and let S be a subset of $V(G)$. Then G is called *S-leaf-connected* if G has a spanning tree T such that S is the set of end-vertices of T. In 1986, Gurgel and Wakabayashi obtained a closure result for S-leaf-connected. This yields a result on a degree sum condition as a corollary. In 2008, Egawa et al. gave a degree sum condition for a graph with high connectivity to be S-leaf-connected. In this paper, we obtain the bipartite analogues of their results. The degree sum conditions of our results are sharp.

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex $x \in V(G)$, we denote the set of vertices adjacent to x in G by $N_G(x)$ and the degree of x in G by $d_G(x)$; thus $d_G(x) = |N_G(x)|$. For $S \subseteq V(G)$, let $N_G(S) = \bigcup_{x\in S} N_G(x)$. For $S \subseteq V(G)$, let $G - S$ denote the subgraph induced by $V(G) \setminus S$ in G.

Ore (1963) obtained a degree sum condition for a graph to be Hamilton-connected. A graph is *Hamilton-connected* if every two vertices are connected by a Hamilton path, i.e., a spanning path.

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Theorem 1.1 (Ore [6]). *Let G be a graph. If $d_G(x) + d_G(y) \geq |G| + 1$ for any two non-adjacent vertices $x, y \in V(G)$, then G is Hamilton-connected.*

Bondy and Chvátal (1976) defined the k-closure of a graph G as the graph obtained from G by recursively joining pairs of non-adjacent vertices with degree sum at least k, until there exists no such a pair. Moreover, they showed that the k-closure of a graph G is well-defined, and obtained the following result.

Theorem 1.2 (Bondy and Chvátal [1]). *A graph G is Hamilton-connected if and only if the $(|G| + 1)$-closure of G is Hamilton-connected.*

Theorem 1.2 implies Theorem 1.1 as a corollary, since the complete graph is Hamilton-connected. Like this, the closure results play an important role in results on degree sum condition. The reader can refer to [1, 2] for details.

In 1986, Gurgel and Wakabayashi investigated S-leaf connectedness as a generalization of Hamilton-connectedness. (In fact, they investigated k-leaf connectedness.) A *leaf* is a vertex of a tree with degree one. For a tree T, let $L(T)$ be the set of leaves of T. For a graph G and a subset S of $V(G)$, G is called *S-leaf-connected* if G has a spanning tree T with $L(T) = S$. A graph G is S-leaf-connected for any $S \subseteq V(G)$ with $|S| = 2$ if and only if G is Hamilton-connected. Hence, we can see that the concept of S-leaf connectedness has a relation to that of Hamilton-connectedness. In this point of view, Gurgel and Wakabayashi generalized Theorem 1.2 as follows.

Theorem 1.3 (Gurgel and Wakabayashi [4]). *Let G be a graph, and let S be a subset of $V(G)$ such that $2 \leq |S| \leq |G| - 1$. Then G is S-leaf-connected if and only if the $(|G| + |S| - 1)$-closure of G is S-leaf-connected.*

Theorem 1.3 yields the following corollary on degree sum condition, since the complete graph is S-leaf-connected for any $S \subseteq V(G)$ with $2 \leq |S| \leq |G| - 1$.

Corollary 1.4. *Let G be a graph, and let S be a subset of $V(G)$ such that $2 \leq |S| \leq |G| - 1$. If $d_G(x) + d_G(y) \geq |G| + |S| - 1$ for any two non-adjacent vertices $x, y \in V(G)$, then G is S-leaf-connected.*

In 2008, Egawa, Matsuda, Yamashita and Yoshimoto weakened the degree sum condition in Corollary 1.4 by adding a necessary condition.

Theorem 1.5 (Egawa et al. [3]). *Let G be a graph, and let S be a subset of $V(G)$ such that $2 \leq |S| \leq |G| - 1$ and $|N_G(S) \setminus S| \geq 2$. Suppose that $G - S'$ is connected for any $S' \subseteq S$. If $d_G(x) + d_G(y) \geq |G| + 1$ for any two non-adjacent vertices $x, y \in V(G) \setminus S$, then G is S-leaf-connected.*

Note that the condition

$$G - S' \text{ is connected for any } S' \subseteq S \qquad (*)$$

is a necessary condition for a graph G to be S-leaf-connected. We will mention it in Section 3. Corollary 1.4 is a corollary of Theorem 1.5, since the degree sum condition in Corollary 1.4 implies the conditions in Theorem 1.5, except for the case $G = K_{k+1}$ and $|S| = k$. (Note that, in the exceptional case, G is S-leaf-connected.)

2 Main theorems

The purpose of this paper is to obtain the bipartite analogies of Theorems 1.3 and 1.5. We denote by $G[A, B]$ a bipartite graph G with partite sets A and B. For a bipartite graph $G[A, B]$, we define

$$\sigma_{1,1}(G) = \min\{d_G(a) + d_G(b) : a \in A, b \in B, ab \notin E(G)\}$$

if $G[A, B]$ is not complete; otherwise $\sigma_{1,1}(G) = \infty$.

It is easy to obtain a $\sigma_{1,1}(G)$ condition for a bipartite graph to be Hamilton-connected. (The reader can refer to [5].)

Theorem 2.1. *Let $G[A, B]$ be a bipartite graph with $1 \le |A| \le |B| \le |A| + 1$. If $|A| = |B|$, then let $u \in A$ and $v \in B$. If $|B| = |A| + 1$, then let $u, v \in B$. If $\sigma_{1,1}(G) \ge |A| + 2$, then G has a Hamilton path connecting u and v.*

We define the k-closure of a bipartite graph G to be the graph obtained from G by recursively joining pairs of non-adjacent vertices contained in different partite sets with degree sum at least k, until no such pair exists. In this paper, we obtain generalizations of Theorem 2.1, which are also the bipartite analogies of Theorems 1.3 and 1.5. We first prove the following bipartite analogy of Theorem 1.3.

Theorem 2.2. *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ such that $|S| \ge 2$. Then G is S-leaf-connected if and only if the $(|A| + |S|)$-closure of G is S-leaf-connected.*

Recall that Theorem 1.3 implies Corollary 1.4. In the same way, we can obtain a result on a degree condition as a corollary of Theorem 2.2. But, this is not obvious because there exists a complete bipartite graph which is not S-leaf connected. Therefore, we need the following proposition. We will also use this proposition in the proof of our theorem.

Proposition 2.3. *Let $G[A, B]$ be a complete bipartite graph and let S be a subset of $V(G)$ with $|S| \ge 2$. Suppose that $G[A, B]$ and S satisfy $1 \le |A \setminus S| \le |B| - 1$ and $1 \le |B \setminus S| \le |A| - 1$. Then G is S-leaf-connected.*

Proof. By the symmetry of A and B, we may assume that $|A \setminus S| \le |B \setminus S|$. First, suppose that $|A \setminus S| = |B \setminus S|$. Then there exists a Hamilton path of $G - S$ because $|A \setminus S| \ge 1$. Note that $S \cap A \ne \emptyset$ and $S \cap B \ne \emptyset$, because $|A \setminus S| \le |B| - 1$ and $|B \setminus S| \le |A| - 1$. Hence we can see that G is S-leaf-connected. Next, suppose that $|A \setminus S| < |B \setminus S|$. Since $|A \setminus S| \ge 1$, there exists a spanning tree T of $G - S$ such that $|L(T)| = |B \setminus S| - |A \setminus S| + 1$ and $L(T) \subseteq B \setminus S$. Since $|B \setminus S| \le |A| - 1$, we obtain $|B \setminus S| - |A \setminus S| + 1 \le |S \cap A|$. Therefore there exists a matching between $L(T)$ and $S \cap A$ covering all vertices of $L(T)$. Hence we can obtain a spanning tree T^* such that $L(T^*) = S$, and so G is S-leaf-connected. □

Note that the condition

$$1 \le |A \setminus S| \le |B| - 1 \text{ and } 1 \le |B \setminus S| \le |A| - 1 \qquad (\sharp)$$

is a necessary condition for a bipartite graph $G[A, B]$ to be S-leaf-connected unless G is a star. We will mention it in Section 3.

By Proposition 2.3, Theorem 2.2 implies the following corollary.

Corollary 2.4. *Let $G[A, B]$ be a bipartite graph with $1 \le |A| \le |B|$, and let S be a subset of $V(G)$ with $|S| \ge 2$. Suppose that $G[A, B]$ and S satisfy the condition ($\sharp$). If $\sigma_{1,1}(G) \ge |A| + |S|$, then G is S-leaf-connected.*

Unfortunately, the lower bound of this degree sum condition is not best possible in the case "$|B \setminus S| \le |A| - 2$ and $|S \cap A| = 1$" and the case "$|S \cap A| \ge 2$". We cannot improve the lower bound of the degree conditions of these cases. In Section 5, we will mention it. Hence, we prove the following theorem, in which the $\sigma_{1,1}$ conditions are best possible.

Theorem 2.5. *Let $G[A, B]$ be a bipartite graph with $1 \le |A| \le |B|$, and let S be a subset of $V(G)$ with $|S| \ge 2$, Suppose that $G[A, B]$ and S satisfy the condition ($\sharp$).*

(i) *If $|S \cap A| = 0$, then we let $\sigma_{1,1}(G) \ge |A| + |S|$.*

(ii) *If $|S \cap A| = 1$ and $|B \setminus S| = |A| - 1$, then we let $\sigma_{1,1}(G) \ge |A| + |S|$.*

(iii) *If $|S \cap A| = 1$ and $|B \setminus S| \le |A| - 2$, then we let $\sigma_{1,1}(G) \ge |A| + |S| - 1$.*

(iv) *If $|S \cap A| \ge 2$, then we let $\sigma_{1,1}(G) \ge |A| + |S| - 1$.*

Then G is S-leaf-connected.

Note that this theorem corresponds to Corollary 1.4. Furthermore, we obtain the following bipartite analogy of Theorem 1.5.

Theorem 2.6. *Let $G[A, B]$ be a bipartite graph with $1 \le |A| \le |B|$, and let S be a subset of $V(G)$ with $|S| \ge 2$. Suppose that $G[A, B]$ and S satisfy the conditions ($*$) and ($\sharp$). If*

$$\sigma_{1,1}(G) > |A| + |S| - 2(|S \cap A| + 1)/3,$$

then G is S-leaf-connected.

Note that the degree sum condition in Theorem 2.6 shows that we can weaken the degree condition in the case $|S \cap A| \ge 3$ or "$|S \cap A| = 1$ and $|B \setminus S| = |A| - 1$" by adding the necessary condition ($*$).

In Section 3, we will discuss the conditions of Theorem 2.5. In Section 4, we will prepare notation and lemmas which are needed in our proofs. In Section 5, we will give a proof of Theorem 2.2. In Section 6, we will give a proof of Theorem 2.5.

3 The conditions of Theorems 2.5 and 2.6

In this section, we discuss the conditions of Theorems 2.5 and 2.6:

(1) The necessary conditions $(*)$ and $(\sharp)$.

(2) The degree sum conditions of Theorems 2.5 and 2.6.

(3) Comparison of the degree sum conditions of Theorems 1.5 and 2.6.

(1) The necessary conditions $(*)$ and $(\sharp)$.

For a bipartite graph $G[A,B]$ and $S \subseteq V(G)$, if G is S-leaf connected, then G and S satisfy $(*)$. Also they satisfy $(\sharp)$ unless G is a star. Let T be a spanning tree of a bipartite graph $G[A,B]$ and let S be the set of leaves of T. We can see that $G - S'$ is connected for any $S' \subseteq S$ because $T - S'$ is connected. Since $T' = T - (S \cap B)$ is a spanning tree in $G - (S \cap B)$ such that $L(T') \subseteq A$, we can obtain $|B \setminus S| = |V(T') \cap B| \leq |V(T') \cap A| - 1 = |A| - 1$. By the same way, we can also obtain $|A \setminus S| \leq |B| - 1$. If $A \setminus S = \emptyset$ or $B \setminus S = \emptyset$, then all the vertices in A or B are leaves. This is possible only if G is a star. Hence $|A \setminus S| \geq 1$ and $|B \setminus S| \geq 1$ hold.

Though the necessity of $(\sharp)$ has an exception, it does not affect the subsequent discussion, including the proof of Theorem 2.2.

Remark 1. *The condition $(*)$ is equivalent to the condition "$G - S$ is connected and $N_G(u) \cap (V(G) \setminus S) \neq \emptyset$ for any $u \in S$". In [3], this expression was used.*

(2) The degree sum conditions (i)–(iv) of Theorems 2.5 and 2.6.

(i) Let k, l and m be integers such that $1 \leq m \leq l \leq k + m - 2$. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = 1$, $|B_1| = k$, $|A_2| = l$ and $|B_2| = m$. Let $G[A,B]$ be a bipartite graph obtained from $G_1 \cup G_2$ by adding all edges in $\{uv : u \in A_2, v \in B_1\}$, where $G_1 \cup G_2$ is a union of G_1 and G_2 (see the left side of Figure 1, where "+" means the join of two graphs). We assume that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, similarly hereinafter. Let $S = B_1$. Then $S \cap A = \emptyset$, $|A| = l + 1 \leq k + m = |B|$, $1 \leq |A \setminus S| = 1 + l \leq k + m - 1 = |B| - 1$, $1 \leq |B \setminus S| = m \leq l = |A| - 1$ and $\sigma_{1,1}(G) = k + l = |A| + |S| - 1$. Since $G - S$ is not connected, G is not S-leaf-connected. Hence the degree sum condition (i) of Theorem 2.5 is best possible.

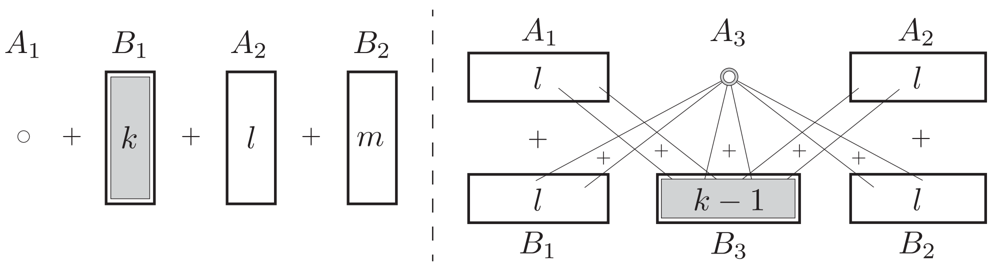

Figure 1: The degree sum conditions (i) and (ii) of Theorem 2.5 are best possible.

(ii) Let k and l be integers such that $k \geq 2$ and $l \geq 1$. For $1 \leq i \leq 3$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = |A_2| = |B_1| = |B_2| = l$, $|A_3| = 1$ and $|B_3| = k-1$. Let $G[A, B]$ be a bipartite graph obtained from $G_1 \cup G_2 \cup G_3$ by adding all edges in $\{uv : u \in A_1 \cup A_2, v \in B_3\} \cup \{uv : u \in A_3, v \in B_1 \cup B_2\}$ (see the right side of Figure 1). Let $S = A_3 \cup B_3$. Then $|S \cap A| = 1$, $|A| = 2l + 1 \leq 2l+k-1 = |B|$, $1 \leq |A \setminus S| = 2l \leq 2l+k-2 = |B|-1$, $1 \leq |B \setminus S| = 2l = |A|-1$ and $\sigma_{1,1}(G) = (l+k-1)+(l+1) = |A|+|S|-1$. Since $G-S$ is not connected, G is not S-leaf-connected. Hence, the degree sum condition (ii) of Theorem 2.5 is best possible.

(iii) Let l, m and n be integers such that $1 \leq m \leq l \leq k + m - 2$. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = 1$, $|B_1| = k - 1$, $|A_2| = l$ and $|B_2| = m$. Let $G[A, B]$ be a bipartite graph obtained from $G_1 \cup G_2$ by adding all edges in $\{uv : u \in A_2, v \in B_1\}$ (see the left side of Figure 2). Let $S = A_1 \cup B_1$. Then $|S| = k$, $|A| = l + 1 \leq k + m - 1 = |B|$, $1 \leq |A \setminus S| = l \leq k + m - 2 = |B| - 1$, $1 \leq |B \setminus S| = m \leq l = |A| - 1$ and $\sigma_{1,1}(G) = k + l - 1 = |A| + |S| - 2$. Since $G - (S \cap B_1)$ is not connected, G is not S-leaf-connected. Hence the degree sum condition (iii) of Theorem 2.5 is best possible.

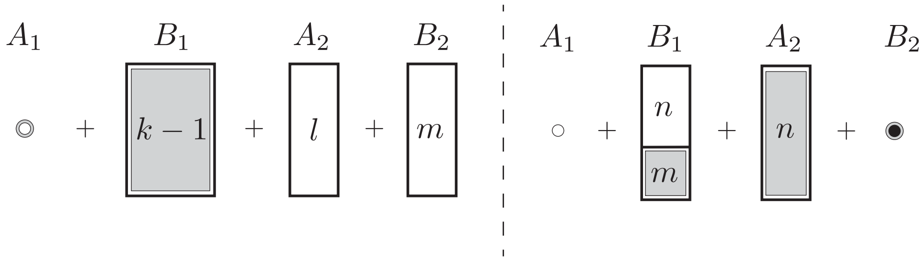

Figure 2: The degree sum conditions (iii) and (iv) of Theorem 2.5 are best possible.

(iv) Let m and n be integers such that $m \geq 0$ and $n \geq 1$. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = 1$, $|B_1| = m + n$, $|A_2| = n$ and $|B_2| = 1$. Let $G[A, B]$ be a bipartite graph obtained from $G_1 \cup G_2$ by adding all edges in $\{uv : u \in A_2, v \in B_1\}$ (see the right side of Figure 2). Let S be a subset of $V(G)$ such that $A_2 \cup B_2 \subseteq S$, $S \cap A_1 = \emptyset$ and $|S \cap B_1| = m$. Then $|A| = n+1 \leq m+n+1 = |B|$, $|A \setminus S| = 1 \leq |B| - 1$, $1 \leq |B \setminus S| = n = |A|-1$ and $\sigma_{1,1}(G) = 2n + m = |A| + |S| - 2$. Since $G - (S \cap A_2)$ is not connected, G is not S-leaf-connected. Hence the degree sum condition (iv) of Theorem 2.5 is best possible.

(v) Let l and m be positive integers. For $0 \leq i \leq 3$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_0| = 3l - 1$, $|B_0| = m$, $|B_1| = l - 1$, $|B_2| = |B_3| = l$ and $A_i = \{a_i\}$ for $1 \leq i \leq 3$. Let $G[A, B]$ be a bipartite graph obtained from $\bigcup_{i=0}^{3} G_i$ by adding two vertices b_1, b_2 and by adding all edges in $\{a_1b_1, a_1b_2, a_2b_2, a_3b_2\}$ and

$$\{uv : u \in A_0, v \in \{b_1, b_2\} \cup B_1 \cup B_2 \cup B_3\} \cup \{uv : u \in \{a_1, a_2, a_3\}, v \in B_0\}$$

(see Figure 3). Let $S = A_0 \cup B_0$. Then, $|A| = 3l + 2 \leq 3l + 1 + m = |B|$, $|A \setminus S| = 3 < |B| - 1$, $|B \setminus S| = 3l + 1 = |A| - 1$, $G - S'$ is connected for any $S' \subseteq S$, and

$$\begin{aligned}\sigma_{1,1}(G) &= l + m + 1 + 3l \\ &= 4l + m + 1 \\ &= |A| + |S \cap A|/3 + |S \cap B| - 2/3 \\ &= |A| + |S| - 2(|S \cap A| + 1)/3.\end{aligned}$$

Suppose that there exists a spanning tree T of G such that $L(T) = S$. Note that $|N_G(b) \cap (V(G) \setminus S)| = 1$ for any $b \in \{b_1\} \cup B_1 \cup B_2 \cup B_3$. This implies that $T' = T - S$ is a spanning tree in $G - S$ such that $\{b_1\} \cup B_1 \cup B_2 \cup B_3 \subseteq L(T')$. Hence, since $N_G(b) \cap S = A_0$ for any $b \in \{b_1\} \cup B_1 \cup B_2 \cup B_3$, we have $|A_0| \geq |\{b_1\} \cup B_1 \cup B_2 \cup B_3|$. This is a contradiction, because $|A_0| = 3l - 1$ and $|\{b_1\} \cup B_1 \cup B_2 \cup B_3| = 3l$. Hence G is not S-leaf connected, and the degree sum condition of Theorem 2.6 is best possible.

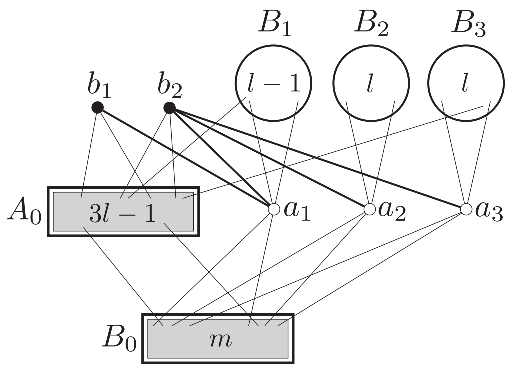

Figure 3: The degree sum condition of Theorem 2.6 is best possible.

(3) Comparison of the degree sum conditions of Theorems 1.5 and 2.6.

Note that the hypothesis in Theorem 1.5 only concerns the degrees of non-adjacent vertices not in S. Therefore, one might expect that the current hypothesis in Theorem 2.6 can be replaced by "$d_G(x) + d_G(y) > |A| + |S| - 2(|S \cap A| + 1)/3$ for any pair of non-adjacent vertices $x \in A \setminus S$ and $y \in B \setminus S$". However, the following example tells us that it is not possible. Let l and m be positive integers. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = m + l$, $|B_1| = m + 2l$, $|A_2| = l + 1$ and $|B_2| = 1$. Let $G[A, B]$ be a bipartite graph obtained from $G_1 \cup G_2$ by adding all edges in $\{vu : u \in A_1, v \in B_2\}$ (see Figure 4). Let S be a subset of $V(G)$ such that $A_2 \subseteq S$, $B_2 \cap S = \emptyset$, $|A_1 \cap S| = l$ and $|B_1 \cap S| = l$. Note that $|A \setminus S| = m \leq m + 2l = |B| - 1$ and $|B \setminus S| = m + l + 1 \leq m + 2l = |A| - 1$. Suppose that there exists a spanning tree T of G such that $L(T) = S$. Then $T' = T - (S \setminus A_2)$ is a spanning tree of $G - (S \setminus A_2)$ such that $|L(T') \cap B_1| \leq |A_1 \cap S| = l$. But, since $|V(T') \cap A_1| = m$ and $|V(T') \cap B_1| = m + l$, we obtain $|L(T') \cap B_1| \geq l + 1$, a contradiction. Hence G is not S-leaf connected. However, G and S satisfy the conditions

"$G - S'$ is connected for any $S' \subseteq S$" and "$d_G(x)+d_G(y) > |A|+|S|-2(|S\cap A|+1)/3$ for any two non-adjacent vertices $x \in A \setminus S$ and $y \in B \setminus S$".

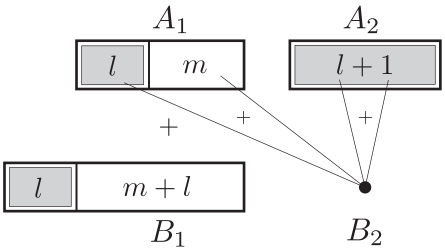

Figure 4: The degree sum condition of Theorem 2.6 cannot be changed into the degree sum condition on $V(G) \setminus S$.

4 Notation and lemmas

In this section, we prepare notation and lemmas which are needed in the proof of our theorems.

Let T be a tree. We denote by $B_R(T)$ the set of vertices of T whose degrees are greater than or equal to three. For $u, v \in V(T)$, there exists the unique path connecting u and v in T, and it is denoted by $P_T[u, v]$. Let T be a rooted tree with root r and $v \in V(T) \setminus \{r\}$. Note that $|N_T(v) \cap V(P_T[v, r])| = 1$. The unique vertex is called the *parent* of v, denoted by $v^{-(T)}$. For a subset U of $V(T) \setminus \{r\}$, let $U^{-(T)} = \{u^{-(T)} : u \in U\}$.

Lemma 4.1. *Let $G[A, B]$ be a bipartite graph and let T be a subtree of G with $V(T) \cap A \neq \emptyset$. Then $|V(T) \cap A| \geq |V(T) \cap B| - |L(T) \cap B| + 1$.*

Proof. Since $V(T) \cap A \neq \emptyset$, there exists a vertex $r \in V(T) \cap A$. We regard T as a rooted tree with root r. Then each $u \in A \cap \big(V(T) \setminus \{r\}\big)$ has the parent. Hence, we can define a function $f : \big(V(T) \setminus \{r\}\big) \cap A \to \big(V(T) \setminus L(T)\big) \cap B$ such that $f(u) = u^{-(T)}$. Then, we can easily see that f is a surjection onto $\big(V(T) \setminus L(T)\big) \cap B$. Therefore $|\big(V(T) \setminus \{r\}\big) \cap A| \geq |\big(V(T) \setminus L(T)\big) \cap B|$, that is, $|V(T) \cap A| \geq |V(T) \cap B| - |L(T) \cap B| + 1$. □

Lemma 4.2. *Let $G[X, Y]$ be a bipartite graph, and let T be a subtree of G with root $u \in X$. Let $v \in Y \setminus V(T)$. Suppose that u, v and T satisfy the following properties:*

(i) $(N_G(u) \cap V(T))^{-(T)} \cap (N_G(v) \cap V(T)) = \emptyset$, *and*

(ii) $(N_G(u) \cap V(T))^{-(T)} \cap (B_R(T) \setminus \{u\}) = \emptyset$.

Then $|N_G(u) \cap V(T)| + |N_G(v) \cap V(T)| \leq |V(T) \cap X| + d_T(u) - 1$.

Proof. By the properties (i) and (ii), we obtain

$$\begin{aligned}
&|N_G(u) \cap V(T)| + |N_G(v) \cap V(T)| \\
&\quad = |(N_G(u) \cap V(T))^{-(T)}| + |N_G(v) \cap V(T)| + d_T(u) - 1 \\
&\quad = |(N_G(u) \cap V(T))^{-(T)} \cup (N_G(v) \cap V(T))| + d_T(u) - 1 \\
&\quad \leq |V(T) \cap X| + d_T(u) - 1.
\end{aligned}$$

□

Lemma 4.3. *Let $G[A, B]$ be a bipartite graph with $1 \leq |A| \leq |B|$, and let S be a subset of $V(G)$ such that $|S| \geq 2$. Suppose that $G[A, B]$ and S satisfy the condition ($\sharp$), and one of the conditions (iii) and (iv) in Theorem 2.5. Then G satisfies the condition ($*$).*

Proof. Suppose that there exists $S' \subseteq S$ such that $G - S'$ is not connected. Let G_1 and G_2 be components of $G - S'$.

Case 1. $|V(G_i)| \geq 2$ for each $i = 1, 2$.

In this case, there exist $u_i \in V(G_i) \cap A$ and $v_i \in V(G_i) \cap B$ for each $i = 1, 2$. Then we have

$$\begin{aligned}
2\sigma_{1,1}(G) &\leq d_G(u_1) + d_G(v_1) + d_G(u_2) + d_G(v_2) \\
&\leq |G_1| + |G_2| + 2|S'| \\
&\leq |G| + |S| \\
&= |A| + (|S \cap B| + |B \setminus S|) + |S| \\
&= |A| + 2|S| + |B \setminus S| - |S \cap A|.
\end{aligned}$$

Since either $|S \cap A| = 1$ and $|B \setminus S| \leq |A| - 2$ or $|S \cap A| \geq 2$, we obtain $|B \setminus S| - |S \cap A| \leq |A| - 3$. Hence $2|A| + 2|S| - 2 \leq 2\sigma_{1,1}(G) \leq 2|A| + 2|S| - 3$, a contradiction.

Case 2. $|V(G_j)| = 1$ for some $j = 1, 2$.

By the symmetry, we may assume that $j = 1$. Suppose first that $V(G_1) \cap A \neq \emptyset$, say $\{u_1\} = V(G_1) \cap A$. Since $|B \setminus S| \geq 1$, we may assume that there exists a vertex $v_2 \in V(G_2) \cap B$. Then

$$\begin{aligned}
\sigma_{1,1}(G) &\leq d_G(u_1) + d_G(v_2) \\
&\leq |S' \cap B| + (|V(G_2) \cap A| + |S' \cap A|) \\
&= |V(G_2) \cap A| + |S'| \\
&\leq |A| - 1 + |S' \setminus A| \\
&\leq |A| + |S| - |S \cap A| - 1,
\end{aligned}$$

which contradicts the $\sigma_{1,1}(G)$ condition, since $|S \cap A| \geq 1$.

Suppose next that $V(G_1) \cap B \neq \emptyset$, say $\{v_1\} = V(G_1) \cap B$. Since $|A \setminus S| \geq 1$, we may assume that there exists a vertex $u_2 \in V(G_2) \cap A$. Therefore, since $|B \setminus S| \leq |A| - 1$, it follows that

$$\begin{aligned}\sigma_{1,1}(G) &\leq d_G(u_2) + d_G(v_1)\\ &\leq |S' \cap A| + (|V(G_2) \cap B| + |S' \cap B|)\\ &\leq |S' \cap A| + \big((|B| - |\{v_1\}| - |S' \cap B|) + |S' \cap B|\big)\\ &\leq |B \setminus S| + |S| - 1\\ &\leq |A| + |S| - 2,\end{aligned}$$

a contradiction. □

Lemma 4.4. *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ such that $|S| \geq 2$. Suppose that G is not S-leaf-connected, and $G + w_1w_2$ is S-leaf-connected for some $w_1 \in A$ and $w_2 \in B$. Let T be a spanning tree in $G + w_1w_2$ such that $L(T) = S$, and let T_1 and T_2 be trees in G obtained from T by deleting w_1w_2. Let w_1 and w_2 be roots of T_1 and T_2, respectively. Moreover, suppose that for $i = 1, 2$, the tuple (w_i, w_{3-i}, T_i) satisfies the property (ii) in Lemma 4.2 as $(u, v, T) = (w_i, w_{3-i}, T_i)$. Then the following statements hold.*

(i) *If G and S satisfy the condition $(*)$, then $d_{T_1}(w_1) \leq 1$ or $d_{T_2}(w_2) \leq 1$ holds.*

(ii) *$d_G(w_1) + d_G(w_2) \leq |A| + |S| - 1$. Especially, the following statements hold.*

  (a) *If "$d_{T_1}(w_1) = 0$ or $d_{T_2}(w_2) = 0$" and "$G[A, B]$ and S satisfy the condition $(*)$", then $d_G(w_1) + d_G(w_2) \leq |A| + |S \cap B| - 1$.*

  (b) *If $d_{T_1}(w_1) = 1$ and $d_{T_2}(w_2) \geq 1$, then $d_G(w_1) + d_G(w_2) \leq |A| + |S \cap B| - |(S \cap B) \cap V(T_1)|$.*

  (c) *If $d_{T_1}(w_1) \geq 2$ and $d_{T_2}(w_2) = 1$, then $d_G(w_1) + d_G(w_2) \leq |A| + |S \cap B| + |(S \cap A) \cap V(T_1)| - 1$.*

(iii) *Suppose that equality holds in the above inequality in the case $d_{T_1}(w_1) = d_{T_2}(w_2) = 1$. And suppose that $N_G(w_1) \cap (V(T_2) \setminus S) = \emptyset$ and $N_G(w_2) \cap (V(T_1) \setminus S) = \emptyset$. Then $N_G(w_1) = (V(T_1) \cup (V(T_2) \cap S)) \cap B$ and $N_G(w_2) = (V(T_2) \cup (V(T_1) \cap S)) \cap A$.*

Proof. Note that $|B \setminus S| \leq |A| - 1$ because $G + w_1w_2$ is S-leaf-connected.

(i) Suppose that G and S satisfy the condition $(*)$, and $d_{T_i}(w_i) \geq 2$ holds for $i = 1, 2$. Then $G - S$ is connected, and hence there exists a spanning tree T' obtained from T_1 and T_2 by adding an edge whose endvertices are not contained in S. Note that $L(T') = S$. Since G is not S-leaf-connected, this is a contradiction. Hence, the statement (i) holds.

(ii) Since G is not S-leaf-connected, the tuple (w_i, w_{3-i}, T_i) satisfies the property (i) in Lemma 4.2 as $(u, v, T) = (w_i, w_{3-i}, T_i)$ for $i = 1, 2$. Hence, by Lemma 4.2, we obtain

$$|N_G(w_1) \cap V(T_1)| + |N_G(w_2) \cap V(T_1)| \leq |V(T_1) \cap A| + d_{T_1}(w_1) - 1 \tag{1}$$

and

$$|N_G(w_1) \cap V(T_2)| + |N_G(w_2) \cap V(T_2)| \leq |V(T_2) \cap B| + d_{T_2}(w_2) - 1. \tag{2}$$

Since G is not S-leaf-connected, if $d_{T_1}(w_1) \geq 2$ then $N_G(w_2) \cap V(T_1) \subseteq (V(T_1) \cap S) \cap A$, and so

$$|N_G(w_1) \cap V(T_1)| + |N_G(w_2) \cap V(T_1)| \leq |V(T_1) \cap B| + |(V(T_1) \cap S) \cap A|, \tag{3}$$

and if $d_{T_2}(w_2) \geq 2$ then

$$|N_G(w_1) \cap V(T_2)| + |N_G(w_2) \cap V(T_2)| \leq |V(T_2) \cap A| + |(V(T_2) \cap S) \cap B|. \tag{4}$$

Case 1. $d_{T_2}(w_2) = 0.$

In this case, note that $w_2 \in S \cap B$. Obviously, we may assume that $d_{T_1}(w_1) \geq 1$. If $d_{T_1}(w_1) = 1$, then by (1) and (2), we obtain

$$d_G(w_1) + d_G(w_2) \leq |V(T_1) \cap A| \leq |A| + |S \cap B| - 1.$$

Hence we may assume that $d_{T_1}(w_1) \geq 2$. Since G is not S-leaf-connected, we have $N_G(w_2) \cap V(T_1) \subseteq L(T_1) \cap A$. This implies that $G[A, B]$ and S do not satisfy the condition $(*)$. Then, by Lemma 4.1, we obtain

$$\begin{aligned} d_G(w_1) + d_G(w_2) &\leq |V(T_1) \cap B| + |L(T_1) \cap A| \\ &\leq |V(T_1) \cap A| + |L(T_1) \cap B| + |L(T_1) \cap A| - 1 \\ &= |A| + |S| - 2. \end{aligned}$$

Hence, the statement (ii) holds in this case.

Case 2. $d_{T_1}(w_1) \leq 1$ and $d_{T_2}(w_2) = 1.$

By Lemma 4.1, $|(V(T_2) \setminus S) \cap B| \leq |V(T_2) \cap A|$. Therefore, by (1) and (2), we obtain

$$\begin{aligned} d_G(w_1) + d_G(w_2) &\leq |V(T_1) \cap A| + |V(T_2) \cap B| + d_{T_1}(w_1) - 1 \\ &= |V(T_1) \cap A| + |(V(T_2) \setminus S) \cap B| + |(V(T_2) \cap S) \cap B| + d_{T_1}(w_1) - 1 \\ &\leq |V(T_1) \cap A| + |V(T_2) \cap A| + |(S \cap B) \cap V(T_2)| + d_{T_1}(w_1) - 1 \\ &= |A| + |S \cap B| - |(S \cap B) \cap V(T_1)| + d_{T_1}(w_1) - 1. \end{aligned}$$

Hence, the statement (ii) holds in this case.

Case 3. $d_{T_1}(w_1) \leq 1$ and $d_{T_2}(w_2) \geq 2.$

By (1) and (4),

$$\begin{aligned} d_G(w_1) + d_G(w_2) &\leq (|V(T_1) \cap A| + d_{T_1}(w_1) - 1) + (|V(T_2) \cap A| + |(V(T_2) \cap S) \cap B|) \\ &= |A| + |S \cap B| - |(S \cap B) \cap V(T_1)| + d_{T_1}(w_1) - 1. \end{aligned}$$

Hence, the statement (ii) holds in this case.

Case 4. $d_{T_1}(w_1) \geq 2$ and $d_{T_2}(w_2) = 1$.

Since $|B \setminus S| \leq |A| - 1$, it follows from (2) and (3) that

$$\begin{aligned} d_G(w_1) + d_G(w_2) &\leq (|V(T_1) \cap B| + |(V(T_1) \cap S) \cap A|) + |V(T_2) \cap B| \\ &= |B| + |(V(T_1) \cap S) \cap A| \\ &\leq (|A| - 1 + |S \cap B|) + |(V(T_1) \cap S) \cap A| \\ &= |A| + |S \cap B| + |(S \cap A) \cap V(T_1)| - 1. \end{aligned}$$

Hence, the statement (ii) holds in this case.

Case 5. $d_{T_1}(w_1) \geq 2$ and $d_{T_2}(w_2) \geq 2$.

By Lemma 4.1, we have $|(V(T_1) \setminus S) \cap B| \leq |V(T_1) \cap A| - 1$. Therefore, by (3) and (4), we obtain

$$\begin{aligned} d_G(w_1) + d_G(w_2) &\leq (|V(T_1) \cap B| + |(V(T_1) \cap S) \cap A|) + (|V(T_2) \cap A| \\ &\qquad + |(V(T_2) \cap S) \cap B|) \\ &= |(V(T_1) \setminus S) \cap B| + |V(T_2) \cap A| + |(V(T_1) \cap S) \cap A| + |S \cap B| \\ &\leq (|V(T_1) \cap A| - 1) + |V(T_2) \cap A| + |(V(T_1) \cap S) \cap A| + |S \cap B| \\ &= |A| + |(V(T_1) \cap S) \cap A| + |S \cap B| - 1 \\ &\leq |A| + |S| - 1. \end{aligned}$$

Hence, the statement (ii) holds in this case.

(iii) We can obtain the statement (iii) by the proof of Case 2 in the statements (ii). □

5 Proof of Theorem 2.2

In this section, we give a proof of Theorem 2.2. In order to prove Theorem 2.2, we have only to show the following proposition.

Proposition 5.1. *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ such that $|S| \geq 2$. Suppose that there exist non-adjacent vertices w_1 and w_2 of G such that $d_G(w_1) + d_G(w_2) \geq |A| + |S|$ and $G + w_1w_2$ is S-leaf-connected. Then G is S-leaf-connected.*

Proof of Proposition 5.1. Suppose that G and S satisfy the assumption of Proposition 5.1. Moreover, suppose that G is not S-leaf-connected. Then there exists a spanning tree T in $G + w_1w_2$ such that $L(T) = S$. By deleting the edge w_1w_2 from T, we obtain two trees T_1 and T_2 with $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$. Choose such two vertices w_1 and w_2 and a spanning tree T so that

$$d_{T_1}(w_1) + d_{T_2}(w_2) \text{ is as large as possible.}$$

We may assume that T_1 and T_2 are rooted trees with roots w_1 and w_2, respectively. By the choice of w_1, w_2 and T, we have $(N_G(w_i) \cap V(T_i))^{-(T_i)} \cap (B_R(T_i) \setminus \{w_i\}) = \emptyset$ for $i = 1, 2$. These imply that G, S, T, T_1, T_2, w_1 and w_2 satisfy the assumption of Lemma 4.4. Hence, $d_G(w_1) + d_G(w_2) \leq |A| + |S| - 1$ by Lemma 4.4 (ii). Since $d_G(w_1) + d_G(w_2) \geq |A| + |S|$, this is a contradiction. □

By Theorem 2.5, we can see that the lower bound of the degree sum condition in Corollary 2.4 is not best possible in the cases (a) $|S \cap A| = 1$ and $|B \setminus S| \leq |A| - 2$ and (b) $|S \cap A| \geq 2$. Therefore, one might expect that we can improve the lower bound of the degree sum condition of Proposition 5.1 in these cases. However, we cannot improve it into $|A| + |S| - 1$ by the existence of the following examples.

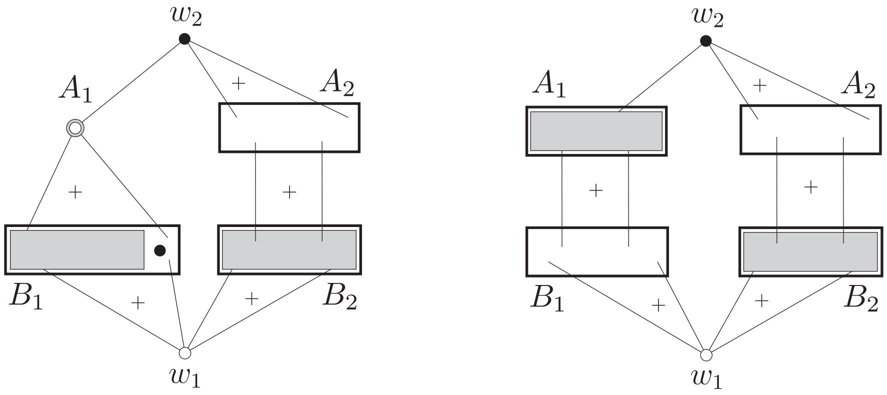

Figure 5: The cases (a) and (b).

(a) $|S \cap A| = 1$ and $|B \setminus S| \leq |A| - 2$ (see the left side of Figure 5).

Let a_2, b_1 and b_2 be positive integers such that $2 \leq a_2 \leq b_2$. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = 1$, $|B_1| = b_1$, $|A_2| = a_2$ and $|B_2| = b_2$. Let $v_1 \in B_1$ and $S = A_1 \cup (B_1 \setminus \{v_1\}) \cup B_2$. Let G be a graph obtained from $G_1 \cup G_2$ by adding two vertices w_1 and w_2 and all edges in $\{w_1 y : y \in B_1 \cup B_2\} \cup \{w_2 x : x \in A_1 \cup A_2\}$, and let $A = A_1 \cup A_2 \cup \{w_1\}$ and $B = B_1 \cup B_2 \cup \{w_2\}$. Then $|B \setminus S| = 2 \leq a_2 = |A| - 2$, $|S \cap A| = 1$ and $d_G(w_1) + d_G(w_2) = (b_1 + b_2) + a_2 + 1 = (a_2 + 2) + (b_1 + b_2) - 1 = |A| + |S| - 1$. Since $b_2 \geq a_2$, $G + w_1 w_2$ is S-leaf connected. But, since $G - S$ is not connected, G is not S-leaf connected.

(b) $|S \cap A| \geq 2$ (see the right side of Figure 5).

Let a_1, a_2, b_1, b_2 be positive integers such that $b_2 \geq a_2$ and $2 \leq a_1 = b_1 \leq a_1 + a_2 - 1$. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph such that $|A_1| = a_1$, $|B_1| = b_1$, $|A_2| = a_2$ and $|B_2| = b_2$. Let $S = A_1 \cup B_2$. Let G be a graph obtained from $G_1 \cup G_2$ by adding two vertices w_1 and w_2 and all edges in $\{w_1 y : y \in B_1 \cup B_2\} \cup \{w_2 x : x \in A_1 \cup A_2\}$. Let $A = A_1 \cup A_2 \cup \{w_1\}$ and $B = B_1 \cup B_2 \cup \{w_2\}$. Then $|B \setminus S| = b_1 + 1 \leq a_1 + a_2 = |A| - 1$, $|S \cap A| = a_1 \geq 2$ and $d_G(w_1) + d_G(w_2) = (b_1 + b_2) + (a_1 + a_2) = (a_1 + a_2 + 1) + (a_1 + b_2) - 1 = |A| + |S| - 1$. Since $b_2 \geq a_2$ and $a_1 = b_1$, $G + w_1 w_2$ is S-leaf connected, but G is not S-leaf connected.

6 Proof of Theorems 2.5 and 2.6

Let $G[A, B]$ be a bipartite graph and let S be a subset of $V(G)$ with $|S| \geq 2$. Suppose that $G[A, B]$ and S satisfy the assumption of Theorem 2.5 or Theorem 2.6. Moreover, suppose that G is not S-leaf-connected. If G and S satisfy the assumption of Theorem 2.5, then we may assume that G satisfies the condition (iii) or (iv) in Theorem 2.5 by Corollary 2.4. Note that $\sigma_{1,1}(G) > |A| + |S| - 2(|S \cap A| + 1)/3$ also holds under the condition (iii) or (iv) in Theorem 2.5. Hence, we have

$$\begin{aligned} \sigma_{1,1}(G) &> |A| + |S| - 2(|S \cap A| + 1)/3 \\ &= |A| + |S \cap A|/3 + |S \cap B| - 2/3. \end{aligned} \tag{5}$$

Note that the condition $(*)$ also holds under the condition (iii) or (iv) in Theorem 2.5 by Lemma 4.3. Since $|A| \leq |B|$ and $|B \setminus S| \leq |A| - 1$, we can see that $S \cap B \neq \emptyset$.

Case 1. There exists a spanning tree T_a such that $B_R(T_a) \cap A \neq \emptyset$, $L(T_a) \cap B = S \cap B$ and $L(T_a) \cap A = (S \cap A) \cup \{a_1\}$, where $a_1 \in A \setminus S$.

In this case, note that

$$d_{T_a}(a_1) = 1 \text{ and } |L(T_a) \cap A| = |S \cap A| + 1.$$

Let $u_0 \in B_R(T_a) \cap A$, and let $v_1, v_2, v_3 \in N_{T_a}(u_0)$. For $1 \leq i \leq 3$, let U_i be a component of $T_a - \{u_0\}$ such that $v_i \in V(U_i)$. We may assume that $a_1 \in V(U_1)$. Choose such a tree T_a, v_1, v_2 and v_3 so that

$$d_{U_1}(v_1) + d_{U_2}(v_2) + d_{U_3}(v_3) \text{ is as large as possible.}$$

We may assume that U_i is a rooted tree with root v_i for $1 \leq i \leq 3$.

Recall that $S \cap B \neq \emptyset$. By the symmetry of U_2 and U_3, we may assume that $(S \cap B) \cap V(U_2) \neq S \cap B$. Let T_a^* be a graph obtained from $T_a - V(U_2)$. Then note that $|(S \cap B) \cap V(T_a^*)| \geq 1$. We may assume that T_a^* is a rooted tree with root a_1. Since G is not S-leaf-connected, the tuple (a_1, v_2, T_a^*) satisfies the property (ii) in Lemma 4.2. By the choice of T_a, the tuple (v_2, a_1, U_2) satisfies the property (ii) in Lemma 4.2. Thus G, S, T_a^*, U_2, a_1 and v_2 satisfy the assumption of Lemma 4.4. Hence, since $|(S \cap B) \cap V(T_a^*)| \geq 1$, it follows from Lemma 4.4 (ii)-(a), (ii)-(b) that

$$\sigma_{1,1}(G) \leq d_G(a_1) + d_G(v_2) \leq |A| + |S \cap B| - 1,$$

which contradicts (5).

Case 2. There exists a spanning tree T_b such that $B_R(T_b) \cap B \neq \emptyset$, $L(T_b) \cap A = S \cap A$ and $L(T_b) \cap B = (S \cap B) \cup \{b_1\}$, where $b_1 \in B \setminus S$.

Then note that

$$d_{T_b}(b_1) = 1 \text{ and } |L(T_b) \cap B| = |S \cap B| + 1.$$

Let $v_0 \in B_R(T_b) \cap B$, and let $u_1, u_2, u_3 \in N_{T_b}(v_0)$. For $1 \leq i \leq 3$, let Q_i be a component of $T_b - \{v_0\}$ such that $u_i \in V(Q_i)$. We may assume that $b_1 \in V(Q_1)$. Choose such a tree T_b, u_1, u_2 and u_3 so that

$$d_{Q_1}(u_1) + d_{Q_2}(u_2) + d_{Q_3}(u_3) \text{ is as large as possible.}$$

We may assume that Q_i is a rooted tree with root u_i for $1 \le i \le 3$ (see the left side of Figure 6).

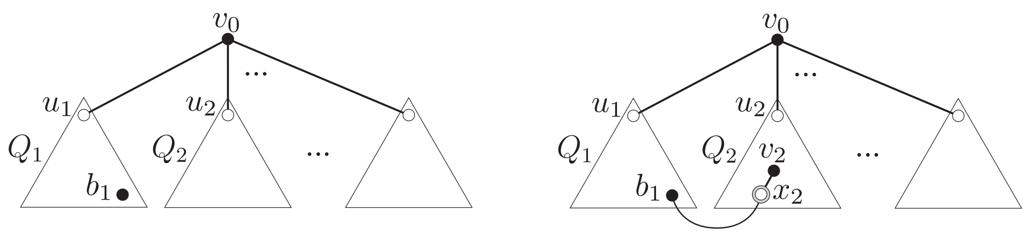

Figure 6: The configuration of Q_1, Q_2 and Q_3.

Let T_b^* be a graph obtained from $T_b - V(Q_2)$. We may assume that T_b^* is a rooted tree with root b_1. Since G is not S-leaf-connected, the tuple (b_1, u_2, T_b^*) satisfies the properties (i) and (ii) in Lemma 4.2. Hence we obtain $b_1u_2 \notin E(G)$ and $|N_G(u_2) \cap V(T_b^*)| + |N_G(b_1) \cap V(T_b^*)| \le |V(T_b^*) \cap B|$ because $d_{T_b^*}(b_1) = 1$. By Lemma 4.1, we have $|V(T_b^*) \cap A| \ge |V(T_b^*) \cap B| - |L(T_b^*) \cap B| + 1$. Therefore

$$|N_G(u_2) \cap V(T_b^*)| + |N_G(b_1) \cap V(T_b^*)| \le |V(T_b^*) \cap A| + |L(T_b^*) \cap B| - 1. \quad (6)$$

Claim 6.1. $|S \cap A| \ge 2$ *and* $|(S \cap A) \cap L(Q_1)| \le (|S \cap A| - 2)/3$.

Proof. Suppose that $|(L(Q_2) \cap S) \cap A| \le |S \cap A|/3$. Since G is not S-leaf-connected and by the choice of T_b, the tuple (u_2, b_1, Q_2) satisfies the properties (i) and (ii) in Lemma 4.2. Hence

$$\begin{aligned} |N_G(u_2) \cap V(Q_2)| + |N_G(b_1) \cap V(Q_2)| &\le |V(Q_2) \cap A| + d_{Q_2}(u_2) - 1 \\ &\le |V(Q_2) \cap A| + |L(Q_2) \cap S| - 1. \quad (7) \end{aligned}$$

Therefore, it follows from the inequalities (6) and (7) that

$$\begin{aligned} \sigma_{1,1}(G) &\le d_G(u_2) + d_G(b_1) \\ &\le (|V(T_b^*) \cap A| + |L(T_b^*) \cap B| - 1) + |V(Q_2) \cap A| + |L(Q_2) \cap S| - 1 \\ &= |A| + |(L(Q_2) \cap S) \cap A| + |L(T_b) \cap B| - 2 \\ &\le |A| + |S \cap A|/3 + |S \cap B| - 1, \end{aligned}$$

which contradicts (5). Hence $|(S \cap A) \cap L(Q_2)| > |S \cap A|/3$. By the same argument, we obtain $|(S \cap A) \cap L(Q_3)| > |S \cap A|/3$. Thus, we deduce $|S \cap A| \ge 2$ and $|(S \cap A) \cap L(Q_1)| \le (|S \cap A| - 2)/3$. ☐

We shaw that $N_G(b_1) \cap (V(Q_2) \setminus S) = \emptyset$. Assume not. Then $d_{Q_2}(u_2) = 1$ because G is not S-leaf connected. Note that G, S, Q_2, T_b^*, u_2 and b_1 satisfy the assumption of Lemma 4.4. Hence, by Lemma 4.4 (ii)-(b),

$$\begin{aligned} \sigma_{1,1}(G) &\le d_G(u_2) + d_G(b_1) \\ &\le |A| + |S \cap B| - |(S \cap B) \cap L(Q_2)|. \end{aligned}$$

By Claim 6.1, we obtain $|S \cap A| \geq 2$, and hence the above inequality contradicts (5).

We now show that $N_G(b_1) \cap (L(Q_2) \cap S) \neq \emptyset$. Assume not. Then it follows from Lemma 4.1 that

$$|N_G(b_1) \cap V(Q_2)| + |N_G(u_2) \cap V(Q_2)| \leq |V(Q_2) \cap B| \leq |V(Q_2) \cap A| + |L(Q_2) \cap B| - 1.$$

Therefore, it follows from this inequality and the inequality (6) that

$$\sigma_{1,1}(G) \leq d_G(b_1) + d_G(u_2) \leq |A| + |S \cap B| - 1,$$

which contradicts (5). Therefore there exists $x_2 \in N_G(b_1) \cap (L(Q_2) \cap S)$. Let $v_2 = x_2^{-(Q_2)}$ (see the right side of Figure 6). Note that $d_{Q_2}(v_2) = 2$ since G is S-leaf-connected. Let R_1 be the tree obtained from Q_1 by adding the vertex x_2 and the edge b_1x_2, and let $R_2 = T_b - V(R_1)$. Then note that $d_{R_2}(v_2) = 1$. We may assume that R_1 and R_2 are rooted trees with roots u_1 and v_2, respectively.

By the construction of R_1 and Claim 6.1, we have

$$|(S \cap A) \cap L(R_1)| = |(S \cap A) \cap L(Q_1)| + |\{x_2\}| \leq (|S \cap A| + 1)/3. \quad (8)$$

Claim 6.2. *The tuples (v_2, u_1, R_2) satisfies the property (ii) in Lemma 4.2.*

Proof. Suppose that there exists $u_R \in (N_G(v_2) \cap V(R_2))^{-(R_2)} \cap B_R(R_2)$. Let R_2^* be the tree obtained from R_2 by adding the edge v_2u_R and by deleting the edge $u_Ru_R^{-(R_2)}$. Let R_{12} be the tree obtained from R_1 and R_2^* by adding the edge v_0u_1. Then R_{12} is a spanning tree such that $L(R_{12}) = S$. This contradicts that G is not S-leaf-connected. Hence $(N_G(v_2) \cap V(R_2))^{-(R_2)} \cap B_R(R_2) = \emptyset$, and so the conclusion holds. □

By the choice of T_b and Claim 6.2, the tuples (u_1, v_2, R_1) and (v_2, u_1, R_2) satisfy the property (ii) in Lemma 4.2. Hence, we can see that G, S, R_1, R_2, u_1 and v_2 satisfy the assumption of Lemma 4.4. Note that $d_{R_1}(u_1) \geq 1$ and $d_{R_2}(v_2) = 1$.

First, suppose that $d_{R_1}(u_1) = 1$ and $d_{R_2}(v_2) = 1$. By Lemma 4.4 (ii)-(b),

$$\sigma_{1,1}(G) \leq d_G(u_1) + d_G(v_2) \leq |A| + |S \cap B| - |(S \cap B) \cap V(R_1)|.$$

By Claim 6.1, this inequality contradicts (5).

Next, suppose that $d_{R_1}(u_1) \geq 2$ and $d_{R_2}(v_2) = 1$. By the inequality (8) and Lemma 4.4 (ii)-(c), we obtain

$$\begin{aligned}\sigma_{1,1}(G) &\leq d_G(u_1) + d_G(v_2) \\ &\leq |A| + |S \cap B| + |(S \cap A) \cap L(R_1)| - 1 \\ &\leq |A| + |S \cap B| + (|S \cap A| - 2)/3,\end{aligned}$$

which contradicts (5).

Case 3. Otherwise (neither Case 1 nor Case 2 holds).

By Proposition 2.3, we may assume that G is an edge-maximal counterexample, that is, $G + uv$ is S-leaf-connected for any non-adjacent vertices $u \in A$ and $v \in B$.

Let w_1 and w_2 be non-adjacent vertices such that $w_1 \in A$ and $w_2 \in B$, and let T be a spanning tree in $G + w_1w_2$ such that $L(T) = S$. By deleting the edge w_1w_2 from T, we obtain two trees T_1 and T_2 with $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$. Choose such two vertices w_1 and w_2 and a spanning tree T so that

$$d_{T_1}(w_1) + d_{T_2}(w_2) \text{ is as large as possible.}$$

We may assume that T_1 and T_2 are rooted trees with roots w_1 and w_2, respectively. By the choice of w_1, w_2 and T, we have $(N_G(w_i) \cap V(T_i))^{-(T_i)} \cap (B_R(T_i) \setminus \{w_i\}) = \emptyset$ for $i = 1, 2$. These imply that G, S, T, T_1, T_2, w_1 and w_2 satisfy the assumption of Lemma 4.4. Since G and S satisfy the condition $(*)$, it follows from Lemma 4.4 (i), (ii)-(a) and (5) that either "$d_{T_1}(w_1) = 1$ and $d_{T_2}(w_2) \geq 2$", "$d_{T_1}(w_1) \geq 2$ and $d_{T_2}(w_2) = 1$" or "$d_{T_1}(w_1) = 1$ and $d_{T_2}(w_2) = 1$" holds.

Claim 6.3. $d_{T_1}(w_1) = 1$ *and* $d_{T_2}(w_2) = 1$

Proof. Suppose that $d_{T_1}(w_1) = 1$ and $d_{T_2}(w_2) \geq 2$. Since G satisfys the condition $(*)$, $G - S$ is connected. Hence there exist vertices $z_1 \in V(T_1) \setminus S$ and $z_2 \in V(T_2) \setminus S$ such that $z_1z_2 \in E(G)$. Note that $z_1 \neq w_1$ since G is not S-leaf-connected. Let T_a be the tree obtained from T_1 and T_2 by adding the edge z_1z_2. Then note that $B_R(T_a) \cap A \cap \{z_1, z_2\} \neq \emptyset$ and $L(T_a) = S \cup \{w_1\}$. Hence T_a satisfies the assumption of Case 1, a contradiction. By the same way, we can obtain a contradiction in the case $d_{T_1}(w_1) \geq 2$ and $d_{T_2}(w_2) = 1$. □

Since G does not satisfy either the assumption of Case 1 or that of Case 2, we can obtain the following claim.

Claim 6.4. $N_G(w_1) \cap (V(T_2) \setminus S) = \emptyset$ *and* $N_G(w_2) \cap (V(T_1) \setminus S) = \emptyset$.

By (5), Lemma 4.4 (ii)-(b) and Claim 6.3,

$$\begin{aligned} |A| + |S \cap B| + |S \cap A|/3 - 2/3 &< \sigma_{1,1}(G) \\ &\leq d_G(w_1) + d_G(w_2) \\ &\leq |A| + |S \cap B| - |(S \cap B) \cap V(T_1)|. \end{aligned}$$

This implies that $(V(T_1) \cap S) \cap B = \emptyset$ and $|S \cap A| \leq 1$. Since $V(T_1) \cap S \neq \emptyset$, we have $|S \cap A| \geq |(V(T_1) \cap S) \cap A| = |V(T_1) \cap S| \geq 1$. Hence, equalities hold in the above inequalities, and we obtain that $V(T_1) \cap S \subseteq A$, $V(T_2) \cap S \subseteq B$ and $|S \cap A| = 1$. Moreover, by Lemma 4.4 (iii) and Claim 6.4, $N_G(w_1) = (V(T_1) \cup (V(T_2) \cap S)) \cap B$ and $N_G(w_2) = (V(T_2) \cup (V(T_1) \cap S)) \cap A$.

Since $G - S$ is connected, there exist $z_1 \in V(T_1) \setminus S$ and $z_2 \in V(T_2) \setminus S$ such that $z_1z_2 \in E(G)$. For $i = 1, 2$, let z_i^+ be a vertex of $V(T_i)$ such that $(z_i^+)^{-(T_i)} = z_i$.

If $z_1 \in A$ and $z_2 \in B$ (see the left side of Figure 7), then $T' = T_1 \cup T_2 + z_1z_2 + w_1z_1^+ + w_2z_2^+ - z_1z_1^+ - z_2z_2^+$ is a spanning tree of G which satisfies $L(T') = S$, a contradiction. Hence $z_1 \in B$ and $z_2 \in A$ (see the right side of Figure 7).

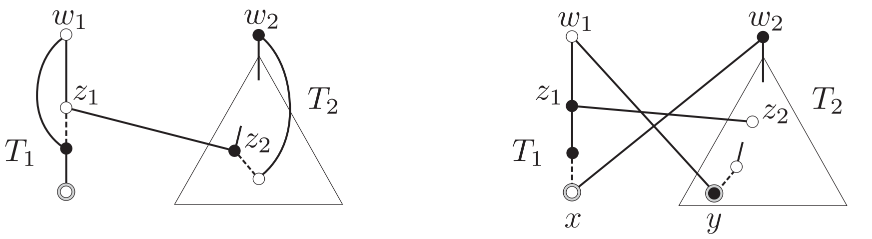

Figure 7: There exist $z_1 \in V(T_1) \setminus S$ and $z_2 \in V(T_2) \setminus S$ such that $z_1z_2 \in E(G)$.

Let $x \in (V(T_1) \cap S) \cap A$ and $y \in (V(T_2) \cap S) \cap B$. Let T_3 be a tree obtained from $T_2 - \{y\}$ by adding the vertex x and the edge xw_2, and let T_4 be a tree obtained from $T_1 - \{x\}$ by adding the vertex y and the edge yw_1. Then T_3 and T_4 are trees such that $(L(T_3) \cup L(T_4)) \setminus \{y^{-(T_2)}, x^{-(T_1)}\} = S$, $V(T_3) \cup V(T_4) = V(G)$ and $V(T_3) \cap V(T_4) = \emptyset$. Since G is not S-leaf-connected, we have $d_{T_3}(y^{-(T_2)}) = 1$ and $d_{T_4}(x^{-(T_1)}) = 1$. Hence, we can see that the tuple $(T_3, T_4, y^{-(T_2)}, x^{-(T_1)})$ plays the same role as the tuple (T_1, T_2, w_1, w_2). Since $z_1 \in V(T_4) \cap B$ and $z_2 \in V(T_3) \cap A$, we can obtain a contradiction by the same argument in the case $z_1 \in A$ and $z_2 \in B$.

This completes the proof of Theorems 2.5 and 2.6. □

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