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Maxiumum packing of inside perfect 8-cycle systems

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Abstract

For an m-cycle C, an inside m-cycle of C is a cycle that is on the same vertex set and edge-disjoint from C. In an m-cycle system, $(\mathcal{X}, \mathcal{C})$, if inside m-cycles can be chosen -one for each cycle- to form another m-cycle system, then $(\mathcal{X}, \mathcal{C})$ is called an inside perfect m-cycle system. Inside perfect cycle systems can be considered as generalisations of i-perfect cycle systems. Cycle packings are generalisations of cycle systems that may have leaves after decomposition. In this paper, we prove that an inside perfect maximum packing of K_n with 8-cycles of order n exists for each $n \geq 8$. We also construct a maximum 8-cycle packing of order n which is not inside perfect for each $n \geq 10$.

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1 Introduction

One of the oldest graph decomposition problems involves decomposing the complete graph K_n into edge disjoint cycles. If all the cycles have uniform length, say m, then this decomposition is called an *m-cycle system*. More formally, we denote an m-cycle system of order n by a pair $(\mathcal{X}, \mathcal{C})$ where $\mathcal{X}$ is an n-element set and $\mathcal{C}$ is a collection of edge-disjoint m-cycles which partitions the edge-set of K_n with vertex set $\mathcal{X}$. To have an m-cycle system of order n, we need n to be odd and m to divide the total number of edges. Namely,

(i) $n \geq m \geq 3$,

(ii) $n \equiv 1 \pmod 2$, and

(iii) $n(n-1)/(2m)$ is an integer.

These conditions are shown to be sufficient when n and m have the same parity in [4], and when n and m have different parity in [16]. The interested reader is referred to [6] for an alternative proof. Given m, the set of all n satisfying these conditions is called the *spectrum* of the m-cycle system.

For a cycle C, let $c(i)$ be the set of edges that connects the vertices that are distance i apart in C. For example, if C is a 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6)$ then $c(2)$ consists of two 3-cycles, namely; (x_1, x_3, x_5) and (x_2, x_4, x_6). If C is a 5-cycle, then $c(2)$ is another 5-cycle. Given an m-cycle system $(\mathcal{X}, \mathcal{C})$, if the collection of $c(i)$ related to each C forms another cycle system (not necessarily an m-cycle system), then $(\mathcal{X}, \mathcal{C})$ is called an *i-perfect m-cycle system*. There are many results on i-perfect m-cycle systems. The interested reader is referred to [1, 3, 5, 7, 8, 11, 14, 15] for results on i-perfect cycle systems.

Observe that when $\gcd(i, m) = 1$, in an i-perfect m-cycle system the $c(i)$s form an m-cycle system too. When $\gcd(i, m) \neq 1$ then the cycles in $c(i)$s cannot be m-cycles. That is why we will generalize the idea of i-perfect systems to inside perfect systems. We want to replace each m-cycle C in the system with another m-cycle C' related with C, and then to see if the collection of the new m-cycles still gives an m-cycle system or not. We can for example relate C' with C by choosing it on the same vertex set. Given an m-cycle C, an *edge-disjoint* m-cycle C' on the same vertex set is called an *inside m-cycle* of C.

An m-cycle system $(\mathcal{X}, \mathcal{C})$ is called an *inside perfect m-cycle system* if it is possible to choose an inside m-cycle from each m-cycle in $\mathcal{C}$, so that the resulting collection of m-cycles is also an m-cycle system. For example, in [13] the authors found the spectrum of inside perfect 6-cycle systems and named them as almost 2-perfect 6-cycle systems.

As seen before, m-cycle systems do not exist for all orders. For all n not in the spectrum of m-cycle systems, we consider maximum packings with m-cycles. A *cycle packing* of the complete graph K_n is a triple $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ where $\mathcal{X}$ is the vertex set, $\mathcal{C}$ is a collection of edge-disjoint cycles from K_n, and the leave $\mathcal{L}$ is the collection of the edges in K_n not belonging to any of the cycles in $\mathcal{C}$. When $|\mathcal{L}|$ is the smallest possible, then $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ is called a maximum packing. If n is not in the spectrum of the m-cycle system, i.e; when it is not possible to decompose K_n completely into m-cycles, we study maximum packings of m-cycles of K_n.

In [13], Lindner et al. also proved the existence of an inside perfect maximum packing of K_n with 6-cycles for each admissible n. In [12], Lindner and Meszka considered the existence of inside perfect minimum coverings of K_n with 6-cycles.

In this paper we study inside perfect 8-cycle systems. Even though for $m = 5$ each 5-cycle has a unique inside 5-cycle, when $m > 5$ there is an increasing number of possible inside cycles for a given cycle. For example, there are three possible inside 6-cycles for each 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6)$, namely, $(x_1, x_3, x_5, x_2, x_6, x_4)$, $(x_1, x_3, x_6, x_4, x_2, x_5)$ and $(x_1, x_4, x_2, x_6, x_3, x_5)$. An 8-cycle has 177 possible inside 8-cycles. Let $(\mathcal{X}, \mathcal{C})$ be an 8-cycle system, and let $\mathcal{C}'$ be a collection of inside 8-cycles, one from each of the cycles in $\mathcal{C}$. If $(\mathcal{X}, \mathcal{C}')$ is an 8-cycle system, we say that $(\mathcal{X}, \mathcal{C})$ is inside perfect. Inside perfect m-cycle decompositions of other graphs may be defined similarly.

Then we extend the problem to inside perfect maximum packings of K_n with 8-cycles by constructing a maximum packing $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ of K_n for every $n \geq 8$ with 8-cycles so that $(\mathcal{X}, \mathcal{C}', \mathcal{L})$ is also a maximum packing, where $\mathcal{C}'$ is a collection of inside 8-cycles of the 8-cycles of $\mathcal{C}$, and $\mathcal{L}$ is the leave of the maximum packing. We call such a maximum packing of order n, an *inside perfect maximum packing* with 8-cycles.

For all n not in the spectrum of 8-cycle systems, we show the existence of inside perfect maximum packings with 8-cycles in Theorem 3.14.

We also answer the opposite problem: given an 8-cycle maximum packing $(\mathcal{X}, \mathcal{C}, \mathcal{L})$, is it always possible to choose an inside 8-cycle for each 8-cycle in $\mathcal{C}$ so that the resulting collection of inside 8-cycles is an 8-cycle maximum packing? In other words, are all the 8-cycle maximum packings inside perfect? The answer to this question is no, except for the orders $n = 8$ and $n = 9$.

In the third section, we construct inside perfect maximum packings of K_n with 8-cycles for all $n \geq 8$, therefore we show that they exist for all admissible n. And in the fourth section, proofs and observations from a comprehensive computer search are given. We construct maximum packings of K_n with 8-cycles that are not inside perfect for all $n \geq 10$.

2 Preliminary results

We start by introducing the results that are used throughout the paper. From now on, inside perfect is abbreviated as IP for brevity. $K_{n,m}$ represents a bipartite graph with parts of size n and m and $K_r \setminus K_s$ represents the graph difference of K_r and K_s, that is the graph obtained from K_r by removing the edges of a subgraph isomorphic to K_s. A bowtie is a 5-vertex connected graph consisting of two triangles with a common vertex.

The following table gives leaves we used for the maximum packings with 8-cycles (see [9] and [10]).

Spectrum for maximum packing with 8-cycles	Leave
1 (mod 16)	$\varnothing$
3 (mod 16)	C_3
5 (mod 16)	K_5
7 (mod 16)	C_5
9 (mod 16)	C_4
11 (mod 16)	$C_3 \cup C_4$
13 (mod 16)	bowtie
15 (mod 16)	$C_4 \cup C_5$
0, 2, 8, 10 (mod 16)	1-factor
4, 6, 12, 14 (mod 16)	$K_4 \cup$ a 1-factor on the remaining vertices

Table 1: Maximum packings with 8-cycles

Throughout the paper when we list the corresponding inside 8-cycles in $\mathcal{C}'$ we obey the order of the original 8-cycles in $\mathcal{C}$.

Lemma 2.1 *There exists an IP 8-cycle decomposition of $K_{4t,4s}$, for all $t, s \in \mathbb{Z}^+$.*

Proof Let $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$ be partitions of the vertex set of $K_{4,4}$. Consider $\mathcal{C} = \{(x_0, y_0, x_1, y_1, x_2, y_2, x_3, y_3), (x_0, y_2, x_1, y_3, x_2, y_0, x_3, y_1)\}$, and the inside cycles as $\mathcal{C}' = \{(x_1, y_2, x_0, y_1, x_3, y_0, x_2, y_3), (x_1, y_1, x_2, y_2, x_3, y_3, x_0, y_0)\}$ to get an IP 8-cycle decomposition of $K_{4,4}$.

Next let $\mathcal{X} = \{x_0, x_1, \ldots, x_{4t-1}\}$ and $\mathcal{Y} = \{y_0, y_1, \ldots, y_{4s-1}\}$ be partitions of the vertex set of $K_{4t,4s}$, where $X_i = \{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, $Y_j = \{y_{4j}, y_{4j+1}, y_{4j+2}, y_{4j+3}\}$ for $i = 0, 1, 2, \ldots, t-1$, $j = 0, 1, \ldots, s-1$ and $t, s \in \mathbb{Z}^+$. Placing an IP 8-cycle decomposition of $K_{4,4}$ on the vertex set $X_i \cup Y_j$ for each pair i, j gives us an IP 8-cycle decomposition of $K_{4t,4s}$. $\square$

Lemma 2.2 *There exists an IP 8-cycle decomposition of $K_{4t,4s+2}$, for all $t, s \in \mathbb{Z}^+$.*

Proof Let $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4, y_5\}$ be partitions of the vertex set of $K_{4,6}$. Consider $\mathcal{C} = \{(x_0, y_4, x_1, y_1, x_2, y_2, x_3, y_3), (x_0, y_0, x_1, y_3, x_2, y_4, x_3, y_5), (x_0, y_1, x_3, y_0, x_2, y_5, x_1, y_2)\}$ and $\mathcal{C}' = \{(x_1, y_2, x_0, y_1, x_3, y_4, x_2, y_3), (x_1, y_4, x_0, y_3, x_3, y_0, x_2, y_5), (x_1, y_1, x_2, y_2, x_3, y_5, x_0, y_0)\}$ to get an IP 8-cycle decomposition of $K_{4,6}$.

Next, let $\mathcal{X} = \{x_0, x_1, \ldots, x_{4t-1}\}$ and $\mathcal{Y} = \{y_0, y_1, \ldots, y_{4s+1}\}$ be partitions of the vertex set of $K_{4t,4s+2}$, where $X_i = \{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, $Y_0 = \{y_0, y_1, y_2, y_3, y_4, y_5\}$, $Y_j = \{y_{4j+2}, y_{4j+3}, y_{4j+4}, y_{4j+5}\}$ for $i = 1, \ldots, t-1$, $j = 1, 2, \ldots, s-1$ and $t, s \in \mathbb{Z}^+$. Placing an IP 8-cycle decomposition of $K_{4,4}$ on the vertex set $X_i \cup Y_j$ for each pair $i = 0, 1, \ldots, t-1$, $j = 1, 2, \ldots, s-1$ and an IP 8-cycle decomposition of $K_{4,6}$ on the vertex set $X_i \cup Y_0$ for each $i = 0, 1, \ldots, t-1$ gives us an IP 8-cycle decomposition of $K_{4t,4s+2}$. $\square$

Lemma 2.3 *If there exist an IP 8-cycle system of order $r+1$ and an IP 8-cycle decomposition of $K_{r,s}$, then there exists an IP 8-cycle decomposition of $K_{r+s+1} \setminus K_{s+1}$.*

Proof Let $\mathcal{X} = \{\infty\} \cup \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$. Placing an IP 8-cycle system of order $r+1$ on $\{\infty\} \cup \{x_1, x_2, \ldots, x_r\}$ and an IP 8-cycle decomposition of $K_{r,s}$ on $\{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$ gives an IP maximum 8-cycle decomposition of $K_{r+s+1} \setminus K_{s+1}$, where the vertex set of K_{s+1} is $\{\infty\} \cup \{y_1, y_2, \ldots, y_s\}$. □

Lemma 2.4 *If there exist an IP maximum 8-cycle packing of order r and of order s with a 1-factor leave and an IP 8-cycle decomposition of $K_{r,s}$, then there exists an IP maximum 8-cycle packing of order $r+s$ with a 1-factor leave.*

Proof Let $\mathcal{X} = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$. Placing an IP maximum 8-cycle packing of order r on $\{x_1, x_2, \ldots, x_r\}$, an IP maximum 8-cycle packing of order s on $\{y_1, y_2, \ldots, y_s\}$, and an IP 8-cycle decomposition of $K_{r,s}$ on $\{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$ gives an IP maximum 8-cycle packing of order $r+s$. The leave is the union of the 1-factor leaves of packings on $\{x_1, x_2, \ldots, x_r\}$ and $\{y_1, y_2, \ldots, y_s\}$, which is clearly a 1-factor. □

Lemma 2.5 *If there exist an IP maximum 8-cycle packing of order r and an IP 8-cycle decomposition of $K_{r,s}$, then there exists an IP maximum 8-cycle packing of $K_{r+s} \setminus K_s$.*

Proof Let $\mathcal{X} = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$. Placing an IP maximum 8-cycle packing of order r on $\{x_1, x_2, \ldots, x_r\}$ and an IP 8-cycle decomposition of $K_{r,s}$ on $\{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_s\}$ gives an IP maximum 8-cycle packing of $K_r \setminus K_s$. □

Now we can present the main construction that we used to construct inside perfect maximum 8-cycle packings.

Main Construction

Let H be a finite set with cardinality h and k be a positive integer. Then, let $X = H \cup \{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq 16\}$.

(1) On $H \cup \{(1,j) \mid 1 \leq j \leq 16\}$, place an IP maximum 8-cycle packing of order $16+h$.

(2) On each set $H \cup \{(i,j) \mid 1 \leq j \leq 16\}$, for $2 \leq i \leq k$, place an IP maximum 8-cycle packing of $K_{16+h} \setminus K_h$.

(3) For each $x, y \in \{1, 2, \ldots, k\}$ with $x < y$, place an IP 8-cycle decomposition of $K_{16,16}$ on $\{(x,j) \mid 1 \leq j \leq 16\} \cup \{(y,j) \mid 1 \leq j \leq 16\}$.

One can easily check that combining (1), (2), and (3) gives an IP maximum 8-cycle packing of order $16k+h$. □

Since a 3-perfect 8-cycle system of order n is an IP 8-cycle system of order n, the case $n \equiv 1 \pmod{16}$ is immediate from [2]. So we analyze the rest of the cases given in Table 1 to prove the existence of an IP maximum 8-cycle packing for each case. We first provide examples for small packings which are then used in the Main Construction.

3 Inside perfect maximum packings with 8-cycles

Example 3.1 *An IP maximum 8-cycle packing of order 8 exists.*

Let $\mathcal{X} = \mathbb{Z}_8$ and consider the maximum 8-cycle packing $\mathcal{C} = \{(1,4,3,6,5,2,0,7), (1,3,2,4,7,5,0,6), (1,2,7,6,4,5,3,0)\}$ with the leave $\mathcal{L} = \{\{1,5\},\{3,7\},\{2,6\}, \{4,0\}\}$. Then the inside cycles $\mathcal{C}' = \{(1,3,2,4,7,5,0,6), (1,0,3,5,4,6,7,2), (1,7,0, 2,5,6,3,4)\}$ with the same leave forms another maximum packing. Hence $(\mathcal{X},\mathcal{C},\mathcal{L})$ is an IP maximum 8-cycle packing.

Example 3.2 *An IP maximum 8-cycle packing of order 9 exists.*

Let $\mathcal{X} = \mathbb{Z}_9$, and consider $\mathcal{C} = \{(0,2,4,1,5,6,7,8), (0,4,3,1,6,8,5,7), (0,5,2,7,3, 8,4,6), (1,7,4,5,3,6,2,8)\}$, with the leave $\mathcal{L} = \{(0,1,2,3)\}$. Then choose the inside cycles as $\mathcal{C}' = \{(0,4,5,2,8,6,1,7), (0,6,4,1,8,7,3,5), (0,2,6,3,4,7,5,8), (1,5,6,7,2,4,8,3)\}$.

Example 3.3 *An IP maximum 8-cycle packing of order 10 exists.*

Let $\mathcal{X}=\mathbb{Z}_{10}$ and consider a maximum packing on $\mathcal{X}$ given by $\mathcal{C}=\{(0,1,3,2,4,5,8,6), (0,2,5,1,4,7,9,3), (0,4,6,3,8,9,2,7), (0,5,3,7,6,9,1,8), (1,6,2,8,4,9,5,7)\}$ with the leave $\mathcal{L} = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,0\}\}$.

Then one can choose the following inside cycles: $\mathcal{C}' = \{(0,4,1,5,3,6,2,8), (0,5,4,9,2,3,7,1), (0,3,9,7,6,8,4,2), (0,7,5,9,8,3,1,6), (1,8,5,2,7,4,6,9)\}$.

Example 3.4 *An IP maximum 8-cycle packing of order 11 exists.*

Let $\mathcal{X} = \mathbb{Z}_{11}$ and consider $\mathcal{C} = \{(0,3,1,4,2,5,7,6), (0,4,6,1,5,8,9,10), (0,5,3,2, 8,10,7,9), (0,7,1,9,2,10,3,8), (1,8,6,9,3,7,4,10), (2,6,10,5,9,4,8,7)\}$ with the leave $\mathcal{L} = \{(0,1,2),(3,4,5,6)\}$.

Then we can choose the following inside cycles: $\mathcal{C}' = \{(0,4,6,2,3,5,1,7), (0,6,8, 4,10,1,9,5), (0,3,7,5,10,2,9,8), (0,9,3,1,8,2,7,10), (1,6,10,3,8,7,9,4), (2,5,8, 10,9,6,7,4)\}$.

Example 3.5 *There exist IP maximum 8-cycle packings of orders $12,14,20$ and 22.*

Applying Lemma 2.5 for $r = 8,10,16$ and 18 with $s = 4$, we get an IP maximum 8-cycle packings of $K_{12} \setminus K_4$, $K_{14} \setminus K_4$, $K_{20} \setminus K_4$ and $K_{22} \setminus K_4$ with 1-factor leaves. But this is an IP maximum 8-cycle packing of orders $12,14,20$ and 22, where the leave is a K_4 and a set of independent edges saturating the remaining vertices.

Example 3.6 *An IP maximum 8-cycle packing of order 13 exists.*

To show this, let $\mathcal{X} = \{\infty\} \cup \{x_1, x_2, \ldots, x_8\} \cup \{y_1, y_2, y_3, y_4\}$. Place an IP maximum 8-cycle packing of order 9 (by Example 3.2) on $\{\infty\} \cup \{x_1, x_2, \ldots, x_8\}$, with the 4-cycle leave (x_1, x_2, x_3, x_4). Then place an IP maximum 8-cycle packing of order 8 (by Example 3.1) on $\{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$ with the 1-factor leave $\{\{x_1, x_3\}, \{x_2, x_4\}, \{y_1, y_3\}, \{y_2, y_4\}\}$. Then, place an IP 8-cycle decomposition of $K_{4,4}$ on $\{x_5, x_6, x_7, x_8\} \cup \{y_1, y_2, y_3, y_4\}$ by Lemma 2.1. The leave is the bowtie $(y_1, y_3, \infty), (y_2, y_4, \infty)$.

Example 3.7 *An IP maximum 8-cycle packing of order 15 exists.*

To see this, let $\mathcal{X} = \mathbb{Z}_{15}$ and consider the maximum packing $\mathcal{C} =$

$$\begin{array}{lll}\{(0,1,8,14,4,5,6,12), & (0,2,11,3,10,13,4,9), & (0,3,6,1,5,12,7,4),\\ (0,5,8,13,7,3,9,6), & (0,8,2,6,13,9,1,10), & (0,11,14,12,2,9,5,13),\\ (9,12,3,1,4,10,2,14), & (1,13,2,4,8,10,5,14), & (0,7,1,11,4,6,10,14),\\ (1,2,3,5,7,11,10,12), & (2,5,11,13,3,8,6,7), & (3,4,12,8,11,9,7,14)\},\end{array}$$

with the leave $\mathcal{L} = \{(6, 11, 12, 13, 14), (7, 8, 9, 10)\}$.

Now we can choose the inside cycles as $\mathcal{C}' =$

$$\begin{array}{lll}\{(0,4,6,8,12,1,14,5), & (0,11,4,10,2,9,13,3), & (0,1,3,4,12,6,5,7),\\ (0,8,3,5,9,7,6,13), & (0,2,1,13,8,10,6,9), & (0,14,9,11,13,2,5,12),\\ (9,1,10,12,14,3,2,4), & (1,8,2,14,10,13,4,5), & (0,6,1,4,14,7,11,10),\\ (1,7,2,12,3,10,5,11), & (2,11,8,5,13,7,3,6), & (3,11,14,8,4,7,12,9)\}.\end{array}$$

Example 3.8 *There exist IP maximum 8-cycle packings of orders 16 and 18.*

There exist IP maximum 8-cycle packings of orders 8 and 10 with 1-factor leaves by Examples 3.1 and 3.3 respectively. There also exist IP 8-cycle decomposition of $K_{8,8}$ and $K_{8,10}$ by Lemmas 2.1 and 2.2, respectively. Considering Lemma 2.4 for $r = 8$ with $s = 8$ for the order 16 and with $s = 10$ for the order 18 gives an IP maximum 8-cycle packing of orders 16 and 18 with a 1-factor leave.

Example 3.9 *An IP maximum 8-cycle packing of order 19 exists.*

Let $\mathcal{X} = \{\infty_1, \infty_2, \infty_3\} \cup \{x_1, x_2, \ldots, x_8\} \cup \{y_1, y_2, \ldots, y_8\}$. Place a copy of an IP maximum 8-cycle packing of order 11 on $\{\infty_1, \infty_2, \infty_3\} \cup \{x_1, x_2, \ldots, x_8\}$ and on $\{\infty_1, \infty_2, \infty_3\} \cup \{y_1, y_2, \ldots, y_8\}$, where the 3-cycle in the leaves is $(\infty_1, \infty_2, \infty_3)$ and the 4-cycles are (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4), respectively. Then place an IP maximum 8-cycle packing of order 8 on $\{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$ with the 1-factor leave $\{\{x_1, x_3\}, \{x_2, x_4\}, \{y_1, y_3\}, \{y_2, y_4\}\}$, and place IP 8-cycle decompositions of $K_{4,4}$ on $\{x_1, x_2, x_3, x_4\} \cup \{y_5, y_6, y_7, y_8\}$, on $\{x_5, x_6, x_7, x_8\} \cup \{y_1, y_2, y_3, y_4\}$ and on $\{x_5, x_6, x_7, x_8\} \cup \{y_5, y_6, y_7, y_8\}$. The necessary examples exist by Lemma 2.1 and Examples 3.1, 3.4.

Example 3.10 *An IP maximum 8-cycle packing of order 21 exists.*

Since an IP 8-cycle decomposition of $K_{16,4}$ exists by Lemma 2.1, considering $r = 16$ and $s = 4$ in Lemma 2.3 gives an IP 8-cycle decomposition of $K_{21} \setminus K_5$, which is the IP maximum 8-cycle packing of order 21 with a K_5 leave on $\{\infty\} \cup \{y_1, y_2, y_3, y_4\}$ as required.

Example 3.11 *An IP maximum* 8*-cycle packing of order* 23 *exists.*

Let $\mathcal{X} = \{\infty\} \cup \{x_1, x_2, \dots, x_8\} \cup \{y_1, y_2, \dots, y_{14}\}$. Place a copy of an IP maximum 8-cycle packing of order 9 on $\{\infty\} \cup \{x_1, x_2, \dots, x_8\}$ (using Example 3.2) and of order 15 on $\{\infty\} \cup \{y_1, y_2, \dots, y_{14}\}$ (using Example 3.7) with the 4-cycle leaves (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) and the 5-cycle leave $(y_9, y_{10}, y_{11}, y_{12}, y_{13})$. Then place an IP maximum 8-cycle packing of order 8 on $\{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$ (by Example 3.1) with the 1-factor leave $\{\{x_1, x_3\}, \{x_2, x_4\}, \{y_1, y_3\}, \{y_2, y_4\}\}$, place an IP 8-cycle decomposition of $K_{4,4}$ on $\{x_1, x_2, x_3, x_4\} \cup \{y_5, y_6, y_7, y_8\}$, on $\{x_5, x_6, x_7, x_8\} \cup \{y_1, y_2, y_3, y_4\}$, and on $\{x_5, x_6, x_7, x_8\} \cup \{y_5, y_6, y_7, y_8\}$, and finally place an IP 8-cycle decomposition of $K_{4,6}$ on $\{x_1, x_2, x_3, x_4\} \cup \{y_9, y_{10}, \dots, y_{14}\}$ and on $\{x_5, x_6, x_7, x_8\} \cup \{y_9, y_{10}, \dots, y_{14}\}$ which exists by Lemma 2.2.

Lemma 3.12 *For every* $n \equiv 3, 5, 7, 9, 11, 13, 15 \pmod{16}$ *with* $n \geq 9$*, there exists an IP maximum* 8*-cycle packing of order* n*.*

Proof

$n \equiv 3 \pmod{16}$: There exists an IP maximum 8-cycle packing of order 19 with a 3-cycle leave given by Example 3.9 which is also an IP 8-cycle decomposition of $K_{19} \setminus K_3$. An IP 8-cycle decomposition of $K_{16,16}$ also exists by Lemma 2.1 as before, and the result follows by the Main Construction considering $h = 3$.

$n \equiv 5 \pmod{16}$: There exists an IP maximum 8-cycle packing of order 21 with a K_5 leave given in Example 3.10. Then the result follows by the Main Construction considering $h = 5$.

$n \equiv 7 \pmod{16}$: There exist an IP maximum 8-cycle packing of order 23 with a 5-cycle leave by Example 3.11 and an IP 8-cycle decomposition of $K_{23} \setminus K_7$ by replacing $r = 16$ and $s = 6$ in Lemma 2.3. Then the result follows by the Main Construction considering $h = 7$.

$n \equiv 9 \pmod{16}$: There exist an IP maximum 8-cycle packing of order 9 with a 4-cycle leave by Example 3.2 and an IP 8-cycle decomposition of $K_{25} \setminus K_9$ by Lemma 2.3 with $r = 16$ and $s = 8$. The result follows by the Main Construction considering $h = 9$.

$n \equiv 11 \pmod{16}$: There exist an IP maximum 8-cycle packing of order 11 with a 3-cycle and a 4-cycle leave by Example 3.4 and an IP 8-cycle decomposition of $K_{27} \setminus K_{11}$ by Lemma 2.3 with $r = 16$ and $s = 10$. The result follows by the Main Construction considering $h = 11$.

$n \equiv 13 \pmod{16}$: There exist an IP maximum 8-cycle packing of order 13 with a bowtie leave by Example 3.6 and an IP 8-cycle decomposition of $K_{29} \setminus K_{13}$ by

Lemma 2.3 with $r = 16$ and $s = 12$. The result follows by the Main Construction considering $h = 13$.

$n \equiv 15 \pmod{16}$: There exist an IP maximum 8-cycle packing of order 15 with a 4-cycle and a 5-cycle leave by Example 3.7 and an IP 8-cycle decomposition of $K_{31} \setminus K_{15}$ by Lemma 2.3 with $r = 16$ and $s = 14$. Then the result follows by the Main Construction considering $h = 15$. □

Lemma 3.13 *For every even n with $n \geq 8$, there exists an IP maximum 8-cycle packing of order n.*

Proof

$n \equiv 0, 2, 8$ and $10 \pmod{16}$: There exist an IP maximum 8-cycle packing of orders $8, 10, 16$ and 18 by Examples 3.1, 3.3, and 3.8, respectively. An IP maximum 8-cycle packing of orders 24 and 26 exist by Lemma 2.4 for $r = 16$ with $s = 8$ and $s = 10$, respectively. For the same r and s in Lemma 2.5 we get IP maximum 8-cycle packings of $K_{24} \setminus K_8$ and $K_{26} \setminus K_{10}$, respectively. $K_{18} \setminus K_2$ comes from the packing of order 18 with a 1-factor leave. Now we have all the ingredients to use the Main Construction with $h = 0, 2, 8$ and 10.

$n \equiv 4, 6, 12$ and $14 \pmod{16}$: There exist an IP maximum 8-cycle packing of order $12, 14, 20$ and 22 by Example 3.5 and an IP maximum 8-cycle packing of $K_{20} \setminus K_4$, $K_{22} \setminus K_6$, $K_{28} \setminus K_{12}$ and $K_{30} \setminus K_{14}$ by Lemma 2.5 for $r = 16$ with $s = 4, 6, 12$, and 14, respectively. Then, the result follows by the Main Construction with $h = 4, 6, 12, 14$. □

Next we have the main result of this section.

Theorem 3.14 *There exists an IP maximum 8-cycle packing of order n for every $n \geq 8$.*

Proof Follows from Lemmas 3.12 and 3.13.

4 8-cycle packings that are not inside perfect

Computer search shows that not all maximum packings with 8-cycles are inside perfect. Even though all maximum packings with 8-cycles for orders 8 and 9 are inside perfect, starting at order 10, there are an increasing number of maximum packings with 8-cycles which do not carry this property. We generated about 2 million maximum packings with 8-cycles of order 10, and only 0.35% of them were not inside perfect. On the other hand, from our computer search for several small cases we observed that when n gets large, the percentage of 8-cycle packings which are not inside perfect tends to increase.

Below we give examples of 8-cycle maximum packings which are not inside perfect for small orders, then use these constructions to obtain 8-cycle maximum packings which are not inside perfect for all orders $n \geq 10$. The following examples were checked by exhaustive computer search.

Example 4.1 *There exist* 8*-cycle maximum packings which are not IP of orders* 10*,* 11*,* 12*,* 13*,* 15*,* 16 *and* 17*.*

These packings of order n are given on the set $\{0, \dots, n-1\}$ for each n.

Order 10: $\mathcal{C} = \{(0,2,1,3,4,6,5,7), (0,3,5,1,4,8,2,9), (0,4,2,7,9,3,6,8), (0,5,8,3,7,1,9,6), (1,6,2,5,9,4,7,8)\}$ with leave $\{\{0,1\}, \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}\}$.

Order 11: $\mathcal{C} = \{(0,3,1,4,2,5,7,6), (0,4,6,1,5,8,9,10), (0,5,3,2,8,10,7,9), (0,7,1,9,2,10,4,8), (1,8,6,9,4,7,3,10), (2,6,10,5,9,3,8,7)\}$ with leave $\{\{0,1\}, \{1,2\}, \{0,2\}, \{3,4\}, \{4,5\}, \{5,6\}, \{3,6\}\}$.

Order 12: $\mathcal{C} = \{(0,11,5,1,7,8,10,9), (0,4,6,2,9,11,3,10), (0,5,2,4,1,6,3,7), (0,6,5,3,4,7,2,8), (1,8,3,9,4,10,2,11), (1,9,5,7,11,8,6,10), (4,8,5,10,7,9,6,11)\}$ with leave K_4 on $\{0,1,2,3\}$ and edges $\{\{4,5\}, \{6,7\}, \{8,9\}, \{10,11\}\}$.

Order 13: $\mathcal{C} = \{(0,11,5,1,7,8,10,9), (0,4,6,2,9,11,3,10), (0,2,4,1,3,5,6,7), (0,3,6,1,8,2,5,12), (0,6,9,1,10,2,12,8), (1,11,2,7,3,8,4,12), (3,9,4,7,5,10,11,12), (4,10,6,8,9,12,7,11), (5,8,11,6,12,10,7,9)\}$ with leave $\{\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{0,5\}\}$.

Order 15: $\mathcal{C} = \{(0,1,8,14,4,5,6,12), (0,2,11,3,10,13,4,9), (0,3,6,1,5,12,7,4), (0,5,8,13,7,3,9,6), (0,8,2,6,13,9,1,10), (0,11,14,12,2,9,5,13), (9,12,3,1,4,10,2,14), (1,13,2,4,8,10,5,14), (0,7,1,11,4,6,10,14), (1,2,3,5,7,11,8,12), (2,5,11,13,3,8,6,7), (3,4,12,10,11,9,7,14)\}$ with leave $\{\{7,8\}, \{8,9\}, \{9,10\}, \{7,10\}, \{11,12\}, \{12,13\}, \{13,14\}, \{14,6\}, \{6,11\}\}$.

Order 16: Place a copy of the 8-cycle packing of order 12 given above on $\{0,\dots,11\}$ with the leave K_4 on $\{0,1,2,3\}$ and edges $\{\{4,5\}, \{6,7\}, \{8,9\}, \{10,11\}\}$, and place a copy of any 8-cycle packing of order 8 on points $\{0,1,2,3,12,13,14,15\}$ with a 1-factor leave. Finally, place a copy of an 8-cycle decomposition of $K_{8,4}$ on the bipartite graph with parts $\{4,5,6,7,8,9,10,11\}$ and $\{12,13,14,15\}$.

Order 17: The cyclic 8-cycle system with base block $\{0,16,1,4,8,13,2,9\}$.

Lemma 4.2 *If an* 8*-cycle packing* $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ *contains a subpacking* $(\mathcal{X}_0, \mathcal{C}_0, \mathcal{L}_0)$ *with* $\mathcal{L}_0 \subseteq \mathcal{L}$ *which is not IP, then the packing* $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ *is also not IP.*

Proof Let $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ be an IP 8-cycle packing with inside 8-cycle packing $(\mathcal{X}, \mathcal{C}', \mathcal{L})$. Let $(\mathcal{X}_0, \mathcal{C}_0, \mathcal{L}_0)$ be a subpacking of $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ and $\mathcal{C}_0'$ be the collection of inside 8-cycles of $\mathcal{C}_0$ in $\mathcal{C}'$. Then $(\mathcal{X}_0, \mathcal{C}_0', \mathcal{L}_0)$ should be a subpacking of $(\mathcal{X}, \mathcal{C}', \mathcal{L})$; as both C_0 and C_0' are on the same vertex set $\mathcal{X}_0$, they both have the same number of 8-cycles and $\mathcal{L}_0 \subseteq \mathcal{L}$. So if $(\mathcal{X}_0, \mathcal{C}_0, \mathcal{L}_0)$ is not IP then $(\mathcal{X}, \mathcal{C}, \mathcal{L})$ cannot be IP. □

Example 4.3 *There exist* 8*-cycle maximum packings of orders* 14, 18, 20, 21 *and* 22 *that are not IP.*

While obtaining an 8-cycle maximum packing of order 18 as in Example 3.8, use an 8-cycle packing of order 10 given in Example 4.1 that is not IP. While obtaining 8-cycle maximum packings of orders $14, 20$ and 22 as in Example 3.5, use 8-cycle packings of orders 10, 16 and 18 that are not IP. And finally, as in Example 3.10, while obtaining an 8-cycle maximum packing of order 21, use an 8-cycle packing of order 16 that is not IP. By Lemma 4.2, these constructions will give 8-cycle maximum packings of orders $14, 18, 20, 21$ and 22 that are not IP.

Example 4.4 *There exist* 8*-cycle maximum packings of orders* 19 *and* 23 *that are not IP.*

When we consider the packings of orders 11 and 15 in Example 4.1, even if we let the edges in the leave be used in the inside cycles in these packings (in other words have different leaves in the two packings), any collection of the inside 8-cycles cannot form an 8-cycle packing of orders 11 or 15. This fact is checked by a computer search. Therefore by replacing the packings of orders 11 and 15 in the constructions of Example 3.9 and Example 3.11, respectively with the packings of orders 11 and 15 given in Example 4.1, one can construct packings that are not IP for the orders 19 and 23.

Theorem 4.5 *There exists an* 8*-cycle packing which is not IP for each* $n \geq 10$.

Proof Using Examples 4.1, 4.3 and 4.4 above for the constructions in Section 2 instead of the examples given in Section 3, one may construct an 8-cycle maximum packing which are not IP for each $n \geq 10$ by Lemma 4.2. □

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