AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **75(2)** (2019), Pages 246–255

Inverting non-invertible weighted trees

Soňa Pavlíková

Institute of Information Engineering, Automation and Mathematics
Faculty of Chemical and Food Technology, Slovak University of Technology
Bratislava, Slovakia

Jozef Širáň

Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, Slovak University of Technology
Bratislava, Slovakia

Abstract

If a graph with non-zero edge weights has a non-singular adjacency matrix, one may use the inverse matrix to define a (weighted) graph that may be viewed as the inverse graph to the original one. It has been known that an adjacency matrix of a weighted tree is non-singular if and only if the tree has a unique perfect matching. In the opposite case one may use a pseudo-inverse (which, in the symmetric case, coincides with Moore-Penrose, Drazin, or group inverse) of the adjacency matrix to 'invert' a tree. A way of calculating entries of a pseudo-inverse follows from the work of Britz, Olesky and van den Driessche (2004), based on a determinant formula for entries of the Moore-Penrose inverse. We give here a different proof of the result for calculating a pseudo-inverse of a weighted tree, based solely on considering maximum matchings and alternating paths.

1 Introduction

We will consider finite, undirected graphs with no multiple edges but we allow every vertex to carry at most one loop (an edge whose both ends are the same vertex); such an object will simply be referred to as a *graph* in this paper. We will further assume that each edge $e = uv$ of a graph G carries a non-zero real *weight* $a(e) = a(uv)$, and the pair (G, a) will be called an *weighted graph*. Let A be the adjacency matrix of (G, a), which means that for any two vertices u, v of G the uv-th entry of A is $a(e)$ if $e = uv$ is an edge of G, and 0 otherwise.

If A is non-singular, the *inverse* of (G, a) is the weighted graph (H, b) determined by the adjacency matrix equal to the inverse A^{-1} of A. We thus assume that G and H share

ISSN: 2202-3518

the same vertex set, and $e = uv$ is an edge of H if the uv-th entry of A^{-1} is non-zero, in which case this uv-th entry is the weight $b(e)$ of e. Obviously, the inverse defined this way is unique up to graph isomorphism preserving weights on edges.

Inverses of weighted graphs as introduced above were studied, for example, in [2, 16, 17, 22]. Edge weights in [2, 17] were even allowed to be elements of a not necessarily commutative ring, and a formula for an inverse graph to (G, a) was given in both papers in the special case of a bipartite graph G with a unique perfect matching and with multiplicatively invertible weights on matched edges.

Returning to real-valued weights, one may further restrict to graphs (G, a) in which all weights on edges are positive. In such a case of a *positively weighted* graph with a non-singular adjacency matrix, its inverse will typically contain both positive as well as negative entries. A positively weighted graph (G, a) with a non-singular adjacency matrix A is then *positively invertible* if A^{-1} is diagonally similar (signable, in the terminology of [1]) to a non-negative matrix. Positively invertible graphs with integral edge weights have been studied in detail in [3, 1, 12] and the last paper also contains a nice survey of development in the study of inverses of graphs.

All this work, however, was initiated by the influential paper [10] on inverses of trees with unit edge weights, extending earlier observations of [9]. By [11], an adjacency matrix of a tree is invertible if and only if the tree has a (unique) perfect matching. In terms of our definition, each such tree is automatically invertible. The much stronger result of [10] says that every tree with a perfect matching is *positively invertible* and its positive inverse is a *simple graph* (containing no loops) with every edge carrying the *unit* weight again. A formula for determining the inverse of a tree with a perfect matching in terms of alternating paths appeared later in [18] and was afterwards extended to bipartite graphs with a unique perfect matching in [3, 1, 2, 12, 17]. For completeness, graphs arising as inverses of trees with a perfect matching, and of bipartite graphs with a unique perfect matching that remain bipartite after contracting the matching, were characterized in [15] and [16], respectively, and self-inverse graphs in the latter family were classified in [20]. Also, Godsil's problem of characterization of positively invertible bipartite graphs with a unique perfect matching was recently solved in [21].

In this situation it is natural to ask what one can do in the case of weighted graphs with a *singular* adjacency matrix. An equally natural move is to consider 'inverting' the matrix by taking one of the generalizations of matrix inverses, such as the Moore-Penrose inverse, or the Drazin inverse, or a special case of the latter known as the group inverse. In the instance of a (square) symmetric matrix A all these inverses coincide and are commonly called a *pseudo-inverse* of A, which we will denote by A^* throughout. The pseudo-inverse of a *symmetric* matrix is easy to introduce as follows. Since a real symmetric $n \times n$ matrix A is orthogonally diagonalizable, there is an orthogonal matrix P such that $PAP^T = D$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ is the diagonal matrix of eigenvalues of A, with $k = \mathrm{rank}(A)$ non-zero eigenvalues $\lambda_1, \ldots, \lambda_k$. Letting $D^* = \mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_k^{-1}, 0, \ldots, 0)$, the *pseudo-inverse* A^* of A is simply given by $A^* = PD^*P^T$, that is, both A and A^* are conjugate to their corresponding diagonal matrices by the *same* orthogonal matrix P. Note that A^* is again symmetric, and A^* coincides with A^{-1} if A is non-singular.

Motivated by this, we define the *pseudo-inverse* of a weighted graph (G, a) with adjacency matrix A to be the weighted graph (G^*, a^*) with adjacency matrix A^*, the pseudo-inverse of A. As before, G and G^* are assumed to have the same vertex set, and $e = uv$ is an edge of G^* if and only if the uv-th entry of A^* is non-zero, and then this entry is also the weight $a^*(e)$ of e. And, again, note that G^* is well defined up to isomorphism preserving edge weights.

Observe that this way of defining pseudo-inverses of weighted graphs is in line with the original motivation of considering graph inverses which comes from chemistry. Namely, there appear to be fewer methods for estimating the smallest positive eigenvalue of a graph in contrast to a larger number of techniques for bounding the largest positive eigenvalue. For graphs representing structure of molecules, however, the smallest positive eigenvalue is a meaningful parameter in quantum chemistry. If such a graph has an inverse, one may hope to increase the number of techniques for estimating its smallest positive eigenvalue by passing to bounds on the largest positive eigenvalue of the inverse graph. This feature remains present also for our pseudo-inverses.

A formula for entries of the adjacency matrix of the pseudo-inverse of a tree with arbitrary non-zero edge weights can be derived from a result of [7] stated in terms of bipartite graphs associated with arbitrary matrices (with the vertex set being the union of row and column indices of a matrix) in the special case when the graphs are acyclic. The proof of the result of [7] is based on a determinant formula for entries of the Moore-Penrose inverse that first appeared in a classical paper [14]; for more recent references see [4] or [5, Appendix A].

In the present paper we give a different proof of a formula for calculating the pseudo-inverse of an arbitrary weighted *tree*. Our proof does not refer to the formulae for entries of the Moore-Penrose inverse and is based solely on considering maximum matchings and alternating paths Also, the length of our completely elementary proof compares well to the length of the proof of [7] combined with the length of the derivation of the associated determinant formula (see e.g. a proof given in [13], using results of [6]).

To state the result we need to introduce a few concepts. Let (T, a) be a weighted tree; for brevity we will often omit the symbol for the weight function in our exposition. For an unordered pair of vertices u, v of distinct vertices of T we let $\mathcal{M}(u, v)$ denote the set of all maximum matchings M of T with the property that edges of the (unique) $u{-}v$ path in T belong alternately to M and not to M, with the condition that both the first and the last edge of the path (that is, those incident to u and v) belong to M. A necessary condition for the set $\mathcal{M}(u, v)$ to be non-empty is that the distance between u and v be odd, but note that this condition does not need to be sufficient; though, if uv is an edge of some maximum matching, then the set $\mathcal{M}(u, v)$ is automatically non-empty. A pair of vertices u, v for which $\mathcal{M}(u, v) \neq \emptyset$ will be called *maximally matchable*.

Further, for any maximally matchable pair of vertices u, v and a maximum matching $M \in \mathcal{M}(u, v)$ let $\alpha_{\overline{u,v}}(M)$ denote the product of all the weights $a(e)$, ranging over all edges e of M that are not contained in the unique $u{-}v$ path P in T. (The line over the pair of vertices u, v in the subscript indicates that edges of $M \cap P$ are not considered in the product; a product over an empty set is considered to be equal to 1.) Also, for the same pair of vertices u, v let $\alpha(u, v)$ be the product of all the values of $a(e)$ taken over all the edges e in the path P (necessarily of odd length), and multiplied by $+1$ or -1 depending on whether the distance between u and v is congruent to $+1$ or -1 mod 4; if u, v are not maximally matchable we set $\alpha(u, v) = 0$. With this in hand we may associate with any maximally matchable pair of vertices u, v of T the value

$$\mu_T(u, v) = \alpha(u, v) \cdot \sum_{M \in \mathcal{M}(u,v)} (\alpha_{\overline{u,v}}(M))^2 \, ; \tag{1}$$

it follows that $\mu_T(u, v) = 0$ if u, v is not a maximally matchable pair (which includes the case $u = v$). Finally, letting $\mathcal{M}$ be the set of all maximum matchings in T, for every $M \in \mathcal{M}$ let $\alpha(M)$ be the product of the weights $a(e)$ taken over all edges e of M, and let

$$m(T) = \sum_{M \in \mathcal{M}} (\alpha(M))^2 \, . \tag{2}$$

In this terminology and notation we have:

Theorem 1 *Let (T, a) be a weighted tree with vertex set V. Then, its pseudo-inverse (T^*, a^*) has two distinct vertices $u, v \in V$ joined by an edge e if and only if u, v is a maximally matchable pair in T, with weight of e given by*

$$a^*(e) = a^*(uv) = \frac{\mu_T(u, v)}{m(T)} \, . \tag{3}$$

We note that this formula is equivalent to the one given in [7] in the language of acyclic bipartite matrices, and that it generalizes the original findings of [18] on inverses of trees with unit edge weights having a (unique) perfect matching.

2 Proof of the pseudo-inversion formula for trees

The way we introduced the pseudo-inverse A^* of a symmetric matrix A is not new and it was used e.g. in [5, Ch. 4.3] in a more general context of diagonalizable matrices, along with the observation that A commutes with A^* as a consequence of simultaneous diagonalization. The latter is exactly the property we will use in deriving a characterization of pseudo-inverses of symmetric matrices (which is likely to be known to specialists).

Proposition 1 *A pair of symmetric matrices A and B of the same dimension are pseudo-inverses of each other if and only if they commute and $ABA = A$.*

Proof. Necessity follows immediately from the way pseudo-inverses of symmetric matrices have been introduced. For sufficiency we invoke the well-known result from linear algebra that a pair of commuting symmetric matrices are simultaneously orthogonally diagonalizable, see e.g. [19, Ch. 2.5]. The simultaneously diagonalized form of the equation $ABA = A$ implies that for every non-zero eigenvalue λ of A its inverse λ^{-1} is an eigenvalue of B and vice versa, including equality of the corresponding eigenspaces. It follows that A and B are mutually pseudo-inverse. □

With this tool in hand the strategy of proving our main result is clear. In the terminology and notation introduced before the statement of Theorem 1, for every weighted tree (T, a) with adjacency matrix A we need to show that if B is a matrix (indexed the same way as A) with uv-th entry equal to $\mu_T(u, v)/m(T)$, then $B = A^*$. In the light of Proposition 1 our task reduces to proving that $AB = BA$ and $ABA = A$, which we will do next. In both auxiliary results that follow we let $N(x)$ be the set of neighbours of a vertex z of our tree T, and we will omit the subscript T in the symbol $\mu_T(u, v)$ in the proofs.

Proposition 2 *Let (T, a) be a weighted tree with vertex set V and with adjacency matrix A, and let B be the matrix (indexed as A) with uv-th entry equal to $\mu_T(u, v)/m(T)$ for every $u, v \in V$. Then, $AB = BA$.*

Proof. The equation $AB = BA$ has the following equivalent form:

$$\sum_{r\in V} a(ur)\mu(r, v) = \sum_{s\in V} \mu(u, s)a(sv) \quad \text{for every} \quad u, v \in V\,. \tag{4}$$

Note that (4) is vacuously true if v is at an odd distance from u, and also in the case when $v = u$. Assume therefore that the vertex v has a positive even distance from u. Let $ux \dots yv$ be the (unique) u–v path in T; note that $y = x$ if u and v are at distance two. Further, let $\mathcal{M}(u, \widehat{v})$ be the set of maximum matchings $M \in \mathcal{M}(u, y)$ not containing v, so that

$$\mathcal{M}(u, y) = \cup_{z\in N(v)\setminus\{y\}}\mathcal{M}(u, z) \cup \mathcal{M}(u, \widehat{v})\,. \tag{5}$$

By repeatedly using (1) and the definition of the values of α on paths, and (5) at an appropriate place, one successively obtains

$$\begin{aligned}
\mu(u, y)\cdot a(yv) \;&=\; a(yv)\left(\alpha(u, y)\sum_{M\in\mathcal{M}(u,y)}(\alpha_{\overline{u,y}}(M))^2\right)\\
&= \alpha(u, y)a(yv)\left(\sum_{z\in N(v)\setminus\{y\}}(a(vz))^2\sum_{M\in\mathcal{M}(u,z)}(\alpha_{\overline{u,z}}(M))^2 + \sum_{M\in\mathcal{M}(u,\widehat{v})}(\alpha_{\overline{u,y}}(M))^2\right)\\
&= -\sum_{z\in N(v)\setminus\{y\}} a(vz)\left(\alpha(u, z)\sum_{M\in\mathcal{M}(u,z)}(\alpha_{\overline{u,z}}(M))^2\right) + \alpha(u, y)a(yv)\sum_{M\in\mathcal{M}(u,\widehat{v})}(\alpha_{\overline{u,y}}(M))^2\\
&= -\sum_{z\in N(v)\setminus\{y\}} a(vz)\mu(u, z) \;+\; \alpha(u, y)a(yv)\sum_{M\in\mathcal{M}(u,\widehat{v})}(\alpha_{\overline{u,y}}(M))^2\,.
\end{aligned}$$

Rearranging terms, the above calculation implies that

$$\sum_{s\in V}\mu(u,s)a(sv)=\alpha(u,y)a(yv)\sum_{M\in\mathcal{M}(u,\widehat{v})}(\alpha_{\overline{u,y}}(M))^2\,. \tag{6}$$

By the same token, just interchanging the roles of u and v (and x and y) and letting $\mathcal{M}(v,\widehat{u})$ denote the set of all $M\in\mathcal{M}(v,x)$ not containing u, one obtains

$$\sum_{r\in V}a(ur)\mu(r,v)=\alpha(v,x)a(xu)\sum_{M\in\mathcal{M}(v,\widehat{u})}(\alpha_{\overline{v,x}}(M))^2\,. \tag{7}$$

Since $\alpha(u,y)a(yv)=\alpha(v,x)a(xu)$, (6) and (7) imply that (4) holds if and only if

$$\sum_{M\in\mathcal{M}(u,\widehat{v})}(\alpha_{\overline{u,y}}(M))^2=\sum_{M\in\mathcal{M}(v,\widehat{u})}(\alpha_{\overline{v,x}}(M))^2\,. \tag{8}$$

Note, however, that there is a bijection $\theta:\ \mathcal{M}(u,\widehat{v})\to\mathcal{M}(v,\widehat{u})$ which, to every maximum matching $M\in\mathcal{M}(u,\widehat{v})$ of T assigns a maximum matching $M^\theta\in\mathcal{M}(v,\widehat{u})$ obtained from M by trading the matched edges on the unique $u{-}v$ path of T by the unmatched ones. In addition, by definition of α on matchings making particular pairs of vertices matchable, it is clear that this bijection satisfies $\alpha_{\overline{u,y}}(M)=\alpha_{\overline{v,x}}(M^\theta)$. This establishes the validity of (8), and hence of Proposition 2. □

Proposition 3 *Let (T,a) be a weighted tree with vertex set V and with adjacency matrix A, and let B be the matrix (indexed as A) with uv-th entry equal to $\mu_T(u,v)/m(T)$ for every $u,v\in V$. Then, $ABA=A$.*

Proof. By Proposition 2 we may equally well prove that $AAB=A$, or, equivalently,

$$\sum_{x\in V}a(ux)\sum_{y\in V}a(xy)\mu(y,v)\ =\ \begin{cases} a(uv)m(T) & \text{if } u\in N(v),\\ 0 & \text{if } u\notin N(v).\end{cases} \tag{9}$$

Let S be the left-hand side of (9). Then, $S=0$ if u is at an even distance from v. In what follows we thus assume that the distance between u and v is odd. Let $uz\ldots v$ be the unique $u{-}v$ path in T, possibly with $v=z$. Now, S may be written in the form $S=S(z)+S(u,\widehat{z})+S(\widehat{u},\widehat{z})$ with the first two terms corresponding to taking $x=z$ in the first sum, and letting $x\in N(u)\backslash\{z\}$ but putting $y=u$ in the second sum in (9), i.e.,

$$S(z)=a(uz)\sum_{y\in N(z)}a(zy)\mu(y,v)\ ,\quad S(u,\widehat{z})=\sum_{x\in N(u)\backslash\{z\}}a(ux)^2\mu(u,v)\,, \tag{10}$$

with the remaining part $$S(\widehat{u},\widehat{z})=\sum_{x\in N(u)\backslash\{z\}}a(ux)\sum_{y\in N(x)\backslash\{u\}}a(xy)\mu(y,v)\,. \tag{11}$$

We begin by evaluating $S(\widehat{u},\widehat{z})$. To do so, for any neighbour $x \in N(u)\backslash\{z\}$ we denote by $\mathcal{M}_x(u,v)$ and $\mathcal{M}_{\widehat{x}}(u,v)$ the sets of maximum matchings $M \in \mathcal{M}(u,v)$ in which the vertex x is matched and unmatched, respectively, so that $\mathcal{M}(u,v) = \mathcal{M}_x(u,v) \cup \mathcal{M}_{\widehat{x}}(u,v)$. Using (1) and the definition of α for the paths $yxuz\ldots v$ we step by step obtain

$$\begin{aligned} S(\widehat{u},\widehat{z}) &= \sum_{x\in N(u)\backslash\{z\}} a(ux) \sum_{y\in N(x)\backslash\{u\}} a(xy)\left(-a(yx)a(xu)\alpha(u,v)\right) \sum_{M\in\mathcal{M}(y,v)} \alpha_{\overline{y,v}}(M)^2 \\ &= -\sum_{x\in N(u)\backslash\{z\}} a(xu)^2\alpha(u,v) \sum_{y\in N(x)\backslash\{u\}} a(yx)^2 \sum_{M\in\mathcal{M}(y,v)} \alpha_{\overline{y,v}}(M)^2 \\ &= -\sum_{x\in N(u)\backslash\{z\}} a(xu)^2\alpha(u,v) \sum_{M\in\mathcal{M}_x(u,v)} \alpha_{\overline{u,v}}(M)^2 . \end{aligned}$$

For the sum $S(u,\widehat{z})$ appearing on the right-hand side of (10) we simply have, by (1),

$$S(u,\widehat{z}) = \sum_{x\in N(u)\backslash\{z\}} a(xu)^2\alpha(u,v) \sum_{M\in\mathcal{M}(u,v)} \alpha_{\overline{u,v}}(M)^2 ,$$

and from the two derivations it follows that

$$S(u,\widehat{z}) + S(\widehat{u},\widehat{z}) = \sum_{x\in N(u)\backslash\{z\}} a(xu)^2\alpha(u,v) \sum_{M\in\mathcal{M}_{\widehat{x}}(u,v)} \alpha_{\overline{u,v}}(M)^2 . \qquad (12)$$

Now let $u \in N(v)$, that is, let $z = v$; we evaluate $S(z) = S(v)$ from the left-hand part of (10). Using the symbol $\mathcal{M}(v)$ to denote the set of all maximum matchings in T in which the vertex v is matched, we obtain

$$S(v) = a(uv) \sum_{y\in N(v)} a(vy) \left(a(yv) \sum_{M\in\mathcal{M}(y,v)} \alpha_{\overline{y,v}}(M)^2 \right) = a(uv) \sum_{M\in\mathcal{M}(v)} \alpha(M)^2 . \qquad (13)$$

The equation (12) for $z = v$ gives, after a rearrangement and with $\alpha(u,v) = a(uv)$,

$$S(u,\widehat{v}) + S(\widehat{u},\widehat{v}) = a(uv) \sum_{x\in N(u)\backslash\{v\}} \left(\sum_{M\in\mathcal{M}_{\widehat{x}}(u,v)} a(xu)^2 \alpha_{\overline{u,v}}(M)^2 \right) . \qquad (14)$$

The set $\mathcal{M}_{\widehat{x}}(u,v)$ consists of the maximum matchings M in T that contain the edge uv but avoid the vertex $x \in N(u)\backslash\{v\}$ on the path xuv. For any such matching M the term $a(xu)^2\alpha_{\overline{u,v}}(M)^2$ in the last sum of (14) is equal to $\alpha(M')^2$ for the maximum matching M' in T obtained from M by trading the edge $uv \in M$ for the edge $xu \in M'$. The assignment $M \to M'$ is clearly a bijection from the set $\mathcal{M}_{\widehat{x}}(u,v)$ onto the set $\mathcal{M}(\widehat{v})$ of the maximum matchings of T that avoid the vertex v. Therefore, the double sum on the right-hand side of (14) is equal to the sum of the squares of the α-values on matchings in $\mathcal{M}(\widehat{v})$, that is,

$$S(u,\widehat{v}) + S(\widehat{u},\widehat{v}) = a(uv) \sum_{M\in\mathcal{M}(\widehat{v})} \alpha(M)^2 . \qquad (15)$$

From (15) and (13) we now obtain, for $z = v$ and hence for $u \in N(v)$,

$$S = S(v)+S(u,\widehat{v})+S(\widehat{u},\widehat{v}) = a(uv)\left(\sum_{M\in\mathcal{M}(v)}\alpha(M)^2+\sum_{M\in\mathcal{M}(\widehat{v})}\alpha(M)^2\right) = a(uv)m(T)$$

which proves the first part of our claim.

It remains to consider the case $u \notin N(v)$, that is, $z \neq v$, in the evaluation of $S(z)$ by (10). As z and v are assumed to be at even distance, we now have a (unique) odd-length path of the form $uzw\ldots v$ in T and we will deal with neighbours $y \in T(z)$, including $y \in \{u, w\}$. For the vertex z we will also use the symbols $\mathcal{M}_z(w,v)$ and $\mathcal{M}_{\widehat{z}}(w,v)$ to denote the set of maximum matchings in T inducing a perfect matching on the path $w\ldots v$ and, respectively, containing z and avoiding z. By calculations similar to the earlier ones we obtain

$$\begin{aligned}\frac{S(z)}{a(uz)} &= a(zw)\mu(w,v) + \sum_{y\in N(z)\setminus\{w\}} a(zy)\mu(y,v)\\ &= a(zw)\mu(w,v) + \sum_{y\in N(z)\setminus\{w\}} a(zy)\left(-a(yz)a(zw)\alpha(w,v)\right)\sum_{M\in\mathcal{M}(y,v)}\alpha_{\overline{y,v}}(M)^2\\ &= a(zw)\alpha(w,v)\sum_{M\in\mathcal{M}(w,v)}\alpha_{\overline{w,v}}(M)^2 \;-\; a(zw)\alpha(w,v)\sum_{M\in\mathcal{M}_z(w,v)}\alpha_{\overline{w,v}}(M)^2\end{aligned}$$

which implies that, for $z \neq v$,

$$S(z) = a(uz)a(zw)\alpha(w,v)\sum_{M\in\mathcal{M}_{\widehat{z}}(w,v)}\alpha_{\overline{w,v}}(M)^2\,. \tag{16}$$

Revisiting the equation (12) and reducing its right-hand side further yields

$$\begin{aligned}S(u,\widehat{z}) + S(\widehat{u},\widehat{z}) &= \sum_{x\in N(u)\setminus\{z\}} a(xu)^2\left(-a(uz)a(zw)\alpha(w,v)\right)\sum_{M\in\mathcal{M}_{\widehat{x}}(u,v)}\alpha_{\overline{u,v}}(M)^2\\ &= -\,a(uz)a(zw)\alpha(w,v)\sum_{x\in N(u)\setminus\{z\}}\;\sum_{M\in\mathcal{M}_{\widehat{x}}(u,v)}\alpha(xu)^2\alpha_{\overline{u,v}}(M)^2\,.\end{aligned}$$

We now argue as done after the equation (12). The set $\mathcal{M}_{\widehat{x}}(u,v)$ consists of the maximum matchings M in T that induce a perfect matching on the path $uzw\ldots v$ but avoid the vertex x on the path $xuzw\ldots v$. Given any such matching M, the product $a(xu)^2\alpha_{\overline{u,v}}(M)^2$ in the last sum above is equal to $\alpha(M')^2$ for the maximum matching M' in T inducing a perfect matching on the path $w\ldots v$, obtained from M by trading the edge $uz \in M$ for the edge $xu \in M'$, so that M' avoids the vertex z. But the latter precisely means that M' belongs to the set $\mathcal{M}_{\widehat{z}}(w,v)$. It can be seen that the assignment $M \to M'$ defines a bijection between the sets $\mathcal{M}_{\widehat{x}}(u,v)$ and $\mathcal{M}_{\widehat{z}}(w,v)$. The last double sum above is thus equal to the sum of the squares of the α-values on matchings in $\mathcal{M}_{\widehat{z}}(w,v)$, that is,

$$S(u,\widehat{z}) + S(\widehat{u},\widehat{z}) = -a(uz)a(zw)\alpha(w,v)\sum_{M\in\mathcal{M}_{\widehat{z}}(w,v)}\alpha_{\overline{w,v}}(M)^2\,. \tag{17}$$

Comparing (17) with (16) shows that $S = 0$ for $u \notin T(v)$, which completes our proof. □

The proof of Theorem 1 now follows immediately from Propositions 1, 2 and 3.

3 Conclusion

We have mentioned equivalence of our formula (3) from Theorem 1 with that of [7], but the reader wanting to check details will have observed that the latter refers to the rank of the adjacency matrix (in a slightly different form) whereas our result makes no such reference. The reason in our case is that, by a formula for evaluation of coefficients of a characteristic polynomial in terms of certain subgraphs of the corresponding weighted graph [8, formula (1.35)'], the rank of the adjacency matrix of a weighted tree is equal to the size of its maximum matchings (independently on the values of the non-zero edge weights).

We believe that our method may furnish extensions of Theorem 1 to broader classes of graphs, at the very least in a similar way Godsil's inversion theorem for trees with a perfect matching [10] and the subsequent formula for inverses of such trees [18] have been extended to bipartite graphs with a unique perfect matching in [1, 2, 3, 12, 17, 21].

Acknowledgements

The authors would like to thank the anonymous referees for carefully reading the original manuscript and making useful comments. Our special thanks go to the referee whose comments on treating null-spaces led (after appropriate modifications) to a substantial shortening of the manuscript.

Both authors acknowledge support by the APVV research grants 15-0220 and 17-0428, and by the VEGA research grants 1/0142/17 and 1/0238/19.

References

[1] S. Akbari and S. J. Kirkland, On unimodular graphs, *Lin. Alg. Appl.* 421 (2007), 3–15.

[2] R. B. Bapat and E. Ghorbani, Inverses of triangular matrices and bipartite graphs, *Lin. Alg. Appl.* 447 (2014), 68–73.

[3] S. Barik, M. Neumann and S. Pati, On nonsingular trees and a reciprocal eigenvalue property, *Lin. Multilin. Alg.* 54 (6) (2006), 453–465.

[4] A. Ben-Israel, The Moore of the Moore-Penrose inverse, *Electron. J. Lin. Alg.* 9 (2002), 150–157.

[5] A. Ben-Israel and T. N. E. Greville, *Generalized inverses*, Theory and Applications, 2nd ed., Springer, 2003.

[6] J. W. Blattner, Bordered matrices, *SIAM J.* 10 (2) (1962), 528–536.

[7] T. Britz, D. D. Olesky and P. van den Driessche, The Moore-Penrose inverse of matrices with an acyclic bipartite graph, *Lin. Alg. Appl.* 390 (2004), 47–60.

[8] D. Cvetković, M. Doob and H. Sachs, 'Spectra of Graphs', Acad. Press, 1980.

[9] D. Cvetković, I. Gutman and S. Simić, On selfpseudo-inverse graphs, *Univ. Beograd Publ. Elektrotechn. Fak. Ser. Mat. Fiz.*, Vol. 602-633 (1978), 111–117.

[10] C. D. Godsil, Inverses of trees, *Combinatorica* 5 (1985), 33–39.

[11] F. Harary, The determinant of the adjacency matrix of a graph, *SIAM Review 4* (1962), 202–210.

[12] C. McLeman and A. McNicholas, Graph invertibility, *Graphs Combin.* 30 (2014), 977–1002.

[13] J. Miao and A. Ben-Israel, Minors of the Moore-Penrose Inverse, *Lin. Alg. Appl.* 195 (1993), 191–208.

[14] E. Moore, On the reciprocal of the general algebraic matrix, *Bull. Amer. Math. Soc.* 26 (1920), 394–395.

[15] M. Neumann and S. Pati, On reciprocal eigenvalue property of a weighted tree, *Lin. Alg. Appl.* 438 (2013), 3817–3828.

[16] S. K. Panda and S. Pati, On the inverse of a class of bipartite graphs with unique perfect matching, *Electron. J. Lin. Alg.* 29 (2015), Art. 8.

[17] S. Pavlíková, A note on inverses of labeled graphs, *Australas. J. Combin.* 67 (2017), 222–234.

[18] S. Pavlíková and J. Krč-Jediný, On the inverse and dual index of a tree, *Lin. Multilin. Alg.* 28 (1990), 93–109.

[19] N. Ravishanker and D. K. Dey, *A first course in linear model theory*, Chapman and Hall, 2002.

[20] R. Simion and D.-S. Cao, Solution to a problem of C.D. Godsil regarding bipartite graphs with unique perfect matching, *Combinatorica* 9 (1989), 85–89.

[21] Y. Yang and D. Ye, Inverses of Bipartite Graphs, *Combinatorica* 38 (5) (2018), 125–1263.

[22] D. Ye, Y. Yang, B. Mandal and D. J. Klein, Graph invertibility and median eigenvalues, *Lin. Alg. Appl.* 513 (2017), 304–323.

(Received 11 Feb 2019; revised 7 July 2019)