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On the Vámos matroid, homogeneous pregeometries and dense pairs

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Abstract

We show that the Vámos matroid is not representable in pregeometries of rank greater than 4 satisfying a certain homogeneity condition, generalizing the result of Ingleton and Main on non-algebraicity of the Vámos matroid. We also show non-representability of the Vámos matroid in the pregeometry of "small closure" in a dense pair of geometric structures, a notion arising in model theory.

1 Introduction and preliminaries

In this short paper, we will be dealing with questions that are at the intersection of matroid theory and model theory. Among the main references on matroid theory and model theory are [13] and [10], respectively.

A set X together with a *closure operator cl* acting on its subsets,

$$cl : \mathcal{P}(X) \to \mathcal{P}(X),$$

is called a *pregeometry* (see, for example, [10], Section 4.6), if

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1. for any $A \subset X$, $A \subset cl(A)$;

2. for any $A, B \subset X$, if $A \subset cl(B)$ then $cl(A) \subset cl(B)$ (note that this implies monotonicity of cl and $cl(cl(A)) = cl(A)$ for any $A \subset X$);

3. for any $A \subset X$, $cl(A) = \bigcup_{B \subset A \text{ finite}} cl(B)$;

4. for any $A \subset X$ and $a, b \in X$, if $a \in cl(A \cup \{b\}) \backslash cl(A)$, then $b \in cl(A \cup \{a\})$.

If, in addition, $cl(\emptyset) = \emptyset$ we say that (X, cl) is a *geometry*. For any pregeometry (X, cl) we can consider its *associated geometry* (X^*, cl^*) by taking classes modulo the equivalence relation $cl(x) = cl(y)$, and letting cl^* be the closure operator on X^* induced by cl. For any pregeometry, (X, cl) and a subset $D \subset X$, we can define the *localization* cl_D of the operator cl over D by $cl_D(A) = cl(A \cup D)$ for any $A \subset X$. For any $D \subset X$, (X, cl_D) is also a pregeometry. When $D = \{d\}$ is a singleton, we denote $cl_{\{d\}}$ as cl_d.

A finite pregeometry is called a *matroid*. A finite geometry is called a *simple matroid*. Model theorists are mostly concerned with infinite pregeometries. The main examples of pregeometries in model theory arise from the algebraic closure operator, denoted acl, in a first-order structure M. Here, for a set $A \subset M$, $acl(A)$ is the set of all solutions of *algebraic* formulas (that is, formulas having finitely many solutions) in one variable with parameters in A. The operator acl always satisfies the properties (1-3) above, and in some "nice" model theoretic settings (e.g. strong mimimality, o-minimality), it also satisfies the "exchange property" (4), thus inducing a pregeometry.

In any pregeometry (matroid) (X, cl), the operator cl gives rise to the natural notion of independence, and one can define the *rank* $r(A)$ of a finite subset $A \subset X$ as the cardinality of the maximal independent subset of A. The definition of rank can be easily extended to finitely generated sets (that is, sets contained in the closure of finitely many of their elements). Note that for a finite A, $a \in cl(A)$ exactly when $r(A \cup \{a\}) = r(A)$. For the rank function associated with the pregeometry (X, cl_D) obtained by localizing over $D \subset X$, we use the notation r_D (or r_d when $D = \{d\}$). Note that if $r(D)$ is finite then $r_D(A) = r(A \cup D) - r(D)$. Rank satisfies the following properties for finitely generated subsets $A, B \subset X$, which can be used as an alternative way to define matroids:

1. $0 \leq r(A) \leq |A|$;

2. if $A \subset B \subset X$ then $r(A) \leq r(B)$;

3. $r(A \cap B) + r(A \cup B) \leq r(A) + r(B)$.

The last property is known as *submodularity*. A closed set (that is, a set A such that $cl(A) = A$) of finite rank is called a *flat*. Flats of rank 1 are called *points*, flat of rank 2 are called *lines* and flats of rank 3 are referred to as *planes*. Any vector

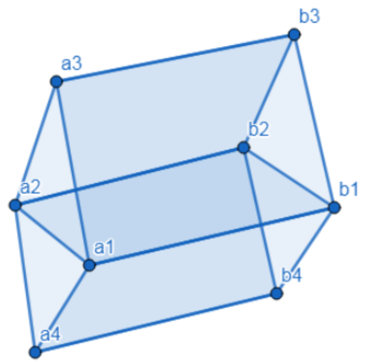

Figure 1.1: The Vámos Matroid.

space pregeometry $(V, Span_F)$ satisfies a stronger property of *modularity*: for any flats (finite dimensional subspaces) A, B in V we have

$$r(A \cap B) + r(A \cup B) = r(A) + r(B).$$

Two important notions used in matroid theory are those of linear and algebraic matroids. In order to define these, and some other notions, we will introduce the following terms. Let (X, cl) and (Y, cl) be pregeometries, and let $f : X \to Y$ be any map form X to Y. We say that f is *closure preserving*, if for any $a \in X$ and $A \subset X$ we have $a \in cl(A)$ if and only if $f(a) \in cl(f(A))$. If f is injective and closure-preserving, then we say that f is an *embedding* of (X, cl) into (Y, cl), and say that (X, cl) *embeds* into (Y, cl), and that (X, cl) is *representable* in (Y, cl). If $f : X \to Y$ is a closure-preserving bijection, we call f an *isomorphism* between the pregeometries (X, cl) and (Y, cl). An isomorphism $f : X \to X$ is called an *automorphism* of the pregeometry (X, cl).

A matroid (X, cl) is called *linear* or *(linearly) representable* over a field $\mathbb{F}$ if (X, cl) can be embedded into $(V, Span_{\mathbb{F}})$ where V is a vector space over a field $\mathbb{F}$ and $Span_{\mathbb{F}}$ is the operator of linear span in V. A matroid (X, cl) is called *algebraic* (over a field $\mathbb{F}$) if (X, cl) embeds into $(K, acl_{\mathbb{F}})$, where K is an algebraically closed extension of the field F and $acl_{\mathbb{F}}$ is the operator of algebraic closure over $\mathbb{F}$, that is, for any $A \subset K$, $acl_{\mathbb{F}}(A) = \overline{\mathbb{F}(A)}$. Note that any matroid linearly representable over $\mathbb{F}$ is also algebraic over $\mathbb{F}$, but the converse is not true (see [13], Section 6.7).

The smallest matroid not (linearly) representable over any field, known as the *Vámos matroid*, was introduced by Peter Vámos in 1968 (see [13], Example 2.1.25). It is a rank 4 matroid defined on a set of 8 elements

$$S = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\},$$

where any set of 3 or fewer elements is independent, and the only dependent sets of 4 elements are

$$\{a_1, b_1, a_2, b_2\}, \{a_1, b_1, a_3, b_3\}, \{a_1, b_1, a_4, b_4\}, \{a_2, b_2, a_3, b_3\}, \{a_2, b_2, a_4, b_4\}$$

(see Fig. 1.1).

In the 1975 paper [11], Ingleton and Main showed that the Vámos matroid is also not algebraic over any field. To show non-algebraicity of the Vámos matroid, the authors proved the following result which we call the *Ingleton-Main Lemma*. It essentially states that in the pregeometry $(K, cl) = (K, acl_{\mathbb{F}})$, any three non-coplanar but pairwise coplanar lines must meet at a common point.

Ingleton-Main Lemma. *Let K be an algebraically closed extension of a field $\mathbb{F}$, and (K, cl) the corresponding pregeometry induced by algebraic dependence over $\mathbb{F}$. Let $a_1, b_1, a_2, b_2, a_3, b_3 \in K$ be such that*

1. $r(a_i, b_i, a_j) = r(a_i, b_i, b_j) = r(a_i b_i a_j b_j) = 3$, *for all* $i \neq j$;
2. $r(a_1, b_1, a_2, b_2, a_3, b_3) = 4$.

Then there exists $e \in K$ such that $r(e) = 1$ and

$$e \in cl(a_1, b_1) \cap cl(a_2, b_2) \cap cl(a_3, b_3).$$

We refer to the configuration of 6 points satisfying the conditions (1) and (2) of the Ingleton-Main Lemma, as the *Ingleton-Main configuration.*

Note that in any pregeometry (matroid), if $r(a, b) = r(a', b') = 2$ then, by submodularity,

$$r(cl(a, b) \cap cl(a', b')) \leq 4 - r(a, b, a', b').$$

Thus, we have:

- two lines that are not coplanar, have no points of intersection (that is, $r(cl(a, b) \cap cl(a', b')) = 0$), and
- two distinct coplanar lines either intersect in a point or have a zero rank intersection (that is, $r(cl(a, b) \cap cl(a', b')) \leq 1$).

If the Vámos matroid were embeddable in $(K, cl) = (K, acl_{\mathbb{F}})$, identifying its elements with the elements of (K, cl), we would have

$$cl(a_1, b_1) \cap cl(a_2, b_2) = cl(e) = cl(a_3, b_3) \cap cl(a_4, b_4),$$

thus implying $r(a_3, b_3, a_4, b_4) = 3$, which contradicts the assumptions.

One of the key ingredients of the proof of Ingleton-Main Lemma is the observation that if each pair of the three lines has a point of intersection, then there is a common point for all three. We will include the proof of this result for completeness.

Lemma 1.1. *Suppose (X, cl) is a pregeometry and $a_1, b_1, a_2, b_2, a_3, b_3 \in X$ satisfy the conditions (1) and (2) of the Ingleton-Main Lemma (that is, form an Ingleton-Main configuration). Suppose $r(cl(a_1, b_1) \cap cl(a_2, b_2)) = 1$. Then*

$$r(cl(a_1, b_1) \cap cl(a_2, b_2) \cap cl(a_3, b_3)) = 1,$$

that is, there exists $e \in X$ *such that* $r(e) = 1$ *and*

$$cl(e) = cl(a_1, b_1) \cap cl(a_2, b_2) \cap cl(a_3, b_3) =$$

$$cl(a_1, b_1) \cap cl(a_2, b_2) = cl(a_1, b_1) \cap cl(a_3, b_3) = cl(a_2, b_2) \cap cl(a_3, b_3).$$

Proof. Let $e \in cl(a_1, b_1) \cap cl(a_2, b_2)$. Then also

$$e \in cl(a_1, b_1, a_3, b_3) \cap cl(a_2, b_2, a_3, b_3) = cl(a_3, b_3),$$

since these are two distinct planes both of which contain the line $cl(a_3, b_3)$. Thus, $e \in cl(a_3, b_3)$, as needed. □

Note that due to modularity, in the pregeometry of a vector space $(V, cl) = (V, Span_{\mathbb{F}})$, any two distinct coplanar lines have a point of intersection. Thus, by Lemma 1.1, the three lines in the Ingleton-Main configuration in a vector space will necessarily meet in a common point. This shows that the Vámos matroid is not linear. In the setting of algebraically closed fields, where modularity fails, the authors in [11] used Chevalley's Place Extension Theorem to establish pairwise intersection of the lines in the Ingleton-Main configuration.

The goal of this paper is to extend the Ingleton-Main Lemma to the wide class of "weakly homogenous" pregeometries that includes both the algebraic (induced by $acl_{\mathbb{F}}$) and vector space (induced by $Span_{\mathbb{F}}$) pregeometries. Our motivation comes from model theory, where homogeneity is quite common. Both algebraically closed fields and (infinite) vector spaces are examples of *strongly minimal structures*, first-order structure where sets defined by formulas in one variable (with parameters) are either finite or co-finite. The model theoretic algebraic closure operator acl (as defined above) in such a structure M induces a pregeometry that is homogeneous in the following sense: for any finitely generated set A and $b, c \notin acl(A)$, there exists an automorphism of M fixing A pointwise and mapping b to c. We will work with a somewhat weaker condition, which we call "weak homogeneity". In Section 2, we show that for any weakly homogeneous pregeometry (X, cl) of rank > 4 a certain version of the Ingleton-Main lemma holds, with the common intersection point found after localizing over a generic point. As a corollary, we show that the Vámos matroid cannot be embedded (represented) in (X, cl).

In Section 3, we introduce the notion of a "dense" subset D of a pregeometry (X, cl) (the notion motivated by the second author's work on unary predicate expansions of geometric theories). We show that the Ingleton-Main lemma holds for the pregeometry (X, cl_D) (known as the pregeometry of "small closure"). As a consequence, the Vámos matroid is not embeddable (representable) in (X, cl_D).

2 Weakly Homogeneous Pregeometries

In this section we introduce the notion of weakly homogeneous pregeometry, and prove that the Vámos matroid is not representable in any such pregeometry of rank greater than 4.

Definition 2.1. Let (M, cl) be a pregeometry (matroid). We say that (M, cl) is *homogeneous*, if for any $a_1, \ldots, a_n, b, c \in M$ such that $b, c \notin cl(a_1, \ldots, a_n)$, there exists an automorphism f of (M, cl) such that $f(b) = c$ and $f(a) = a$ for all $a \in cl(a_1, \ldots, a_n)$.

It is well-known that for any strongly minimal first-order structure M, the pregeometry (M, acl) is homogeneous. In particular, for any vector space V over a field (or a division ring) $\mathbb{F}$, $(V, Span_{\mathbb{F}})$ is homogeneous, and for any algebraically closed field K extending a field $\mathbb{F}$, the pregeometry $(K, acl_{\mathbb{F}})$ is homogeneous. Although vector spaces and algebraically closed fields are the main classical examples, the class of strongly minimal structures is much wider. For example, it includes "exotic" structures obtained by a variant of the Fraisse limit known as the Hrushovski construction that are nontrivial but do not allow any definable group operation. Thus, homogeneity is not necessarily associated with the classical algebraic structures. We will be working with a slightly weaker version of homogeneity sufficient for our purposes.

Definition 2.2. Let (M, cl) be a pregeometry (matroid). We say that (M, cl) is *weakly homogeneous*, if for any $a_1, \ldots, a_n, b, c \in M$ such that $b, c \notin cl(a_1, \ldots, a_n)$, there exists a closure preserving bijection $f : cl(a_1, \ldots, a_n, b) \to cl(a_1, \ldots, a_n, c)$ such that $f(b) = c$ and $f(a) = a$ for all $a \in cl(a_1, \ldots, a_n)$.

Proposition 2.3. *Suppose (M, cl) is a weakly homogeneous pregeometry of rank > 4. Then the Vámos matroid is not embeddable (representable) in (M, cl).*

Proof. For the sake of contradiction, assume the Vámos matroid is embeddable in (M, cl). So there exists a subset $S = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\}$ of M which has the following properties:

1. $r(a_i, b_i, a_j) = r(a_i, b_i, b_j) = r(a_i, b_i, a_j, b_j) = 3$ for all $i \neq j$ except for $(i, j) = (3, 4)$;
2. $r(a_3, b_3, a_4, b_4) = 4$;
3. $r(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4) = 4$.

Choose $d \in (M, cl)$ such that $d \notin cl(S)$. This is possible since (M, cl) has rank greater than 4 and $r(S) = 4$. Note that $r(d) = 1$. Note also that since $d \notin cl(S)$, $r_d(A) = r(A)$ for any $A \subset S$, so the properties (1-3) above hold for r_d as well.

Note that each of $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ and $\{a_1, b_1, a_2, b_2, a_4, b_4\}$ forms an Ingleton-Main configuration with respect to both cl and cl_d.

Consider the subset $\{a_1, b_1, a_2, b_2\}$ of S. Since $d, b_3 \notin cl(a_1, b_1, a_2, b_2)$, by weak homogeneity, there exists a bijection

$$f : cl(a_1, b_1, a_2, b_2, b_3) \to cl(a_1, b_1, a_2, b_2, d)$$

preserving cl such that $f(b_3) = d$ and $f(x) = x$ for any $x \in cl(a_1, b_1, a_2, b_2)$. Let $e = f(a_3)$; then we have

$$a_3 \in cl(a_1, b_1, b_3);$$
$$e = f(a_3) \in cl(f(a_1), f(b_1), f(b_3)) = cl(a_1, b_1, d) = cl_d(a_1, b_1);$$
$$e \in cl_d(a_1, b_1).$$

Similarly,

$$a_3 \in cl(a_2, b_2, b_3);$$
$$e = f(a_3) \in cl(f(a_2), f(b_2), f(b_3)) = cl(a_2, b_2, d) = cl_d(a_2, b_2);$$
$$e \in cl_d(a_2, b_2).$$

Thus, $e \in cl_d(a_1, b_1) \cap cl_d(a_2, b_2)$. Note also that $a_3 \notin cl(b_3)$ and hence $e = f(a_3) \notin cl(f(b_3)) = cl(d)$, that is, $r_d(e) = 1$.

By Lemma 1.1, $e \in cl_d(a_3, b_3)$ and $e \in cl_d(a_4, b_4)$. Thus,

$$r_d(cl_d(a_3, b_3) \cap cl_d(a_4, b_4)) = 1,$$

and, therefore,

$$r_d(a_3, b_3, a_4, b_4) \le r_d(a_3, b_3) + r_d(a_4, b_4) - r_d(cl_d(a_3, b_3) \cap cl_d(a_4, b_4)) = 2 + 2 - 1 = 3.$$

Thus, $r_d(a_3, b_3, a_4, b_4) = 3$ and since r coincides with r_d on subsets of S, we conclude that $r(a_3, b_3, a_4, b_4) = 3$, contradicting $r(a_3, b_3, a_4, b_4) = 4$. □

Remark 2.4. We have essentially shown the following variant of the Ingleton-Main Lemma for a weakly homogeneous geometry (M, cl): if the subset $S = \{a_1, b_1, a_2, b_2, a_3, b_3\} \subset M$ forms an Ingleton-Main configuration, then for any $d \in M \backslash cl(S)$, there exists $e \in M$ such that $r_d(e) = 1$ and

$$\begin{aligned} cl_d(e) &= cl_d(a_1, b_1) \cap cl_d(a_2, b_2) \cap cl_d(a_3, b_3) \\ &= cl_d(a_1, b_1) \cap cl_d(a_2, b_2) \\ &= cl_d(a_1, b_1) \cap cl_d(a_3, b_3) \\ &= cl_d(a_2, b_2) \cap cl_d(a_3, b_3). \end{aligned}$$

In other words, the three lines in the Ingleton-Main configuration meet after localizing over a "generic point".

Remark 2.5. In the case when (M, cl) is the pregeometry induced by the algebraic closure in a strongly minimal structure M, the non-representability of the Vámos matroid in (M, cl) follows from Lemma 6(b) in [15]. The following quick proof in the strongly minimal case, using canonical bases, was suggested to the second author by David Evans (see [14] for the notions of canonical base and U-rank). First, using the notation from the proof of Proposition 2.3 and working in M^{eq}, we have

$$c = Cb(a_3b_3/a_1a_2b_1b_2) = Cb(a_3b_3/a_1b_1) = Cb(a_3b_3/a_2b_2).$$

Note that $U(c) \ge 1$ and $c \in acl^{eq}(a_1b_1) \cap acl^{eq}(a_2b_2)$. It follows that $U(c) = 1$. Then also $c \in acl^{eq}(a_3b_3) \cap acl^{eq}(a_4b_4)$, contradicting $U(a_3b_3a_4b_4) = 4$.

3 Dense pairs

In this section we will prove an analogue of the Ingleton-Main Lemma for the localization of a pregeometry (M, cl) over a subset $D \subset M$ satisfying a certain "richness" property we call *density*. The notion of dense subset in the sense of pregeometry is motivated by the model theoretic notion of *lovely pairs of geometric structures*, as introduced in [1]. A first-order theory T is called *geometric* if in all of its models the algebraic closure operator *acl* satisfies the exchange property (hence, induces a pregeometry) and the theory eliminates the quantifier $\exists^\infty$, that is, for every formula $\phi(x, \vec{y})$ in the language of T there exists a number n such that if $\vec{a}$ is a tuple in a model M of T, and $\phi(x, \vec{a})$ has more than n solution in M, then it has infinitely many solutions in M. Models of geometric theories are called *geometric structures*. Examples include strongly minimal structures (e.g. pure infinite set, vector spaces, algebraically closed fields), o-minimal structures (e.g. real closed field, its reducts and some "tame" expansions), pseudofinite fields, and the field of p-adics. We say that a subset D of a geometric structure M is *dense* in M, if any infinite definable (with parameters in M) subset $\phi(M, \vec{a})$ of M intersects D. We call D *co-dense* if no such $\phi(M, \vec{a})$ is contained in $acl(\vec{a}D)$. If, in addition, $D = acl(D)$ (and the pair (M, D) is sufficiently saturated), we refer to (M, D) as a *lovely pair*. In the o-minimal setting, the notions of density and co-density are equivalent to the classical notions density and co-density defined in terms of order. Dense pairs in this context were studied in [7]. Dense/co-dense expansions have been studied actively in recent years, both in o-minimal [6, 8, 9] and general geometric contexts [3, 4, 5]. With any pair (M, D) where $D \subset M$ and M is geometric, we can associate the pregeometry (M, scl) of the *small closure* operator $scl(-) = acl(- \cup D)$ (that is, the localization of *acl* over D). The (pre)geometry of small closure was of special interest in the study of the linear/nonlinear dichotomy. In particular, it was shown in [2] that the property of *weak one-basedness* of M (the appropriate linearity notion in the general context of geometric theories) is equivalent to (M, scl) being modular whenever D is a dense co-dense subset of M (and (M, D) is sufficiently saturated). Moreover, in this case, the associated geometry of (M, scl) is either trivial, or splits into a disjoint union of infinite dimensional projective geometries over division rings (and possibly, a trivial geometry). The role of (M, scl) is less clear in the nonlinear case (e.g. in the field setting).

In the proof of non-algebraicity of the Vámos matroid in [11], the main argument can, in fact, be formulated in the language of pairs and derived using model theory. Namely, given an Ingleton-Main configuration $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ in an algebraically closed extension K of a field $\mathbb{F}$ (where $\mathbb{F}$ can also be assumed to be algebraically closed), one can find $c \in \mathbb{F}$ and $d \in K \setminus \mathbb{F}$ with the property that $d \in acl(a_1, b_1, c, \vec{f}) \cap acl(a_2, b_2, c, \vec{f})$, witnessed by the same polynomials as the ones that witness $a_3 \in acl(a_1, b_1, b_3, \vec{f}) \cap acl(a_2, b_2, b_3, \vec{f})$, where $\vec{f}$ is a tuple in $\mathbb{F}$. The existence of such c and d can be derived from the model theoretic fact that all proper pairs of algebraically closed fields of the given characterstic are elementarily equivalent, and therefore, $(K, \mathbb{F})$ is elementarily equivalent to a pair where the smaller field has an infinite transcendence degree, thus, any $c \notin acl(a_1, b_1, a_2, b_2, \vec{f})$ will work in that pair. Also, note that the closure operator in the Ingleton-Main lemma is precisely the small closure operator in the pair $(K, \mathbb{F})$, and $\mathbb{F}$ is dense in K.

These observations motivate us to examine whether Ingleton-Main lemma will hold for *scl* in an arbitrary geometric structure expanded with a dense subset. However, for the sake of generality, we will work in the context of pregeometries rather than first-order structures, and therefore, will consider the version of density appropriate for the context of pregeometry. This property is satisfied by (M, acl) for any sufficiently saturated pair (M, D) where $D = acl(D)$ is dense and M is geometric (in this case, D is an elementary substructure of M).

Definition 3.1. We say that a subset D of any pregeometry M is *dense* in M if for any finite subset A of M and any $b \notin cl(A)$, there exists $b' \in D$, and a bijection

$$f : cl(Ab) \to cl(Ab'),$$

such that

1. $f(a) = a$ for all $a \in A$;
2. $f(b) = b'$;
3. f is closure preserving.

Note that if (M, cl) is a weakly homogeneous pregeometry and $D = cl(D) \subset M$ has an infinite rank, then D is dense in (M, cl).

Proposition 3.2. *Let D be a dense subset of a pregeometry M. Let $a_1, b_1, a_2, b_2, a_3, b_3 \in M$ be such that*

1. $r_D(a_i, b_i, a_j) = r_D(a_i, b_i, b_j) = r_D(a_ib_ia_jb_j) = 3$, *for all* $i \neq j$;
2. $r_D(a_1, b_1, a_2, b_2, a_3, b_3) = 4$.

Then there exists $e \in M$ such that $r_D(e) = 1$ and

$$e \in cl_D(a_1, b_1) \cap cl_D(a_2, b_2) \cap cl_D(a_3, b_3).$$

Proof. We may assume $D = cl(D)$. Let $\vec{d}$ be a tuple in D such that $a_3 \in cl(a_1, b_1, b_3, \vec{d})$ and $a_3 \in cl(a_2, b_2, b_3, \vec{d})$. By density of D, there exists $d' \in D$ and a bijection $f : cl(a_1, b_1, a_2, b_2, b_3, \vec{d}) \to cl(a_1, b_1, a_2, b_2, d', \vec{d})$ such that:

1. f fixes $a_1, b_1, a_2, b_2, \vec{d}$ pointwise;
2. $f(b_3) = d'$;
3. f preserves closure.

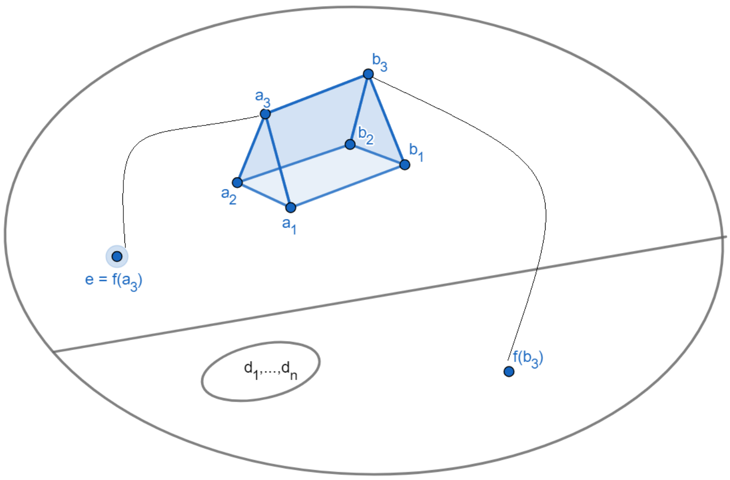

Figure 3.1: Ingleton-Main Lemma for localization over a dense subset.

We know:

$$a_3 \in cl(a_1, b_1, b_3, \vec{d});$$
$$a_3 \in cl(a_2, b_2, b_3, \vec{d}).$$

We can now apply the bijection f. Recall that $f(b_3) = d'$. Let $e = f(a_3)$. Then we have:

$$e = f(a_3) \in cl(a_1, b_1, d', \vec{d});$$
$$e = f(a_3) \in cl(a_2, b_2, d', \vec{d}).$$

Now, since $d' = f(b_3) \in D$ and $\vec{d} \in D$ we have:

$$e \in cl_D(a_1, b_1) \cap cl_D(a_2, b_2).$$

By Lemma 1.1, $e \in cl_D(a_3, b_3)$.

We need to show that $r_D(e) = 1$, or, equivalently, $e \notin D$ (recall that $D = cl(D)$). For contradiction, suppose $e \in D$. By the exchange property,

$$a_1 \in cl(b_1, a_3, b_3, \vec{d});$$
$$a_3 \in cl(a_1, b_1, b_3, \vec{d}) \backslash cl(b_1, b_3, \vec{d}).$$

Applying f again, we get:

$$f(a_1) = a_1 \in cl(f(a_3), f(b_1), f(b_3), f(\vec{d}\,));$$
$$a_1 \in cl(e, b_1, d', \vec{d}).$$

Note that $e, d', \vec{d} \in D$, which means $a_1 \in cl_D(b_1)$, a contradiction. Therefore, $e \notin D$. Hence, $r_D(e) = 1$. □

Corollary 3.3. *Suppose (M, cl) is a pregeometry of rank > 4. Let D be a dense subset of M. Then the Vámos matroid is not embeddable (representable) in (M, cl_D).*

Proof. If a Vámos matroid is embeddable in (M, cl_D) then there exists a subset $S = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\}$ of M, which has the following properties:

1. $r_D(a_i, b_i, a_j) = r_D(a_i, b_i, b_j) = r_D(a_i, b_i, a_j, b_j) = 3$ for all $i \neq j$ except for $(i, j) = (3, 4)$;
2. $r_D(a_3, b_3, a_4, b_4) = 4$;
3. $r_D(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4) = 4$.

Then by Proposition 3.2, there exists

$$e \in cl_D(a_1, b_1) \cap cl_D(a_2, b_2) \cap cl_D(a_3, b_3)$$

with $r_D(e) = 1$. By Lemma 1.1, we also have $e \in cl_D(a_4, b_4)$. Then, as in the proof of Proposition 2.3, $r_D(a_3, b_3, a_4, b_4) = 3$, a contradiction.

□

4 Open questions

Note that the proof of non-representability of the Vámos matroid in a weakly homogeneous pregeometry requires the rank to be greater than 4. Thus, the following is a natural question.

Question 4.1. *Is the Vámos matroid representable in a weakly homogeneous pregeometry of rank 4?*

Although the Vámos matroid is not itself homogeneous, it is not obvious that we cannot extend it to a homogeneous matroid of rank 4 by adding (possibly infinitely many) points to the original 8 element set. Note also that for any weakly homogeneous pregeometry (M, cl) and a closed subset A of M, and the restriction $cl|_A$ of the operator cl to the subsets of A, $(A, cl|_A)$ also forms a weakly homogeneous pregeometry. Thus, a negative answer to the question above would give another proof of Proposition 2.3.

A natural question regarding the pregeometry (M, cl_D) where D is dense in (M, cl) is how far this pregeometry is from being homogeneous. In the case when T is a weakly one-based geometric theory (see [2]) we know that for any dense-codense pair (M, P) of models of T (also known as a "lovely pair"), the associated geometry of $(M, scl) = (M, acl_{P(M)})$ splits in a disjoint union of homogeneous geometries (either trivial or projective over a division ring).

Question 4.2. *Suppose (M, P) is a dense-codense pair of geometric structures. Is the associated geometry of (M, scl) a disjoint union of homogeneous geometries?*

We end this section with the following question.

Question 4.3. *Does there exist a (finite) non-algebraic matroid representable in a (weakly) homogeneous pregeometry?*

In [12], Lindström shows that the (rank 3) "non-Desargues" matroid (see [13], p.167, Fig. 6.4) is not algebraic. His proof works for any pregeometry satisfying the Ingleton-Main lemma and the following "limited" homogeneity condition: for any line L (closed set of rank 2) and a point a not on L, there exists a' not in the plane generated by L and a and a closure preserving bijection

$$f : cl(La) \to cl(La')$$

fixing L pointwise. Clearly, these conditions hold in (M, cl_D), where M is any homogeneous pregeometry and D is a closed subset of M of infinite rank. It follows that the non-Desargues matroid is not representable in any infinite dimensional homogeneous geometry.

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