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Bad drawings of small complete graphs

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Abstract

We show that for K_5 (respectively, $K_{3,3}$) there is a drawing with i independent crossings, and no pair of independent edges cross more than once, provided i is odd with $1 \leq i \leq 15$ (respectively, $1 \leq i \leq 17$). Furthermore, using the deleted product cohomology, we show that for K_5 and $K_{3,3}$, if A is any set of pairs of independent edges, and A has odd cardinality, then there is a drawing in the plane for which each element in A cross an odd number of times, while each pair of independent edges not in A cross an even number of times. For K_6 we show that there is a drawing with i independent crossings, and no pair of independent edges cross more than once, if and only if $3 \leq i \leq 40$.

1 Introduction

We consider planar drawings of finite simple graphs in which vertices are represented as points, the edges are smooth arcs, joining distinct vertices, that do not self-intersect or pass through any vertex, and when distinct edges meet they only do so at common vertex endpoints, or at transverse crossings and in the latter case only have finitely many such crossings. Recall that two edges are said to be *independent* if they are distinct and not adjacent, and a drawing is *good* if no pair of adjacent edges cross one another, and each pair of independent edges cross at most once [15]. A crossing of two independent edges is called an *independent crossing.*

Definition 1.1 We say that a graph drawing is *bad* if it is not good, but that it is *tolerable* if no pair of independent edges cross more than once.

Note that in tolerable drawings, pairs of dependent edges are allowed to cross any number of times. Obviously, all good drawings are tolerable and as we will see, there are more tolerable drawings than good ones. For example, it is easy to see that in any good drawing of K_4, there is at most one crossing; it follows that good drawings of

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K_n have at most $\binom{n}{4}$ crossings [3] and this upper bound is attained for a straight-line drawing with the vertices at the vertices of a regular n-gon. But there are tolerable drawings of K_4 in which all 3 pairs of independent edges cross; see Figure 1.

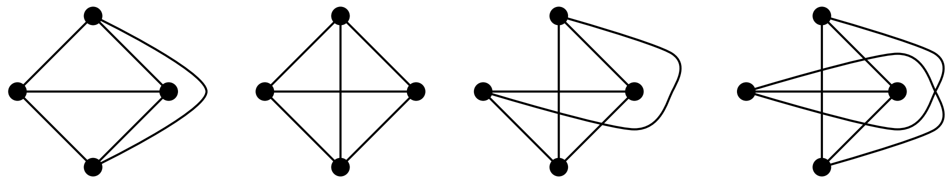

Figure 1: Tolerable drawings of K_4 with zero to three independent crossings respectively

As another example, consider K_5, which has 15 pairs of independent edges. We will show below in Section 4, there is a tolerable drawing of K_5 in which all 15 pairs of independent edges cross exactly once. To set this in context, note that K_5 cannot be drawn as a *thrackle* in the plane [7]; that is, it cannot be drawn in the plane so that each pair of independent edges cross exactly once, and adjacent edges do not cross. Moreover, K_5 cannot be drawn as a *generalized thrackle* in the plane [8]; that is, it cannot be drawn in the plane so that each pair of independent edges cross an odd number of times, and adjacent edges cross an even number of times. Also, K_5 cannot be drawn as a *superthrackle* in the plane [4]; that is, it cannot be drawn in the plane so that each pair of edges (independent or not) cross exactly once.

Good drawings of small complete bipartite graphs have also been considered in the literature; see for example [9].

In this paper we study bad drawings and tolerable drawings of small complete graphs, and small complete bipartite graphs. We present two kinds of results. The first kind concerns the existence of tolerable drawings having a certain number of independent crossings.

Theorem 1.1

(a) For each odd integer i with $1 \leq i \leq 15$, there is a tolerable drawing of K_5 with i independent crossings.

(b) For each odd integer i with $1 \leq i \leq 17$, there is a tolerable drawing of $K_{3,3}$ with i independent crossings.

(c) For each integer i with $3 \leq i \leq 40$, there is a tolerable drawing of K_6 with i independent crossings.

The existence of drawings described in the above theorem is presented explicitly in Section 4. Conversely, we will see below that for each of these graphs, there are no tolerable drawings having a number of independent crossings other than those indicated in Theorem 1.1. In order to explain this in greater detail, we require some terminology.

Definition 1.2 For a graph G, let PG denote the set of pairs of independent edges of G. We will say that a subset A of PG is *2-realisable* if there is a drawing of G in the plane for which each element in A cross an odd number of times, while each element of $PG\backslash A$ cross an even number of times. Further, we say that such a drawing 2-*realises* A.

To avoid any confusion, let us emphasise that in the above definition we impose no restrictions on the numbers of crossings of pairs of adjacent edges. (Specifications for sets of arbitrary pairs of edges, whether adjacent or not, have been previously considered; see for example [19, 21]). For a given graph G and given subset A of PG, it is natural to ask whether A is 2-realisable, and if so, is there a tolerable, or even good, drawing that 2-realises A. For example, for K_4, there are 6 edges and the set PK_4 of independent pairs has three elements. So there are $2^3 = 8$ possible subsets of PK_4. However, exploiting the S_4 symmetry, one easily sees that up to a relabelling of the vertices, there are just 4 essentially different subsets of PK_4, having 0, 1, 2, 3 elements respectively. As shown in Figure 1, these subsets are all 2-realisable. In fact, the subsets having 0 or 1 element have a good drawing, while the subsets having 2 or 3 elements have tolerable drawings, but no good ones.

If G is $K_5, K_{3,3}$ or K_6, and $A \subset PG$ is 2-realisable, we will see that the cardinality of A satisfies the corresponding inequality in Theorem 1.1. For K_5 and $K_{3,3}$, this result is immediate from Kleitman's Theorem (see Section 6). We present the stronger statement (see Theorem 6.2):

Theorem 1.2 *If G is K_5 or $K_{3,3}$ and $A \subset PG$, then A is 2-realisable if and only if its cardinality satisfies the corresponding condition of Theorem 1.1(a) or (b).*

A similar result does not hold for K_6; for example, as we explain in Section 7, there are 3-element subsets of PK_6 that are 2-realisable and there are 3-element subsets of PK_6 that are not 2-realisable. However, one does have:

Theorem 1.3 *If $A \subset PK_6$ is 2-realisable, then the cardinality of A satisfies the condition in Theorem 1.1(c).*

Our proofs of Theorems 1.2 and 1.3 use the deleted product cohomology machinery. We recall this briefly in Section 5.

Notice for $G = K_5$ and $G = K_{3,3}$, the above results do not claim that tolerable drawings exist for every 2-realisable set $A \subset PG$. Indeed, we strongly suspect that this is not the case; see Remark 3.4. In Section 3 we give examples of graphs G and subsets $A \subset PG$ which are 2-realisable but not 2-realised by any tolerable drawing.

In order to give the reader further familiarity with the concepts, we begin in the following section with a complete account of the graph $K_{2,3}$, where the situation is sufficiently simply that calculations can be readily done by hand. We show that every subset of $PK_{2,3}$ is 2-realised by a tolerable drawing.

Remark 1.1 Since submitting this paper we learnt of a preprint of Jan Kynčl which, while its approach is different, has results having nontrivial intersection with those of this paper; the preprint [20] was posted to the arXiv in 2016. In particular, the notion of 2-realisable sets was introduced in [20]; there, 2-realisability is called *independent* $\mathbb{Z}_2$*-realizability.* The K_5 part of the above Theorem 1.2 is an immediate consequence of [20, Theorem 3]. We make a further comment on Kynčl's work in Remark 7.1 below.

2 The complete bipartite graph $K_{2,3}$

The graph $K_{2,3}$ has 6 edges. For each edge, there are two independent edges, so this gives 6 pairs of independent edges. Thus $PK_{2,3}$ has $2^6 = 64$ subsets. For $G = K_{2,3}$, the symmetry group $S_3 \times \mathbb{Z}_2$ has order 12. It acts naturally on $PK_{2,3}$ and one can compute the number of orbits using the "lemma that is not Burnside's" [24]. It is clear that by taking complements, the number of orbits of subsets of $PK_{2,3}$ of cardinality i is the same as the number of orbits of subsets of cardinality $6 - i$. Furthermore, the group $S_3 \times \mathbb{Z}_2$ clearly acts transitively on the set of subsets having just one element. So it suffices to consider the action of $S_3 \times \mathbb{Z}_2$ on the set of subsets of $PK_{2,3}$ of cardinality 2, and on the set of subsets of cardinality 3. The number of fixed points for the action is given in Table 1; the vertices in the two parts of $K_{2,3}$ are labelled 1, 2, 3 and 4, 5 respectively. In the first column we have typical elements of conjugacy classes; in the second, the number of elements in the conjugacy class. From this we have that for subsets of cardinality 2 there are $36/12 = 3$ orbits; representatives for these orbits are as follows:

$$\{(14)(25), (14)(35)\}, \{(14)(25), (24)(35)\}, \{(14)(25)\ (15)(24)\}.$$

Similarly, for subsets of cardinality 3 there are also $36/12 = 3$ orbits, and representatives for these orbits are as follows:

$$\{(14)(25), (14)(35), (24)(35)\}, \{(14)(25), (15)(24), (24)(35)\},$$
$$\{(14)(25), (14)(35), (15)(34)\}.$$

Hence, up to relabelling, there are 13 essentially distinct subsets A of $PK_{2,3}$; there are 1, 1, 3, 3, 3, 1, 1 such subsets having 0, 1, 2, 3, 4, 5, 6 elements respectively. Each of these subsets is 2-realised by a tolerable drawing, as shown in Figure 2; note that the cardinalities of A in this Figure are not in increasing order. The first 5 of these are good drawings. The other subsets cannot be 2-realised by good drawings. Indeed, there are only 6 good drawings of $K_{2,3}$ up to isomorphism [17], and two of these (with 3 crossings) correspond to the same subset of $PK_{2,3}$.

Remark 2.1 Notice that in all the drawings in Figure 2, one can draw a vertical edge between the two left-most vertices without crossing any other edge. The resulting graph is $K_{1,1,3}$ and the new edge is not part of any independent pair. Thus Figure 2 shows that every subset of $PK_{1,1,3}$ can be 2-realised by a tolerable drawing.

typical element	# elements	# fixed subsets of card. 2	# fixed subsets of card. 3
id	1	$\binom{6}{2} = 15$	$\binom{6}{3} = 20$
(12)	3	3	2
(123)	2	0	2
(45)	1	3	0
(12)(45)	3	3	2
(123)(45)	2	0	0
	12	36	36

Table 1.

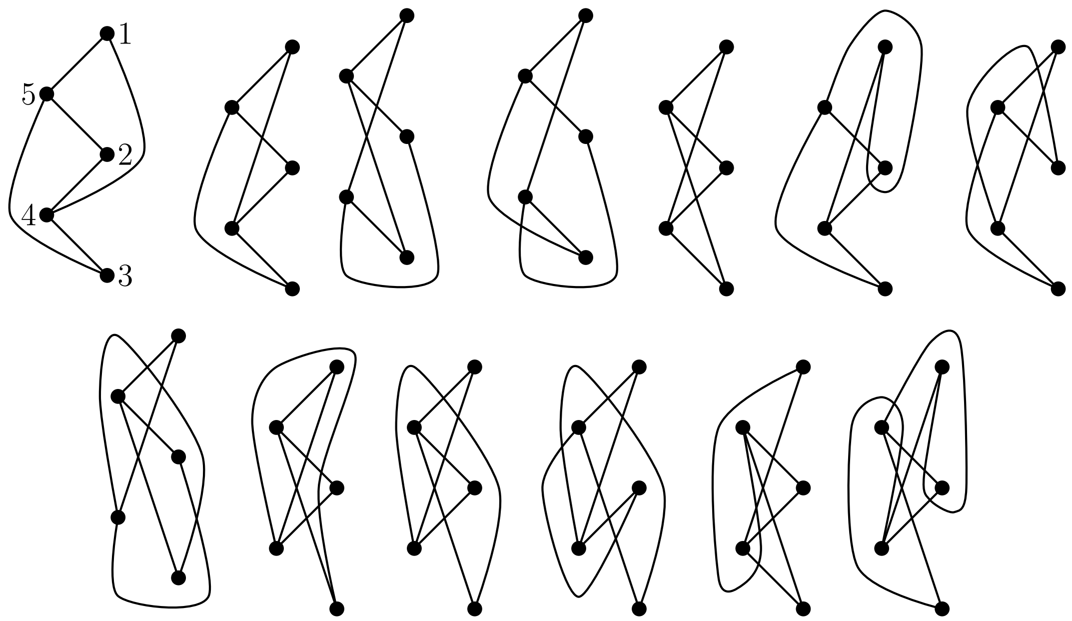

Figure 2: Tolerable drawings of $K_{2,3}$

3 Intolerably bad examples

In this section, we exhibit a graph G and a subset of PG that is 2-realisable but cannot be 2-realised by a tolerable drawing of G. Our graph G will be a disjoint union of N edges. Since G has no adjacent edges, we have full control on its crossing pattern, but having constructed such an example we will be able to construct a connected example, as explained in Remark 3.1 below.

Lemma 3.1 *If G is a disjoint union of edges, then any subset A of the set PG of pairs of its independent edges is* 2*-realisable.*

Proof: First draw the edges of G as disjoint line segments on the plane. Next, we work through the elements of A successively and for every pair of edges $\{e, f\} \in A$, we join an interior point of e to one of the endpoints v of f by a simple curve γ whose interior does not meet any edge or vertex and γ only meets previously drawn curves at transverse crossings and that there are at most a finite number of such crossings, and furthermore, no curves starting on a common edge meet at all. Having drawn a curve γ for each element of A, then replace each curve γ by a curve passing along the boundary of a thin strip centred on γ and a small semicircle centred at v, as in Figure 3. □

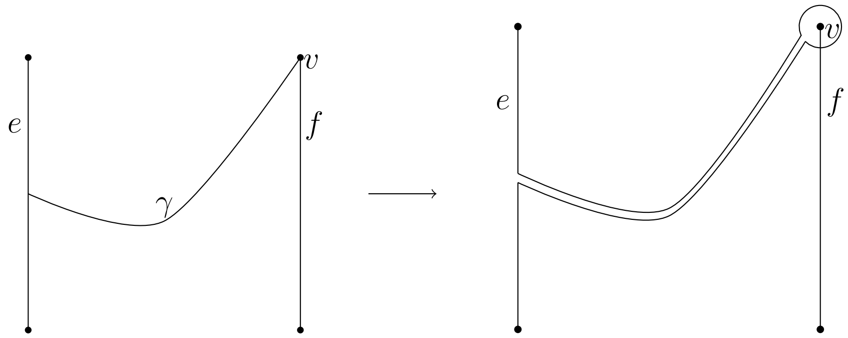

Figure 3: Construction of 2-realisable drawings

Remark 3.1 Suppose we have a graph G which is a disjoint union of N edges and a subset $A \subset PG$ that cannot be 2-realised by a tolerable drawing. Take a cycle G' of length $2N$ with G being its subgraph of even labelled edges (in some cyclic order). Consider an arbitrary 2-realisation of A and add to it odd labelled edges of G' arbitrarily to get a drawing of G'. Now define the subset A' of pairs of independent edges of G' from the drawing (a pair belongs to A' if the edges cross an odd number of times). Then A' is 2-realisable by design, but the drawing is not tolerable, because if it were, then the drawing of G corresponding to A would have also been tolerable.

For $n, m \in \mathbb{N}$, denote $I = \{1, 2, \ldots, n\}$ and let S be a set, of cardinality m, of subsets of I. We define G to be the union of $N = n + m$ pairwise disjoint edges which we label e_i, $i \in I$, e_s, $s \in S$. We then define A to be the set of pairs $\{i, s\}$, where $i \in s$. In a tolerable drawing, the first n edges must be pairwise disjoint, the last m edges must be pairwise disjoint, and then one of the first n edges e_i crosses one of the last m edges e_s if and only if $i \in s$.

Lemma 3.2 *In the above notation, for $n = 5$ and $S = \{I\} \cup \{(i, j) : 1 \leq i < j \leq n\}$, there is no tolerable drawing that 2-realises A.*

Proof: Suppose there exists a tolerable drawing that 2-realises A. The graph G here is the disjoint union of 16 edges. Up to isotopy, the drawing of the edges $e_1, \ldots, e_5, e_I$ looks like the one in Figure 4, where we relabel the edges $e_1, e_2, \ldots, e_5$ if necessary in such a way that the edge e_I crosses them in the increasing (or decreasing) order of labels.

We can ignore the edges $e_{k,k+1}$, $k = 1, 2, 3, 4$, as they can be inserted at any stage. This leaves us with the six edges, $e_{1,3}, e_{1,4}, e_{1,5}, e_{2,4}, e_{2,5}$ and $e_{3,5}$. Now for every

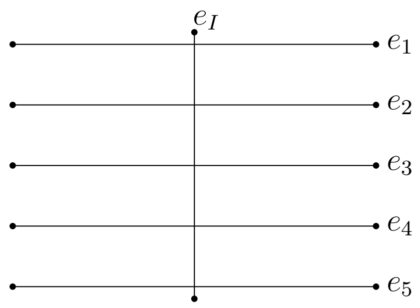

Figure 4: Drawing of the edges $e_1, \ldots, e_5, e_I$

$s = \{i, j\} \in S$, $i \neq j$, the edge e_s crosses the edges e_i and e_j, once each, and does not cross any other edges of G. Referring to Figure 4 we say that the edge e_s is *left* (respectively *right*) if the crossings of both e_i and e_j with e_s occur to the left (respectively to the right) of their crossings with e_I. Otherwise call the edge e_s *mixed*. Consider $e_{1,5}$. We need only consider the two possibilities that $e_{1,5}$ is either mixed or left since we can use the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by the reflection about the line containing e_I and the reflection (followed by relabelling the e_i's in the reverse order) about the line containing e_3. If $e_{1,5}$ is mixed, then $e_{2,4}$ must be right (up to symmetry), and then $e_{1,3}$ is left, and then $e_{2,5}$ is right, and then $e_{1,4}$ and $e_{3,5}$ cross. If $e_{1,5}$ is left, then there could be no more than two other left edges, and up to reflection, we can have either $e_{1,3}, e_{1,4}$, or $e_{1,3}, e_{3,5}$, or $e_{1,4}, e_{2,4}$. It is then easy to see that in each of the three cases, we get unwanted crossings one way or another after adding the remaining three edges. If apart from $e_{1,5}$, there is only one or no left edge, a contradiction is also readily found. □

Remark 3.2 Note that Lemmas 3.1 and 3.2 give an example of a graph G and a set $A \subset PG$ that is 2-realisable but for which there is no tolerable drawing. As pointed out by the referee, similar examples are known in the language of string graphs ("there exist graphs which are not string graphs").

Remark 3.3 In contrast to the above lemma, when $n = 4$, a tolerable drawing exists even when we take for S the set of all subsets of $\{1, 2, 3, 4\}$ as shown in Figure 5.

Remark 3.4 We have seen above that for G equal to K_4, $K_{2,3}$ or $K_{1,1,3}$, every 2-realisable subset $A \subset PG$ can be 2-realised by a tolerable drawing. Consider the drawing of K_5 in Figure 6. It is a 2-realisation of the set $A = \{(12)(34), (13)(24), (14)(23)\}$, and it fails to be tolerable only because edge (13) crosses edge (45) twice (in opposite directions); the other edges adjacent to vertex 5 do not cross independent edges of the K_4 having vertices $1, 2, 3, 4$, and the drawing of this K_4 is tolerable. We suspect that there is no tolerable drawing of K_5 that 2-realises the set A, but we have not been able to prove this. The following argument for which we are thankful to one of the referees proves that the problem is finite, for any graph G and any subset

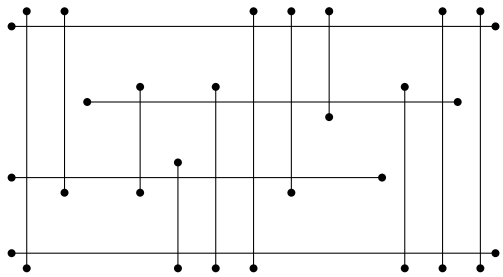

Figure 5: Tolerable drawing for $n = 4$ and S the power set of $\{1, 2, 3, 4\}$, with the subsets of cardinality 0 and 1 omitted

$A \subset PG$, but the resulting number of cases, even for such a small graph, is probably beyond the reach of a computer or by-hand proof. Given a tolerable drawing which 2-realises a subset $A \subset PG$, the pairs of adjacent edges may apriori cross any number of times. To control this number we temporarily replace every crossing of a pair of edges in A with a dummy vertex. Then by [30, Theorem 3.2], we can redraw the resulting graph G' in such a way that the set of pairs of crossing edges of G' does not increase and that the number of crossings is bounded by a number $N(G')$ which depends (exponentially) only on the number of edges of G'; moreover, the analysis of the proof of [30, Theorem 3.2] shows that the process of redrawing does not affect the rotation diagrams in the vertices. Hence turning the dummy vertices back into crossings we obtain a tolerable drawing of G which 2-realises A and which has no more than $N(G')$ crossings between the pairs of adjacent edges in total.

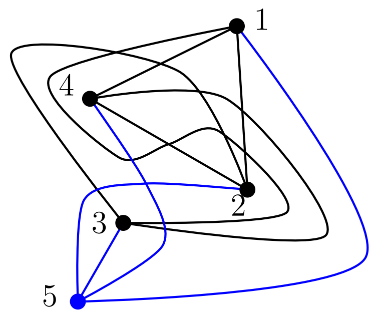

Figure 6: A bad drawing of K_5

Remark 3.5 As we remarked in the introduction, for K_4 there are 3-element sets $A \subset PK_4$ that are 2-realisable but for none of these sets A is there a good drawing. A natural open question is as follows: does there exist a graph G and an integer i for which there exist 2-realisable subsets of PG of cardinality i but for none of these is there a tolerable drawing? One might say that such an integer i is *intolerable* for G. The results of this paper show that no such phenomenon exists for any of $K_5, K_{3,3}$ or K_6. Furthermore, there can be no intolerable integers for graphs that are disjoint union of edges. To see this, consider all our edges drawn as straight line segments passing through a single point, then replace each of them by a nearby parallel segment, so that all the crossing points are pairwise distinct, and then remove the crossings one by one by shortening the segments until we get the desired number of crossings.

4 Tolerable drawings for the graphs $K_5, K_{3,3}$ and K_6

The complete graph K_5 has 10 edges and PK_5 has 15 elements. (We note in passing that for the complete graph K_n, the set PK_n is the edge set of the Kneser graph $KG_{n,2}$). Up to relabelling, only 5 of these subsets of PK_5 can be 2-realised by good drawings; see [26, Figure 3.1] or [29, Figure 1.7]. They each have 1, 3 or 5 crossings. The first 3 drawings of Figure 7 are good, and have 1, 3 and 5 crossings respectively. The remaining drawings of Figure 7 are tolerable and have 7, 9, 11, 13 and 15 independent crossings respectively.

As far as the 2-realisable subsets of PK_5 are concerned, note that PK_5 has 2^{15} subsets, and that the group acting here is the symmetry group S_5 of K_5. There is no difficulty in conducting the kind of symmetry reduction we employed for $K_{2,3}$ is Section 2. For example, one finds that up to relabelling, there are 9 essentially distinct subsets of PK_5 of cardinality 3. They are:

$$\begin{gathered}\{(12)(34),\ (13)(24),\ (14)(23)\},\quad \{(12)(34),\ (13)(24),\ (12)(35)\},\\ \{(12)(35),\ (13)(24),\ (14)(23)\},\quad \{(12)(34),\ (12)(35),\ (12)(45)\},\\ \{(12)(34),\ (12)(35),\ (15)(24)\},\quad \{(12)(34),\ (13)(25),\ (14)(25)\},\\ \{(12)(34),\ (15)(23),\ (14)(25)\},\quad \{(12)(34),\ (15)(34),\ (12)(45)\},\\ \{(12)(34),\ (13)(45),\ (15)(24)\}.\end{gathered}$$

By Theorem 1.2, each of these subsets is 2-realisable. However, the difficulty is in determining whether or not a given set can be 2-realised by a tolerable drawing. For K_5, we have not been able to resolve this problem, even for subsets of PK_5 of cardinality 3; see Remark 3.4 above, in which the subset A in Figure 6 corresponds to the first subset above.

The graph $K_{3,3}$ has 9 edges and 18 pairs of independent edges. Harborth [17] determined that there are 102 good drawings of $K_{3,3}$ up to isomorphism; there are 1, 9, 33, 48, and 11 good drawings with 1, 3, 5, 7, and 9 crossings, respectively. In Figure 8, the first 5 drawings are good. The remaining drawings are tolerable and have 11, 13, 15 and 17 independent crossings, respectively.

The graph K_6 has 15 edges and 45 pairs of independent edges. It is known that K_6 only has good drawings with i crossings for $3 \leq i \leq 12$ and for $i = 15$; see [26, 14]. Figure 9 gives examples of such good drawings. Figures 10 through 12 give tolerable drawings having i independent crossings for $i = 13, 14$ and $16 \leq i \leq 40$. Note that in the drawings in Figures 11 and 12, the idea is that one extends the red lines out and connects them up to a 6th vertex (at infinity, if one likes). It is perhaps easiest to keep track of the independent crossings in these diagrams by comparing each drawing

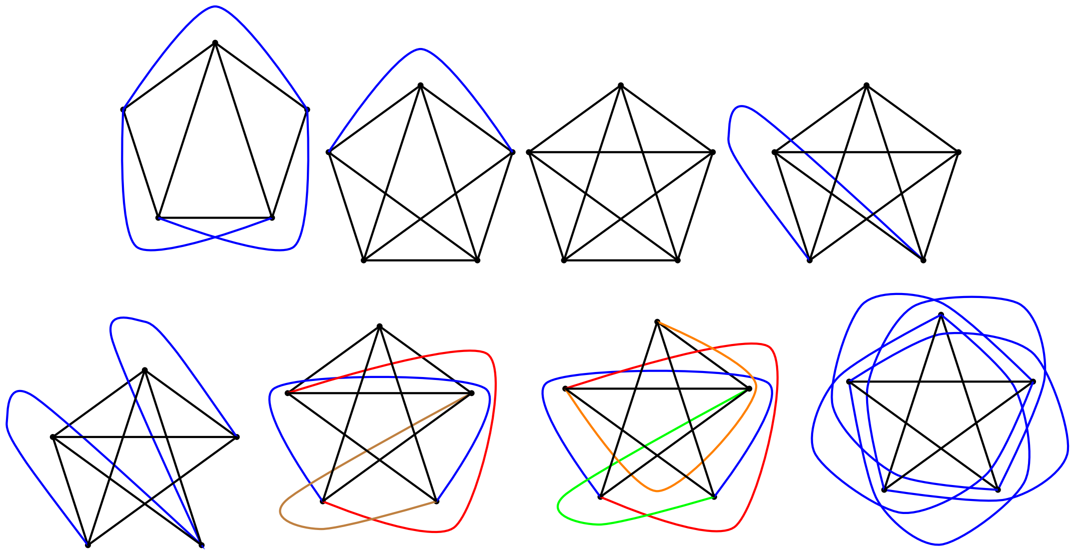

Figure 7: Tolerable drawings of K_5

having i independent crossings with the drawing having $i+3$ independent crossings. Note that the first drawing in Figure 11 and the last drawing in Figure 12 have a 5-fold symmetry and are easy to understand; in each of these drawings the blue and black edges give a tolerable drawing of K_5 with 15 independent crossings. In the first case, each red line has 2 independent crossings with blue edges, while in the second case, each red line has 3 independent crossings with blue edges and 2 independent crossings with black edges.

5 Deleted product cohomology and the van Kampen–Wu invariant

Consider a graph G, which we consider as a *cell complex*; its "cells" are just its vertices and its edges. The *deleted product space* G^* of G is the subcomplex of the cell complex $G \times G$ obtained by deleting all cells having nontrivial intersection with the diagonal. A 1-cell in G^* is of the form (v, e) or (e, v), where v is a vertex of G and e is an edge that is not incident to v. For ease of presentation, we will denote these 1-cells ve and ev respectively. A 2-cell in G^* is given by the pair (e_1, e_2), where e_1, e_2 are independent edges. We will denote this 2-cell e_1e_2. Notice that the group $\mathbb{Z}_2$ acts on G^*; the nontrivial involution is determined by the map τ on $G \times G$ sending (x, y) to (y, x). Since the map τ is fixed point free, the $\mathbb{Z}_2$ action is free. So the quotient $\overline{G^*} := G^*/\mathbb{Z}_2$ is also a cell complex. For further information on the deleted product space of a graph, see Mark de Longueville's excellent text [12], which provides a clear and clean exposition of this material.

We will be working with the cohomology of $\overline{G^*}$, or equivalently, with the cohomology of the $\mathbb{Z}_2$-invariant cocycles on G^*; we say that these cocycles are *symmetric*. Specifically, we work with the cohomology with coefficients in $\mathbb{Z}_2$. So a 2-cochain is given by a function from the set of 2-cells to $\mathbb{Z}_2$; that is, it is just a marking of the

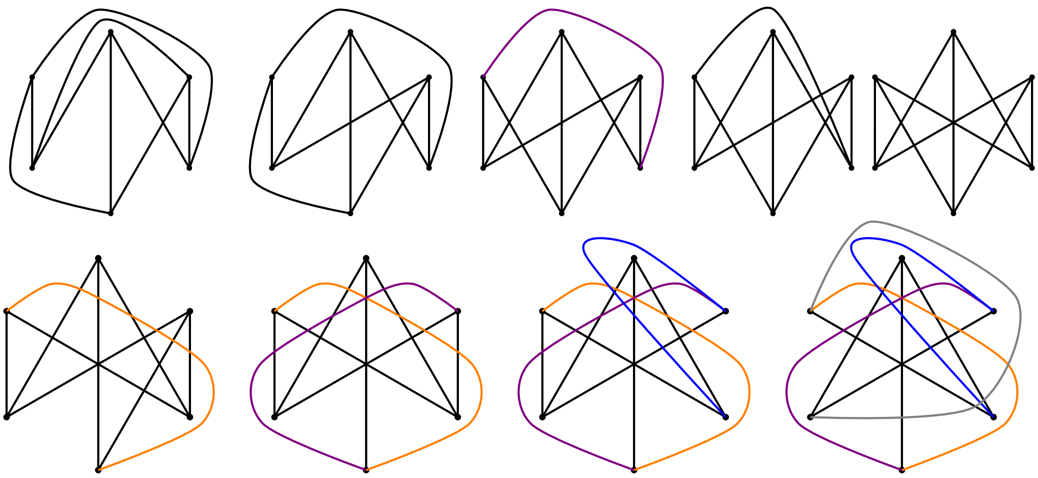

Figure 8: Tolerable drawings of $K_{3,3}$

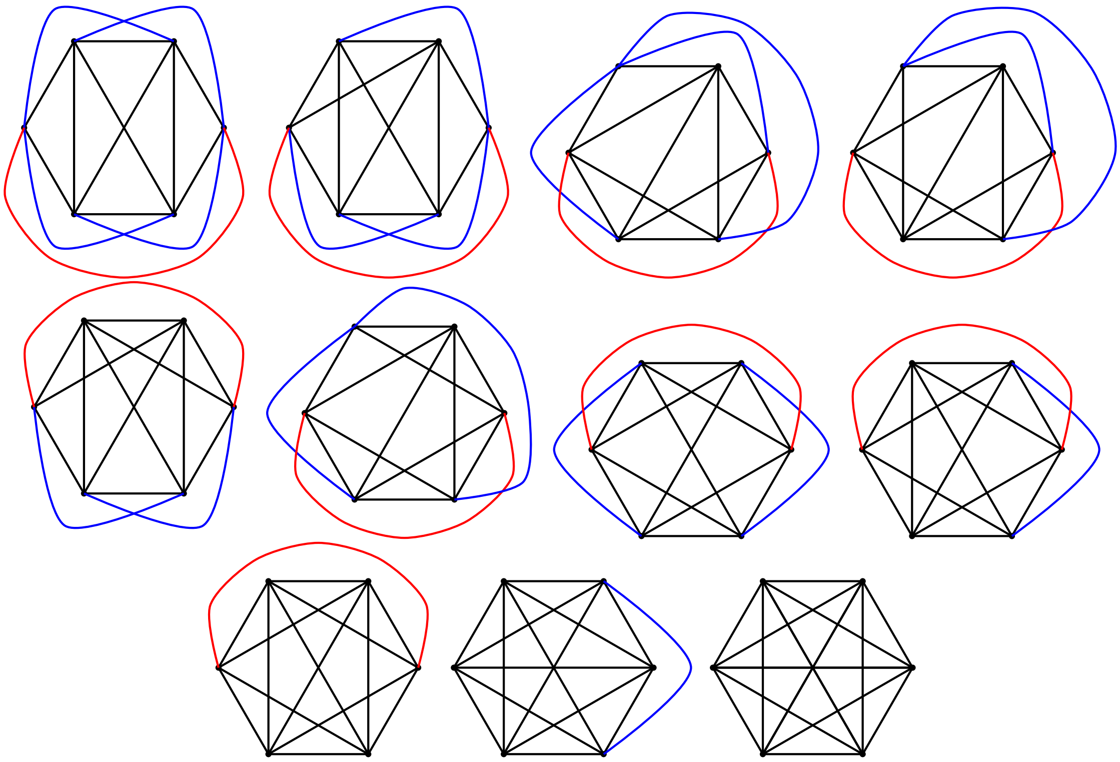

Figure 9: Good drawings of K_6 with $3 \le i \le 12$ and $i = 15$ crossings.

2-cells with the symbols 0 or 1, or, if you like, it is determined by a subset of the set of 2-cells (given by the 2-cells labelled 1), or again as a formal sum over $\mathbb{Z}_2$ of 2-cells. Of course, a symmetric cochain is just a symmetric labelling. So, for example a symmetric 2-cochain is a sum of terms of the form $e_1e_2 + e_2e_1$, where e_1, e_2 are independent edges. Similarly, a 1-cochain is a formal sum over $\mathbb{Z}_2$ of 1-cells, so a symmetric 1-cochain is a sum of terms of the form $ev + ve$.

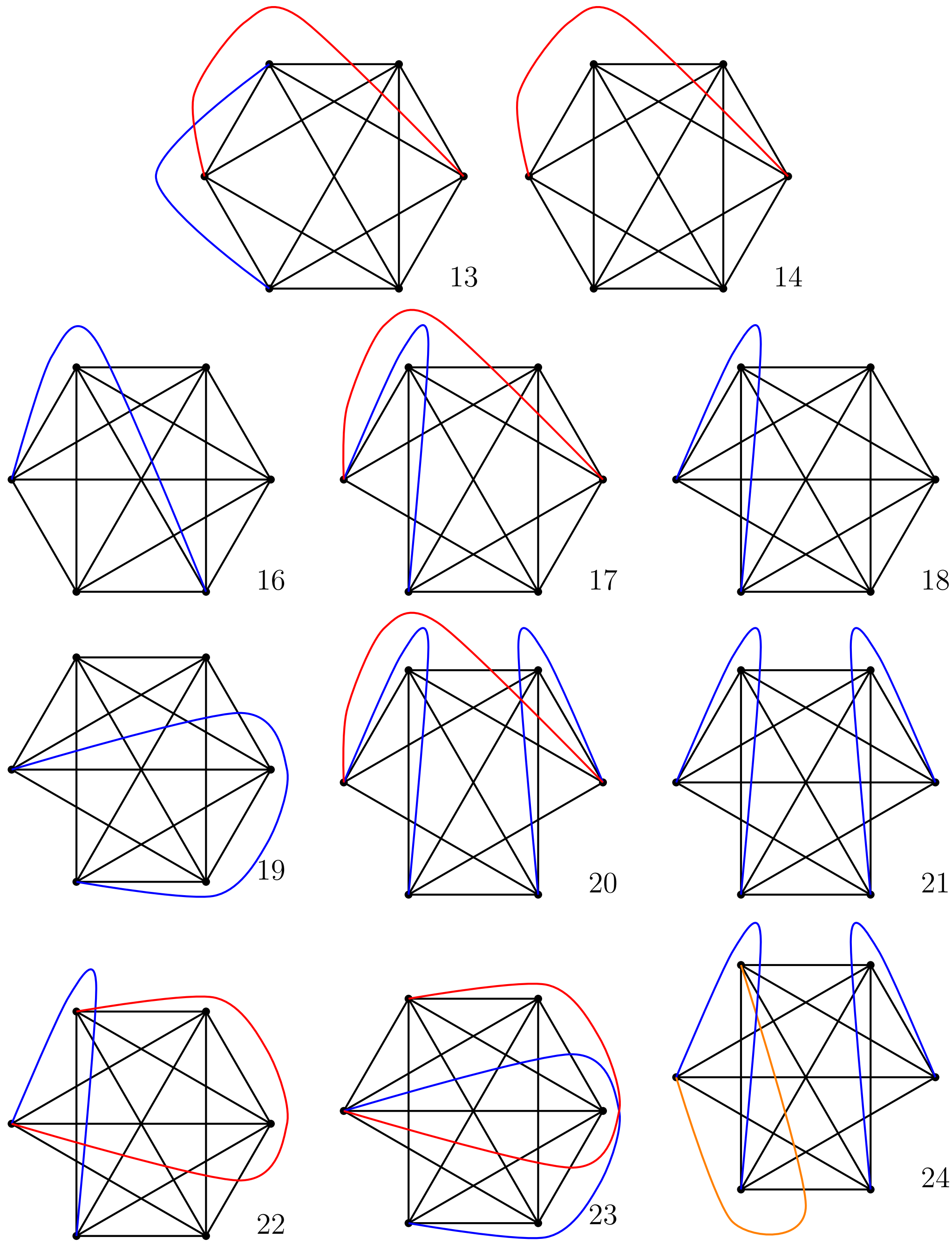

Figure 10: Tolerable drawings of K_6 with $i = 13, 14$ and $16 \leq i \leq 24$ independent crossings.

The differential of a 1-cochain ev (respectively ve) is the sum of the 2-cochains of the form ee' (respectively $e'e$) where e, e' are independent and e' is incident to v. The differential of any 2-cochain is 0 (just because we are working with a cell complex of dimension 2), so every 2-cochain is a 2-cocycle. A 2-cocycle is *exact* if it is the differential of a 1-cochain.

For a given drawing f of G in the plane, we define the symmetric 2-cocycle $\Phi_f(G)$

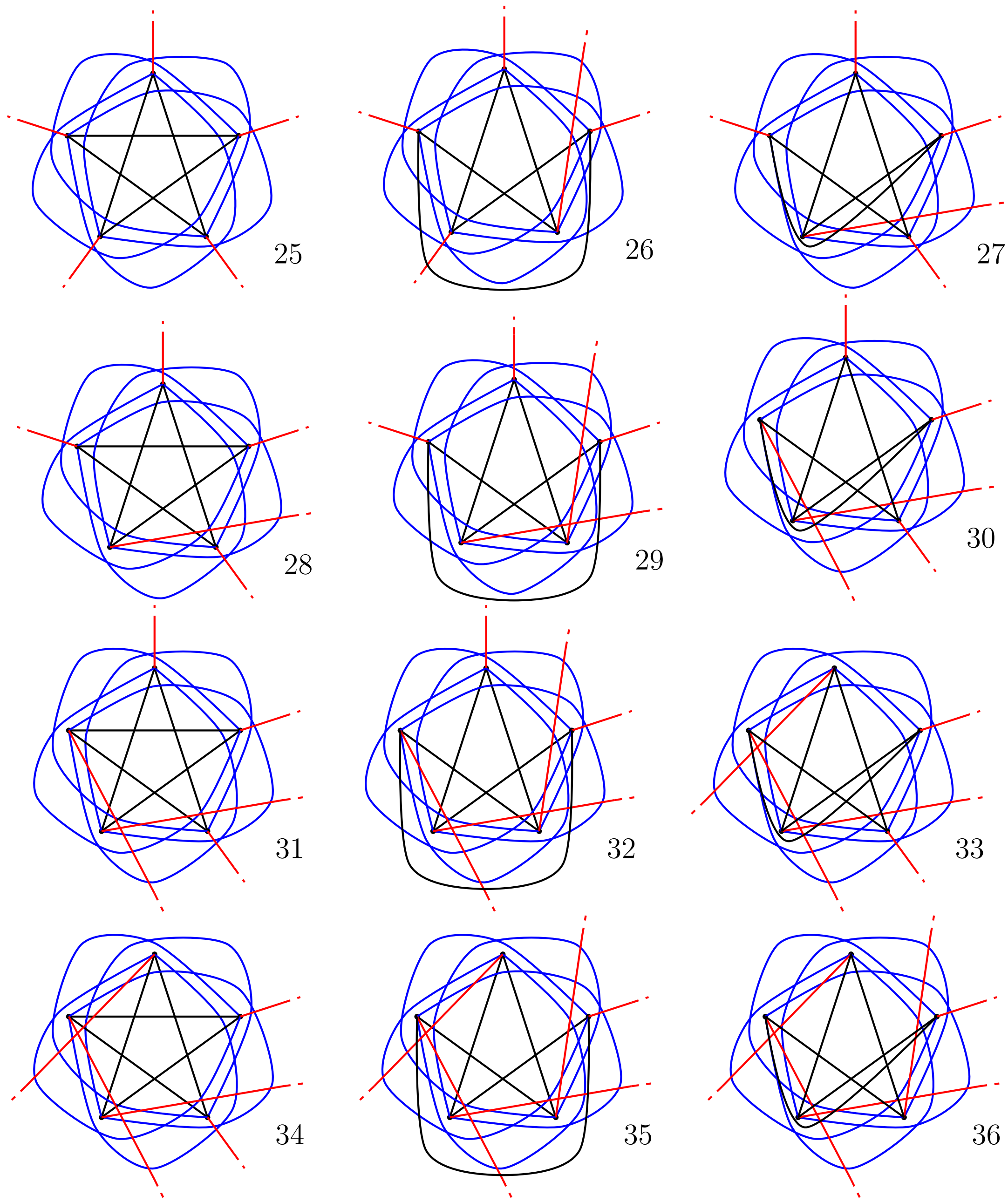

Figure 11: Tolerable drawings of K_6 with 25 to 36 independent crossings.

as follows: if e_1, e_2 are independent edges, we assign the number 1 to the 2-cells e_1e_2 and e_2e_1 if e_1, e_2 cross an odd number of times, and 0 otherwise.

Definition 5.1 The *van Kampen obstruction* is the symmetric cohomology class $\mathfrak{o}(G)$ of $\Phi_f(G)$.

Where useful, we will also consider the corresponding form $\overline{\Phi}_f(G)$ on $\overline{G^*}$, and identify the symmetric cohomology class $\mathfrak{o}(G)$ with the element $[\overline{\Phi}_f(G)] \in H^2(\overline{G^*}, \mathbb{Z}_2)$.

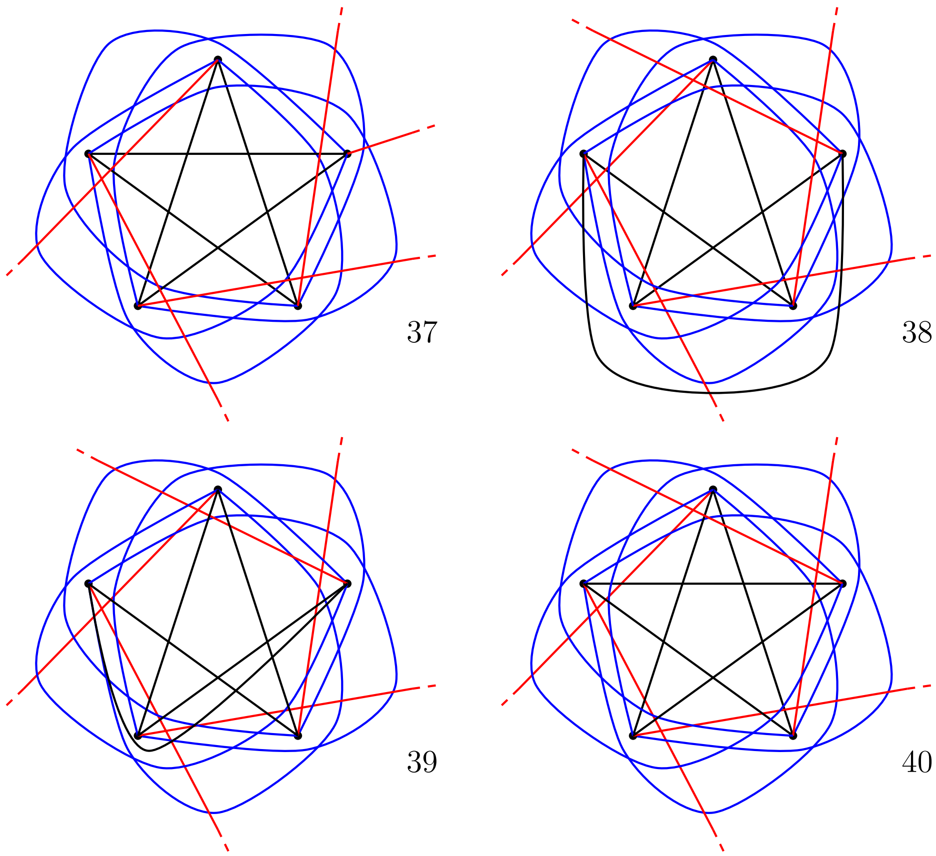

Figure 12: Tolerable drawings of K_6 with 37 to 40 independent crossings.

The van Kampen–Shapiro–Wu Theorem *The class $\mathfrak{o}(G)$ is well defined, independent of the drawing. Moreover, if $\mathfrak{o}(G) = [\alpha]$ for some symmetric 2-cocycle α, then there is a drawing f of G in the plane with $\alpha = \Phi_f(G)$.*

Notice that in the obvious manner, every subset $A \in PG$ determines a symmetric 2-cocycle, and every symmetric 2-cocycle determines a subset $A \in PG$. The van Kampen–Shapiro–Wu Theorem can be reformulated in terms of 2-realisable subsets:

Theorem 5.1 *Consider a graph G with van Kampen symmetric cohomology class $\mathfrak{o}(G) \in H^2(\overline{G^*}, \mathbb{Z}_2)$. Then a subset A of PG is a 2-realisable crossing set if and only if the element of the symmetric cohomology group $H^2(\overline{G^*}, \mathbb{Z}_2)$ corresponding to A is equal to $\mathfrak{o}(G)$.*

A key application of this machinery is:
The Wu–Tutte Theorem $\mathfrak{o}(G) = 0$ *if and only if G is planar.*

Remark 5.1 We should mention that it is well-known that the early literature on the deleted product cohomology contained a number of errors. Errors in [25] were observed by Ummel [34] and Barnett and Farber [6]. An error in [11] was discussed by Sarkaria [27] and Barnett and Farber [6]. Sarkaria's paper [27] is a very attractive and readable work, but it also has errors that have been discussed by Skopenkov [32], van der Holst [36], Barnett [5], and Schaefer [28].

Remark 5.2 The history of this material might also merit a few comments, since there are confusions in the literature, possibly in part reflecting the divide between the fields of topology and combinatorics. The original notions came from van Kampen's 1932 paper [37], just 2 years after Kuratowski's famous 1930 Theorem and 2 years prior to Hanani's version [10] of what is now known as the Hanani–Tutte Theorem. The "deleted product" obstruction was introduced by van Kampen for measuring the non-embeddability of an n-dimensional simplicial complex in $\mathbb{R}^{2n}$, but only for $n \geq 3$. (Related results were published by Flores in 1933 [13]). This work was later clarified, reformulated in cohomological language, and extended to dimension 2 in 1955 by Wu (subsequently translated into English [38, 39, 40] in 1958-1959, and later elaborated with a slightly different argument in his 1965 book [41], in English) and in 1957 by Shapiro [31]. In fact, Wu was a well-known topologist, in part because of his 1950 work on what is known as the *Wu class*, and Wu's 1955–1959 embedding work was widely read by topologists at the time; for example, see [16]. Later in 1970, as Levow [22] writes,"Tutte [33] rediscovered the van Kampen–Shapiro–Wu characterization of planar graphs". It seems that Tutte's paper brought this topic to the attention of combinatorialists, and motivated much of the subsequent investigations; although Tutte's paper uses topological arguments and some topological language (*chain* and *coboundary*), it never uses the word *cohomology*, or even *differential*. The theorem known as the *Hanani–Tutte Theorem*, which says that a graph is planar if it can be drawn in the plane so that each pair of independent edges cross an even number of times, is an immediate consequence of the result we have called the Wu–Tutte Theorem, and many would regard the two as being essentially the same result. Wu presented the Wu–Tutte theorem in [39] and also proved it in [41, p. 210]; he called it *Kuratowski's Theorem*!

6 Drawings of K_5 and $K_{3,3}$

We will require the following beautiful result, which was stated without proof in [2], and proved in Abrams' thesis [1, Theorem 5.1].

Theorem 6.1 *For K_5 and $K_{3,3}$, the deleted product is a closed surface. Moreover, K_5 and $K_{3,3}$ are the only graphs for which the deleted product is a closed surface.*

As the proof of this result is quite short, we include it for the convenience of the reader.

Proof: In order for the deleted product G^* of a graph G to be a closed surface, one requires that each edge (i.e., 1-cell) in G^* be incident with exactly two faces. This occurs precisely when, for each edge e in G and each vertex $v \in G$ that is not incident with e, there are exactly two edges in G that are incident with v and are independent of e. Rephrasing this yet again we obtain the following necessary and sufficient condition:

(∗) for each edge e in G, the graph $G - e$ obtained by deleting e and the interior of all adjacent edges, is a union of disjoint cycles.

Now K_5 and $K_{3,3}$ satisfy condition (∗), and so their deleted products are closed surfaces. On the other hand, if a graph G satisfies (∗), it must have at least 5 vertices. It is easy to verify that if G has 5 vertices and satisfies (∗), then it is K_5, and if G has 6 vertices and satisfies (∗), then it is $K_{3,3}$. One checks readily that no graph with more than 6 vertices can satisfy (∗). □

Kleitman proved that for odd m, n, any two drawings of $K_{m,n}$ (or of K_n) have equal numbers of independent crossings, mod 2 [18]. The result was independently proved for good drawings by Harborth [17], and another proof, again for good drawings, was given by McQuillan and Richter [23]. In particular, all drawings of K_5 and $K_{3,3}$ have an odd number of independent crossings. The following result, which is a rewording of Theorem 1.2, is a converse to Kleitman's Theorem:

Theorem 6.2 *For G equal to K_5 or $K_{3,3}$, every odd subset A of PG is 2-realisable.*

Proof: By Theorem 6.1, $\overline{G^*}$ is a closed surface. Consequently, the cohomology space $H^2(\overline{G^*}, \mathbb{Z}_2)$ has dimension one. Hence the exact 2-cocycles form a codimension one vector subspace of the space Z of 2-cocycles in $\overline{G^*}$. So exactly half the 2-cocycles are exact, and half are not exact. Because $\overline{G^*}$ is a closed surface, the differential of each 1-cell is the sum of two faces. It follows that all exact 2-cocycles are the sum of an even number of faces. Hence, because of their number, the set of exact 2-cocycles is precisely the set of 2-cocycles that are the sum of an even number of faces. Now consider the first drawings f of G in Figures 7 and 8 respectively. It has only 1 independent crossing. So the corresponding cocycle $\overline{\Phi}_f(G)$ in $\overline{G^*}$ is a single face. It follows that $\mathfrak{o}(G) \neq 0$. Thus for any drawing g of G in the plane, the corresponding cocycle $\overline{\Phi}_g(G)$ in $\overline{G^*}$ is the sum of an odd number of faces; i.e., the number of independent crossings is odd. Conversely, by the van Kampen–Shapiro–Wu Theorem (see Theorem 5.1), every such cocycle is obtained from such a drawing. □

7 Drawings of K_6

According to our calculations, $\overline{K_6^*}$ has 45 faces and 60 edges, and the differential from the space of symmetric 1-cochains has rank 35. Using Theorem 5.1, a 2-realisable crossing set of K_6 is given by a cochain of the form $\Phi_f(K_6) + \alpha$, for some drawing f of K_6, where α is an element of the image of the differential. Choose f to be the first drawing of Figure 9, and consider the 2^{35} possible elements α. For each cochain $\Phi_f(K_6) + \alpha$, one just adds the numbers of 1's to obtain the "cardinality". Performing this on a personal computer using Mathematica, we found that there is no 2-realisable crossing set for K_6 with cardinality in $\{0, 1, 2, 41, 42, 43, 44, 45\}$. This establishes Theorem 1.3. For completeness, we will give below a proof of Theorem 1.3 that does not rely on computer computations, or the use of the deleted product cohomology.

We conclude this study with some further comments on the 2-realisable crossing set of K_6. First notice that there are tolerable drawings of K_6 with just 3 independent crossings, but not all subsets $A \subseteq P$ with 3 elements are 2-realisable. Indeed, in any drawing, every pair of independent 3-cycles must cross each other an even number of times, by the Jordan curve theorem. Since $\overline{K_6^*}$ has 45 faces and the differential has rank 35, the symmetric cohomology $H^2(\overline{K_6^*}, \mathbb{Z}_2)$ has dimension 10. Notice also that there are 10 ways of separating the vertices of K_6 into two 3-element subsets. For each such partition, the two 3-element subsets give a copy of the disjoint union $2K_3$ of two 3-cycles; there are 9 potential crossing of the edges of one 3-cycle with the edges of the other, but as we just remarked, there must be an *even* number of such crossings. This gives 10 conditions on $H^2(\overline{K_6^*}, \mathbb{Z}_2)$ for $\mathfrak{o}(K_6)$. These conditions do not uniquely determine $\mathfrak{o}(K_6)$; for instance, the zero element satisfies them all, but $\mathfrak{o}(K_6) \neq 0$ as K_6 is not planar. Consider the induced $K_{3,3}$ subgraphs in K_6. For a 2-realisable set $A \subset PK_6$, the intersection $A \cap PK_{3,3}$ must have an odd number of elements, by Kleitman's theorem. There are 10 such induced $K_{3,3}$ subgraphs, so there are 10 conditions of this kind. However, it turns out that these conditions are not independent, so they also do not by themselves uniquely determine $\mathfrak{o}(K_6)$. Similarly, for the induced K_5 subgraphs in K_6, for a 2-realisable set $A \subset PK_6$, the intersection $A \cap PK_5$ must have an odd number of elements. This gives 6 conditions on $H^2(\overline{K_6^*}, \mathbb{Z}_2)$ for $\mathfrak{o}(K_6)$. Together, the above conditions are sufficient. One has:

Theorem 7.1 *The subset A of PK_6 is 2-realisable if and only if the following three conditions are satisfied:*

(a) for each induced K_5 subgraph of K_6, the intersection $A \cap PK_5$ has an odd number of elements,

(b) for each induced $K_{3,3}$ subgraph of K_6, the intersection $A \cap PK_{3,3}$ has an odd number of elements,

(c) for each induced $2K_3$ subgraph of K_6, the intersection $A \cap P(2K_3)$ has an even number of elements.

The forward direction of Theorem 7.1 is clear from the above discussion, and remains true if we replace K_6 by an arbitrary graph (in which case, one may replace in condition (c) the disjoint union of two 3-cycles by a disjoint union of any two cycles). We established the sufficiency of the conditions directly by computer computation. The result is closely related to van der Holst's theorem [35, Theorem 4], which holds for arbitrary graphs G and which says that the symmetric deleted product cohomology $H^2(\overline{G^*}, \mathbb{Z}_2)$ is generated by subdivisions of K_5's and $K_{3,3}$'s, and by 2-tori resulting from pairs of disjoint cycles. Note that K_6 has subgraphs which are nontrivial subdivisions of K_5 but it was not necessary to consider these in Theorem 7.1.

Proof of Theorem 1.3: We use the forward direction of Theorem 7.1 (which did not require computer computation). Suppose that A is a 2-realisable crossing set for K_6. One needs to show that $3 \leq \#A \leq 40$. If A has only 0, 1 or 2 elements then it is easy to find an induced $K_{3,3}$ subgraph for which $A \cap PK_{3,3}$ is empty, contradicting condition (b) of Theorem 7.1. So it remains to show that $\#A \leq 40$. Indeed, one has the following general result.

Lemma 7.1 *If $n \geq 6$ and A is a 2-realisable crossing set for K_n, then $\#A \leq \lfloor \frac{8}{3}\binom{n}{4} \rfloor$.*

Proof: By condition (c) of Theorem 7.1, for a plane drawing of a complete graph K_n, the number of crossings of any two *independent 3-cycles* (i.e., 3-cycles which have no common vertices) must be even. Therefore if a subset $A \subset PK_n$ is 2-realisable, then from the nine pairs of independent edges defined by a pair of independent 3-cycles, at least one pair (and in general, an odd number of pairs) does not belong to A. For $n \geq 6$ one has $\frac{1}{2}\binom{n}{3}\binom{n-3}{3}$ pairs of independent 3-cycles. A pair of independent edges belongs to $(n-4)(n-5)$ pairs of independent 3-cycles, and so the complement A' to A in PK_n must be of cardinality at least

$$\frac{\frac{1}{2}\binom{n}{3}\binom{n-3}{3}}{(n-4)(n-5)} = \frac{1}{3}\binom{n}{4}.$$

Thus, since $\#PK_n = 3\binom{n}{4}$, we have $\#A \leq 3\binom{n}{4} - \frac{1}{3}\binom{n}{4} = \frac{8}{3}\binom{n}{4}$. □

In the case $n = 6$, the above lemma gives $\#A \leq 40$, as required. □

Remark 7.1 In [20], Kynčl gives a somewhat stronger result than Theorem 7.1. Indeed, according to [20], the pair of above conditions (a) and (c) are sufficient, but that individually, neither of these conditions is sufficient by itself. In fact, by computer computation we have verified that in Theorem 7.1, any two of the three conditions is sufficient, but no single condition is sufficient by itself. Note that in the above proof of Theorem 1.3 we have only used conditions (b) and (c).

8 Future work

As a direct extension of this paper, it would be interesting to know the cardinality of the elements of the 2-realisable crossing sets of K_7; it has a drawing with only 9 independent crossings, and so by Kleitman's Theorem, all 2-realisable crossing sets of K_7 are odd; do all odd numbers between 9 and $105 - x$ occur, for suitable x? More specifically, from our considerations, the pertinent questions seem to be:

1. Is it the case that for each odd integer i with $9 \leq i \leq 91$, there is a 2-realisable subset of PK_7 of cardinality i?

2. Is there a tolerable drawing of K_7 with 91 independent crossings?

Note that a positive answer to (1) together with a negative answer to (2) would resolve the question posed in Remark 3.5: does there exist a graph G and an integer i for which there exist 2-realisable subsets of PG of cardinality i but for none of these is there a tolerable drawing?

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