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On covering the square flat torus by congruent discs

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Abstract

We consider coverings of the square flat torus (the quotient of the plane by the lattice generated by two unit perpendicular vectors) by congruent discs of minimal radius. These are periodic discs coverings of the Euclidean plane by congruent discs of minimal radius. Let $r(k)$ be the greatest lower bound of the radius of k congruent discs such that the square flat torus can be covered by these discs. It will be proved that $r(1) = \sqrt{2}/2$, $r(2) = 1/2$, $r(3) = 5\sqrt{2}/18$ and $r(4) \leq 5/16$.

1 Introduction

The unit discs $D_1, \dots, D_k$ *cover* the planar body $\mathcal{B}$ if the body $\mathcal{B}$ is contained by the union of the discs. It is a classical problem to determine the smallest radius of k equal circles that can cover the unit circle, the equilateral triangle, the unit square or, alternatively, a rectangle. Optimality proofs exist only for a few cases. The computational complexity can be measured by the fact that the standard discretizations of similar problems are $\mathcal{NP}$-complete [9].

Let $r_{\mathcal{B}}(k)$ be the minimal radius of k congruent discs that cover the body $\mathcal{B}$.

The covering problem for the unit circle has been solved by K. Bezdek [1] for $k = 5, 6$ and by Fejes Tóth [8] for $k = 8, 9, 10$. The known values can be found in Table 1.

For the case of the equilateral triangle, the covering problem has been solved by A. Bezdek and K. Bezdek [2] for $k \leq 6$, $k = 9$, $k = 10$. Melissen [16] rediscovered the results for $k \leq 6$. The known values can be found in Table 2.

The covering problem for the unit square has been solved in [2] for $k \leq 5$ and $k = 7$. The known values can be found in Table 3. Nurmela and Östergard presented arrangements in several cases [20].

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k	$r_{\mathcal{B}}(k)$	approximation	source
2	$1/2$	0.5	elementary
3	$\sqrt{3}/2$	0.8660 . . .	elementary
4	$1/\sqrt{2}$	0.5	elementary
5	0.60938 . . .	0.60938 . . .	[1]
6	0.555 . . .	0.555 . . .	[1]
7	$1/2$	0.5	elementary
8	$1/(1+2\cos(2\pi/7))$	0.4450 . . .	[8]
9	$1/(1+2\cos(2\pi/8))$	0.4142 . . .	[8]
10	$1/(1+2\cos(2\pi/9))$	0.3949 . . .	[8]

Table 1: The known values of $r_{\mathcal{B}}$ if $\mathcal{B}$ is the unit circle.

k	$r_{\mathcal{B}}(k)$	approximation	source
2	$1/2$	0.5	elementary
3	$1/(2\sqrt{3})$	0.2886 . . .	[2], [16]
4	$1/(2+\sqrt{3})$	0.2679 . . .	[2], [16]
5	$1/4$	0.25	[2], [16]
6	$\sqrt{3}/9$	0.1924 . . .	[2], [16]
9	$1/6$	0.1666 . . .	[2]
10	$\sqrt{3}/12$	0.1443 . . .	[2]

Table 2: The known values of $r_{\mathcal{B}}$ if $\mathcal{B}$ is the equilateral triangle of side length 1.

Heppes and Melissen [14] solved the problem for a general rectangle for $k \leq 5$. Moreover Heppes and Melissen [14] have found the best radius for $k = 7$ if the aspect ratio of the rectangle is either between 1 and 1.34457 . . ., or larger than 3.43017. Melissen and Schuur [17] extended the first range of the aspect ratio to the range between 1 and 1.422202580 . . .. Melissen and Schuur [17] presented coverings of rectangles in the case $k = 6$ for any aspect ratio and proved the optimality if the aspect ratio is on the interval $[3.118\ldots, 3.464\ldots]$.

k	$r_{\mathcal{B}}(k)$	approximation	source
2	$\sqrt{5}/4$	0.5590 . . .	elementary
3	$\sqrt{65}/16$	0.5038 . . .	[2]
4	$\sqrt{2}/4$	0.3535 . . .	[2]
5	0.3261 . . .	0.3261 . . .	[2]
7	$1/(1+\sqrt{7})$	0.2742 . . .	[2]

Table 3: The known values of $r_{\mathcal{B}}$ if $\mathcal{B}$ is the unit square.

There are results about partial covering of the unit disc with three [22] (Szalkai), four [12] (Tarnai, Gáspar, Hincz) or five [10, 11] (Tarnai, Gáspar, Hincz) discs.

Let $\mathbb{R}^2$ be the Euclidean plane and let Λ^2 be a 2-rank lattice in $\mathbb{R}^2$. A *torus* may be regarded geometrically as a quotient $\mathbb{R}^2/\Lambda^2$ of the Euclidean plane by a rank 2 lattice Λ^2. Let $\mathbb{Z}^2$ be the lattice generated by the vectors $(1, 0)$ and $(0, 1)$. The *square flat torus* is the quotient of the Euclidean plane by the lattice $\mathbb{Z}^2$. In Figure 1 (Figure 2, respectively), a covering of the square flat torus by three (four, respectively) congruent discs can be found.

Numerous results can be found in the literature about packing circles in a flat torus; see e.g. [3–6, 13, 15, 17, 19, 22] and in higher dimensions [23]. To our knowledge the present work is the first to consider coverings of the square flat torus. Our aim is to cover the square flat torus by congruent discs of minimal radius. This is the dual problem of packing circles on a flat torus. As usual it can be realized that to prove the covering problem is more difficult than to prove the packing problem. The result is the following.

Theorem 1 *If $r(k)$ is the greatest lower bound of the radius of k congruent discs covering the square flat torus, then*

$$r(1) = \frac{\sqrt{2}}{2}, \quad r(2) = \frac{1}{2}, \quad r(3) = \frac{5\sqrt{2}}{18} = 0.3928\ldots.$$

Observe the covering in Figure 1 is a covering of the Euclidean plane but it is different from the most efficient covering of the Euclidean plane presented by Fejes Tóth [7]; moreover $r(3) < r_{\mathcal{B}}(3)$ if $\mathcal{B}$ is the unit square.

The sketch of the proof of $r(3)$ is the following. The arrangement in Figure 1 is a candidate for the optimal covering. Using particular points and assuming that three congruent discs of radius less than $5\sqrt{2}/18$ can cover the torus, three cases are distinguished. In each case can be found a point on the torus outside the discs, thus the covering assumption is contradicted.

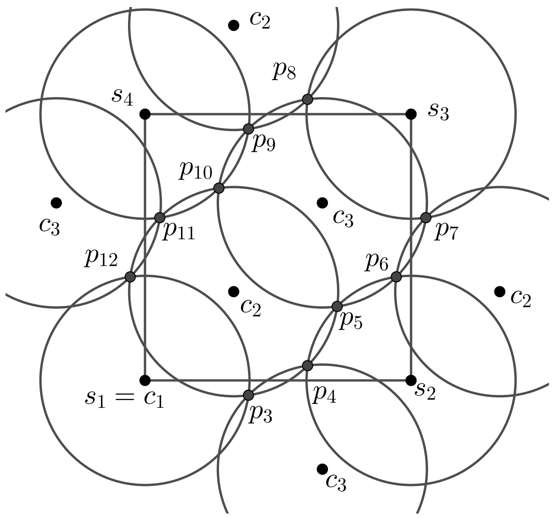

Figure 1: An optimal arrangement for $k = 3$.

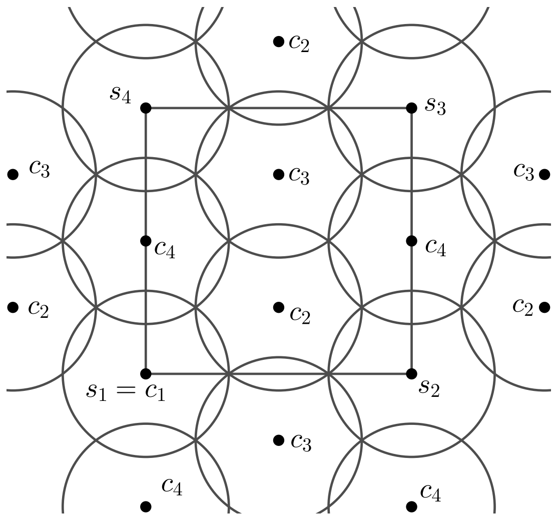

Figure 2: A conjecture for $k = 4$.

The arrangement in Figure 2 is a candidate for the optimal covering for $k = 4$. Of course from this arrangement comes $r(4) \leq 5/16$.

Conjecture 1 *If $r(k)$ is the greatest lower bound of the radius of k congruent discs covering the square flat torus, then $r(4) = 5/16$ (Figure 2).*

2 Notation

Let $\mathcal{T} = [0,1)^2$ be the 2-dimensional square flat torus, $S = [0,1]^2$ the unit square, $C(c,r)$ the circle of radius r center c and $D(c,r)$ the disc of radius r center c. For convenience points and vectors are identified. Let s_1 (s_2, s_3, s_4 respectively) be the point $(0,0)$ ($(1,0)$, $(1,1)$, $(0,1)$ respectively) on the Euclidean plane. In general, the coordinates of the point b_1 is denoted by (b_{1x}, b_{1y}).

If the side t_1t_2 of a triangle $t_1t_2t_3$ is changed for a circular arc t_1t_2, then the planar convex body which vertices are t_1, t_2, t_3 and sides are the circular arc t_1t_2 and the segments t_2t_3, t_3t_1 is called *semi-triangle* (e.g. $s_2p_5p_6$ is a semi-triangle in Figure 1). *Semi-quadrangle* can be defined similarly.

Throughout this paper, ab will also denote the length of the segment ab.

A *lift* of a point q on $\mathcal{T}$ is any point $\bar{q}$ on the plane that maps to q under the universal covering map. Observe the fundamental domain can be the region in the unit square with vertices s_1, s_2, s_3 and s_4.

If there is no confusion, then we will not write where the arrangement is considered (on $\mathbb{R}^2$ or on $\mathcal{T}$).

3 Preliminaries

The following lemma will be used several times.

Lemma 1 *Let $0 < r < 1/2$ be a fixed number and $D(s_1, r)$, $D(c_2, r)$ discs on $\mathcal{T}$. If $0 \le c_{2x} < 1 - 2r$ or $0 \le c_{2y} < 1 - 2r$ or $2r < c_{2x} < 1$ or $2r < c_{2y} < 1$, then the remaining part of $\mathcal{T}$ can not be covered by a disc of radius r.*

Proof. Using the symmetry of the square flat torus, it may be assumed that $0 \le c_{2x} < 1 - 2r$. Let p_{21} (p_{22} respectively) be the intersection point of $C(s_2, r)$ ($C(s_3, r)$ respectively) and the segment s_1s_2 (s_3s_4 respectively) (on the Euclidean plane) (Figure 3). Since the interior of the disc $D(c_2, r)$ (on $\mathcal{T}$) and the segment $p_{21}p_{22}$ (on $\mathbb{R}^2$) are disjoint, a disc of radius r can not cover the vertical segment $p_{21}p_{22}$. □

4 The Proof of Theorem 1

Let $D(c_1, r(k))$, …, $D(c_k, r(k))$ be the k discs on $\mathcal{T}$. By translation, it may be assumed that $c_1 = s_1$ (Figure 4). Thus s_1, s_2, s_3, s_4 are the same point and the center of the disc $D(c_1, r(k))$ on $\mathcal{T}$.

The proof of $r(1) = \sqrt{2}/2$ is trivial.

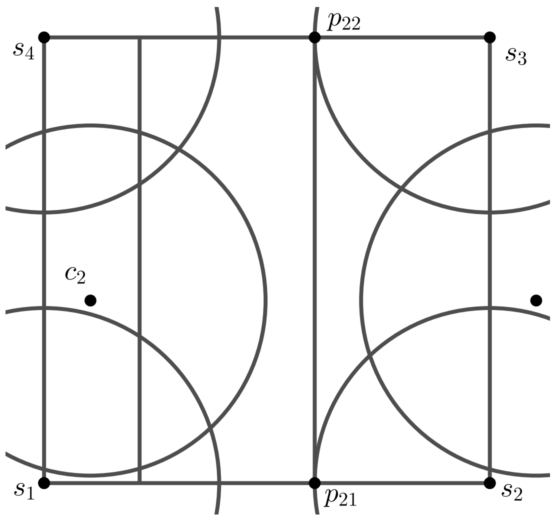

Figure 3: Lemma 1.

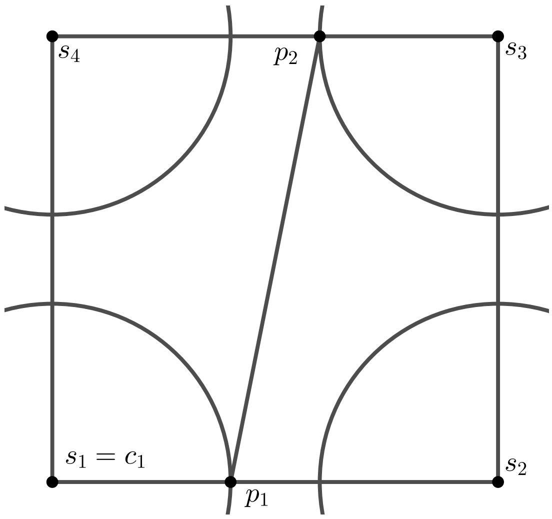

Figure 4: The square S.

For completeness, it will be proved that $r(2) = 1/2$ as well. If $k = 2$, then the arrangement $C(s_1, 1/2)$ and $C((1/2, 1/2), 1/2)$ shows that $r(2) \leq 1/2$. Let it be assumed that there is a radius r_2' such that $r_2' < 1/2$ and the discs $D(s_1, r_2')$, $D(c_2', r_2')$ cover $\mathcal{T}$. Let p_1 (p_2 respectively) be the intersection point of the circle $C(s_1, r_2')$ ($C(s_3, r_2')$ respectively) and the segment s_1s_2 (s_3s_4 respectively) (on the Euclidean plane). Since the segment p_1p_2 (on $\mathbb{R}^2$) and the interior of $D(s_1, r_2')$ (on $\mathcal{T}$) are disjoint and $p_1p_2 > 1$, this segment can not be covered by a disc of radius r_2', a contradiction. Thus $r(2) = 1/2$.

It will be proved that $r(3) = \frac{5\sqrt{2}}{18}$. Let $r_3 = \frac{5\sqrt{2}}{18}$. Let $c_2(1/3, 1/3)$ and $c_3(2/3, 2/3)$. The arrangement $D(s_1, r_3)$, $D(c_2, r_3)$ and $D(c_3, r_3)$ shows that $r(3) \leq r_3$ (Figure 1). Let $p_3, \ldots, p_{12}$ be points on the Euclidean plane as in Figure 1. For completeness $p_3(2/3 - r_3/\sqrt{2}, -1/3 + r_3/\sqrt{2})$, $p_4(1/3 + r_3/\sqrt{2}, 1/3 - r_3/\sqrt{2})$, $p_5(1 - r_3/\sqrt{2}, r_3/\sqrt{2})$, $p_6(2/3 + r_3/\sqrt{2}, 2/3 - r_3/\sqrt{2})$, $p_7(4/3 - r_3/\sqrt{2}, 1/3 + r_3/\sqrt{2})$, $p_8(1/3 + r_3/\sqrt{2}, 4/3 - r_3/\sqrt{2})$, $p_9(2/3 - r_3/\sqrt{2}, 2/3 + r_3/\sqrt{2})$, $p_{10}(r_3/\sqrt{2}, 1 - r_3/\sqrt{2})$, $p_{11}(1/3 - r_3/\sqrt{2}, 1/3 + r_3/\sqrt{2})$ and $p_{12}(-1/3 + r_3/\sqrt{2}, 2/3 - r_3/\sqrt{2})$. Observe $p_3p_{10} = 2r_3$, $p_4p_{11} = 2r_3$, etc. Since $p_3 = p_9 - (0, 1)$, p_3 is a lift of the point p_9. Similarly p_7 (p_8, p_{12} respectively) is a lift of the point p_{11} (p_4, p_6 respectively).

Let it be assumed that there is a radius r_3' such that $r_3' < r_3$ and three discs of radius r_3' cover $\mathcal{T}$. Let $D_1' = D(s_1, r_3')$, $D_2' = D(c_2', r_3')$ and $D_3' = D(c_3', r_3')$ be the three discs in this covering on $\mathcal{T}$. The points $p_3, \ldots, p_{12}$ can not be covered by D_1' on $\mathcal{T}$. It may be assumed that p_{11} lies in D_2'. From now on through the paper on the pictures can be seen circles of radius r_3. Assuming that p_{11} is in D_2' there are three possibilities: Case 1 where p_3 covered by D_2', Case 2 where p_5 is covered by D_2', and Case 3 where p_3 and p_5 are not covered by D_2'.

Case 1. The point p_3 is covered by the disc D_2'.
To cover the points p_{11} and p_3 on the torus with disc D_2', lift them to the Euclidean plane and analyze the arrangement. When lifting it may be assumed that p_{11} is in

the fundamental domain and it must be figured out which lifts of p_3 are less than $2r_3$ from p_{11} so that they both can possibly be covered by disc D'_2. The four possible lifts for p_3 are

1. p_3
2. $p_3 + (0, 1) = p_9$
3. $p_3 + (-1, 1)$
4. $p_3 + (-1, 0)$

The last case is eliminated because the distance from p_{11} to $p_3+(-1, 0)$ is the same as the distance p_3p_7 and $p_3p_7 > 2r_3$ which is too large. The first case and the third case are the same because the relationship between $p_3+(-1, 1)$ and p_{11} is the same p_9 and p_7 ('translate' both by vector $(1, 0)$) and this is the same at the relationship between p_{11} and p_3 (reflect over the line s_4s_2). This leads to two distinguished subcases: Subcase 1.1 where D'_2 contains p_{11} and p_3 and Subcase 1.2 where D'_2 contains p_{11} and p_9.

Subcase 1.1. The center c'_2 lies in the intersection of the discs $D(p_3, r_3)$ and $D(p_{11}, r_3)$ (or in the intersection of the discs $D(p_7, r_3)$ and $D(p_9, r_3)$) (Figure 5).
Let p_{111} be the intersection point of the circles $C(p_3, r_3)$ and $C(p_{11}, r_3)$ as in Fig 5. Observe $p_{111}(1/3, 1/3)$. Thus p_4, p_5 and p_{10} are covered neither by D'_1 nor by D'_2. After lifting it can be realized that by Lemma 1, c'_3 must lie in $D(p_4, r_3)$ or $D(p_4 + (0, 1), r_3)$. Similarly c'_3 must lie in $D(p_{10}, r_3)\cap D(p_5, r_3)$ or $D(p_{10}, r_3)\cap D(p_5+(0, 1), r_3)$ or $D(p_{10}-(0, 1), r_3)\cap D(p_5, r_3)$ or $D(p_{10}, r_3)\cap D(p_5-(1, 0), r_3)$ or $D(p_{10}+(1, 0), r_3)\cap D(p_5, r_3)$. Since neither $D(p_{10}, r_3)\cap D(p_5-(1, 0), r_3)$ nor $D(p_{10}+(1, 0), r_3)\cap D(p_5, r_3)$ lie in $D(p_4, r_3)$ or $D(p_4 + (0, 1), r_3)$, there are two distinguished subcases: Subcase 1.1.1 where the center c'_3 lies in the intersection of the discs $D(p_4, r_3)$ and $D(p_{10}, r_3)$, and Subcase 1.1.2 where the center c'_3 lies in the intersection of the discs $D(p_5, r_3)$ and $D(p_{10} - (0, 1), r_3)$.

Subcase 1.1.1. The center c'_3 lies in the intersection of the discs $D(p_4, r_3)$ and $D(p_{10}, r_3)$ (Figure 6).
Let p_{1111} (p_{1112} respectively) be the intersection point of the circles $C(p_4, r_3)$ and $C(p_{10}, r_3)$ ($C(s_3, r_3)$ and $C(p_{1111}, r_3)$ respectively) as in Figure 6. Observe the point p_{1112} is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 1.1.2. The center c'_3 lies in the intersection of the discs $D(p_5, r_3)$ and $D(p_{10} - (0, 1), r_3)$ (Figure 7).
Let A_1 (A_2 respectively) be the segment $(0, 2r_3)(1, 2r_3)$ ($(0, 1 - 2r_3)(1, 1 - 2r_3)$ respectively). By Lemma 1, $1 - 2r_3 \leq c'_{3y} \leq 2r_3$, a contradiction.

Subcase 1.2. The center c'_2 lies in the intersection of the discs $D(p_9, r_3)$ and $D(p_{11}, r_3)$ (Figure 8).
By Lemma 1, $c'_{2x} \geq 1-2r_3$ and $c'_{2y} \leq 2r_3$. Observe p_5 is covered neither by D'_1 nor by D'_2. Let p_{121} be the intersection point of the circles $C(p_9, r_3)$ and $C(p_{11}, r_3)$ as in Figure 8. Let p_{122} (p_{123} respectively) be the point $(1/3, 2/3)$ ($(1-2r_3, 2r_3)$ respectively).

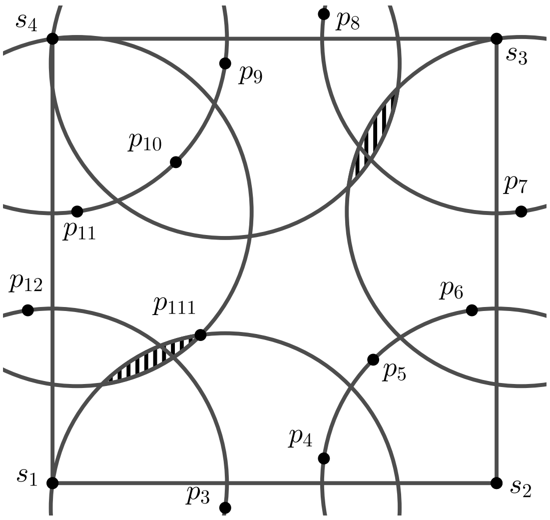

Figure 5: Subcase 1.1.

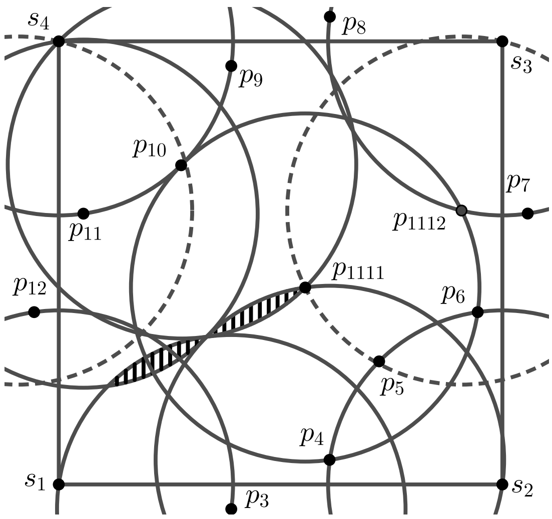

Figure 6: Subcase 1.1.1.

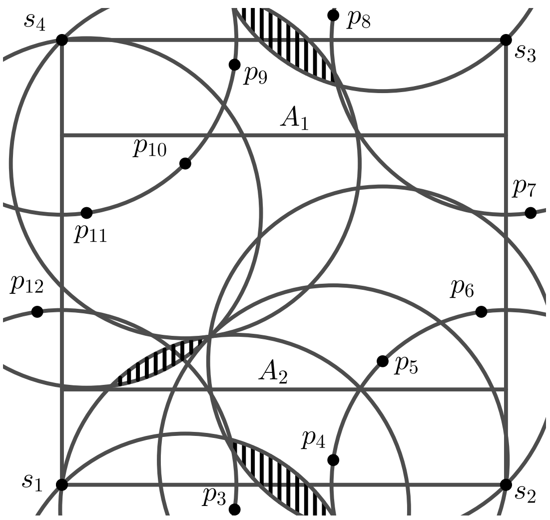

Figure 7: Subcase 1.1.2.

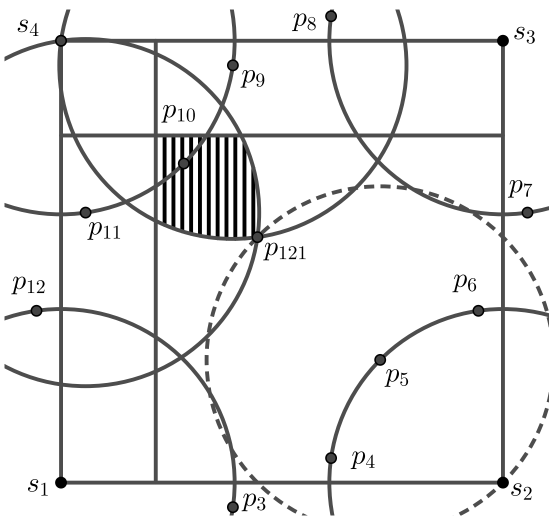

Figure 8: Subcase 1.2.

Let p_{124} (p_{125} respectively) be the intersection point of the circle $C(p_9, r_3)$ and the segment $(1-2r_3, 0)(1-2r_3, 1)$ ($(1/3, 0)(1/3, 1)$ respectively) (Figure 9). Since there is reflective symmetry over the segment s_2s_4 in $\mathcal{T}$, it may be assumed $c'_{2y} \leq 1 - c'_{2x}$. There are two distinguished subcases. Subcase 1.2.1 where c'_2 lies in the semi-triangle $p_{121}p_{123}p_{124}$ and not to the right of $x = 1/3$ and Subcase 1.2.2 where c'_2 lies in the semi-triangle $p_{121}p_{123}p_{124}$ and not to the left of $x = 1/3$.

Subcase 1.2.1. The center c'_2 lies in the semi-quadrangle $p_{122}p_{123}p_{124}p_{125}$ (Figure 10).

Let p_{1211} and p_{1212} be the intersection points of the circles $C(p_{122}, r_3)$ and $C(s_3, r_3)$ as in Figure 10. Since $p_{123}p_{122}p_{1212}\angle = \pi/2$ and $p_{125}p_{122}p_{1211}\angle > \pi/2$, the point p_{122} is the point in the semi-quadrangle $p_{122}p_{123}p_{124}p_{125}$ that is closest to points p_{1211} and p_{1212}. Thus the points p_5, p_{1211} and p_{1212} are covered neither by D'_1 nor by D'_2. Since

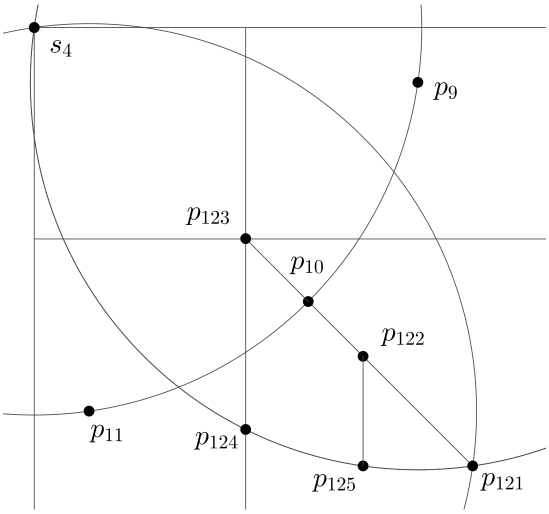

Figure 9: The points

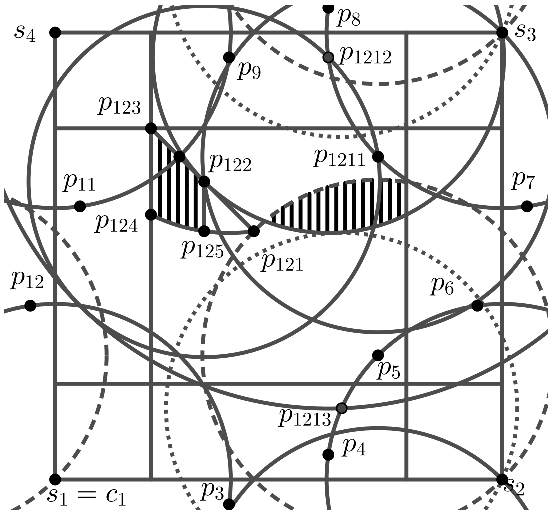

Figure 10: Subcase 1.2.1.

$p_{1211}(p_{1212} - (0,1)) = 2r_3$ and by Lemma 1, $1 - 2r_3 \leq c'_{3y} \leq 2r_3$, the center c'_3 lies in the intersection of the discs $D(p_{1212}, r_3)$ and $D(p_5, r_3)$.
By Lemma 1, $c'_{3x} \leq 2r_3$. Let p_{1213} be the intersection point of the circles $C(p_{1212}, 2r_3)$ and $C(s_2, r_3)$ as in Figure 10. Observe p_{1213} is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 1.2.2. The center c'_2 lies in the semi-triangle $p_{121}p_{122}p_{125}$ (Figure 11). Let p_{1221} (p_{1222} respectively) be the intersection point of the circles $C(p_{122}-(0,1), r_3)$

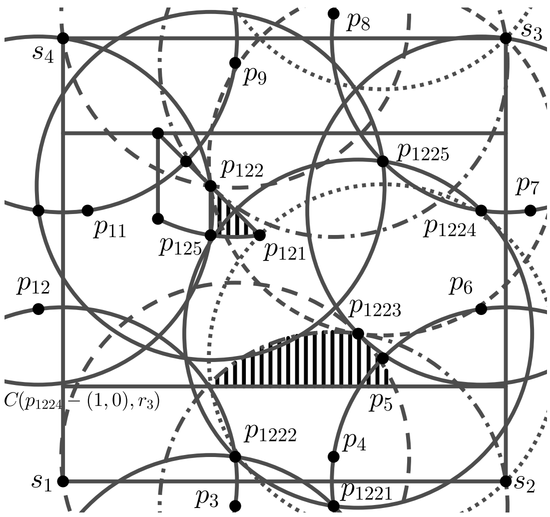

Figure 11: Subcase 1.2.2

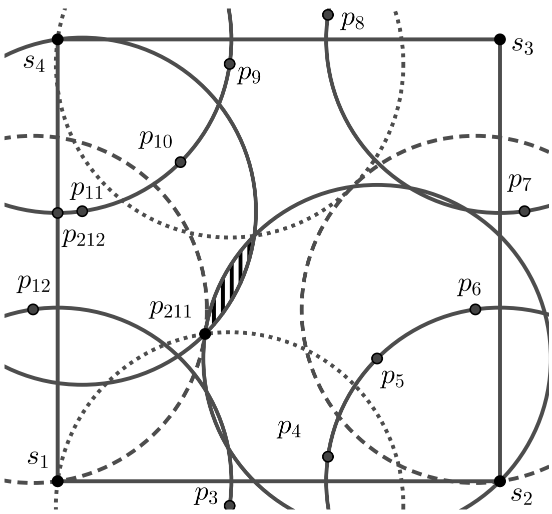

Figure 12: Case 2.1.

and $C(s_2, r_3)$ ($C(s_1, r_3)$ respectively) as in Figure 11. The points p_5, p_{1221} and p_{1222} are covered neither by D'_1 nor by D'_2 ($\angle p_{121}p_{122}(p_{1221}+(0,1)) = \frac{\pi}{2}$ and $p_{121}p_{122}(p_{1222}+(0,1)) > \frac{\pi}{2}$). Since $(p_{1221}+(0,1))p_{1222} > 2r_3$, $p_{1221}(p_{1222}+(1,0)) = 2r_3$ and $p_5(p_{1222}+(0,1)) > 2r_3$, the center c'_3 lies in the intersection of the discs $D(p_5, r_3)$, $D(p_{1221}, r_3)$ and $D(p_{1222}, r_3)$. By Lemma 1, $1-2r_3 \leq c'_{3y} \leq 2r_3$. Let p_{1223} be the intersection point of the circles $C(p_{1221}, r_3)$ and $C(p_{1222}, r_3)$ as in Figure 11. Observe $p_{1223}(2/3, 1/3)$. Let p_{1224} and p_{1225} be the intersection points of the circles $C(p_{1223}, r_3)$ and $C(s_3, r_3)$ as in Figure 11. Since $p_{1221}p_{1225} = 2r_3$ and $p_{1222}p_{1224} = 2r_3$, the points p_{1224} and p_{1225} are covered neither by D'_1 nor by D'_3. The distance between the points p_{121} and p_{1224} is greater than r_3. In order to cover p_{1224} by D'_2 the center c'_2 lies in the disc $D(p_{1224} - (1, 0), r_3)$. Since $(p_{1224} - (1, 0))p_{1225} = 2r_3$, p_{1225} is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Case 2. The point p_5 is covered by the disc D'_2.
After lifting it may be assumed that p_{11} is in the fundamental domain. The three possible lifts for p_5 are p_5 or $p_5-(1, 0)$ or $p_5+(-1, 1)$. This leads to three distinguished subcases: Subcase 2.1 where the center c'_2 lies in $D(p_5, r_3) \cap D(p_{11}, r_3)$ and Subcase 2.2 where the point c'_2 lies in $D(p_5, r_3) \cap D(p_7, r_3)$ and Subcase 2.3 where the point c'_2 lies in $D(p_5 + (0, 1), r_3) \cap D(p_7, r_3)$.

Subcase 2.1. The center c'_2 lies in the intersection of the discs $D(p_5, r_3)$ and $D(p_{11}, r_3)$ (Figure 12).
Let p_{211} be the intersection point of the circles $C(p_5, r_3)$ and $C(p_{11}, r_3)$ as in Figure 12. Observe $p_{211}(1/3, 1/3)$. The points p_6 and p_9 are covered neither by D'_1 nor by D'_2.

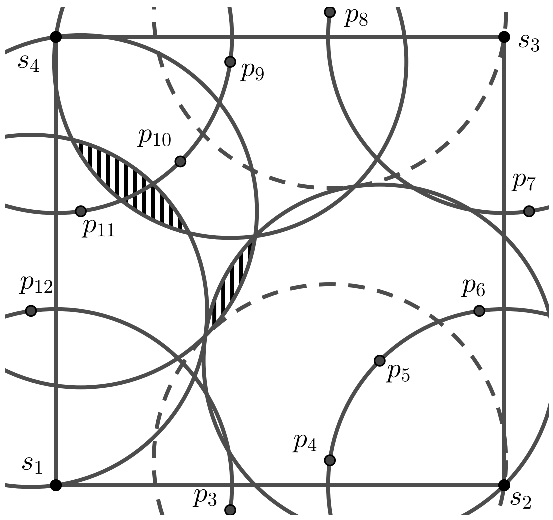

Figure 13: Subcase 2.1.1

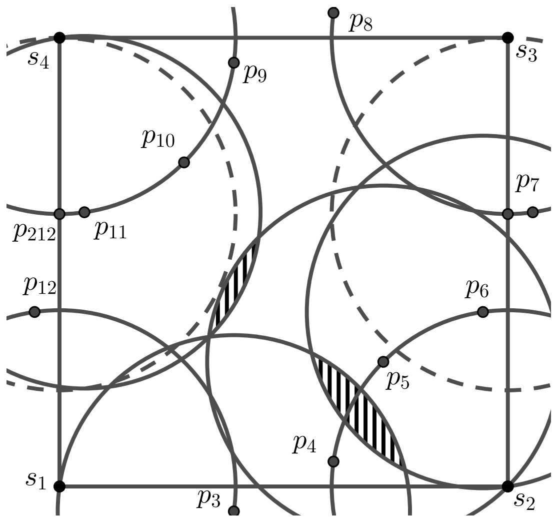

Figure 14: Case 2.1.2.

Let p_{212} be the intersection point of the circle $C(s_4, r_3)$ and the segment s_1s_4. After lifting it may be assumed that p_9 lies in the fundamental domain. The three possible lifts for p_6 are p_{12} or $p_6 + (0, 1)$ or $p_6 + (-1, 1)$. This leads to three distinguished subcases: Subcase 2.1.1 where the point c'_3 lies in $D(p_9, r_3) \cap D(p_{12}, r_3)$, Subcase 2.1.2 where the point c'_3 lies in $D(p_3, r_3) \cap D(p_6, r_3)$ and Subcase 2.1.3 where the point c'_3 lies in $D(p_3, r_3) \cap D(p_{12}, r_3)$.

Subcase 2.1.1. The point c'_3 lies in the intersection of the discs $D(p_9, r_3)$ and $D(p_{12}, r_3)$ (Figure 13).
In this case the point p_4 is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 2.1.2. The point c'_3 lies in the intersection of the discs $D(p_3, r_3)$ and $D(p_6, r_3)$ (Figure 14).
In this case the point p_{212} is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 2.1.3. The point c'_3 lies in the intersection of the discs $D(p_3, r_3)$ and $D(p_{12}, r_3)$ (Figure 15).
Let p_{2131} (p_{2132} respectively) be the intersection point of the circles $C(p_5, r_3)$ and

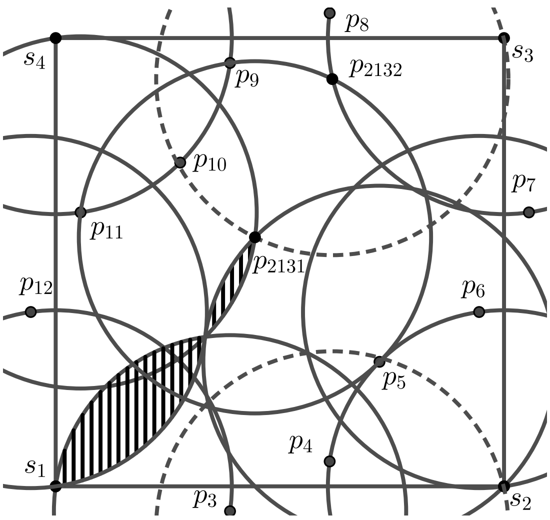

Figure 15: Subcase 2.1.3.

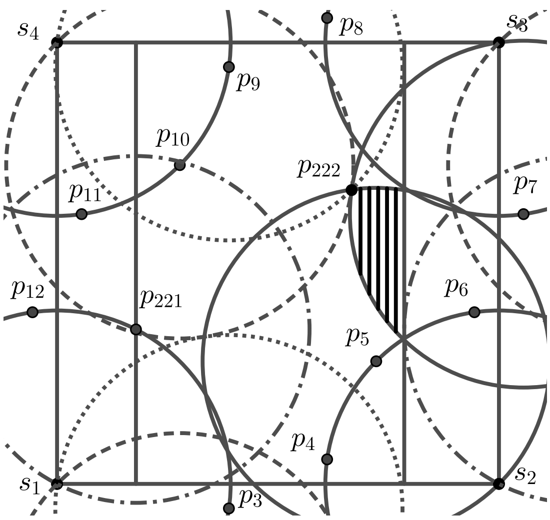

Figure 16: Case 2.2.

$C(p_{11}, r_3)$ ($C(p_{2131}, r_3)$ and $C(s_3, r_3)$ respectively) as in Figure 15. In this case the point p_{2132} is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 2.2. The point c'_2 lies in the intersection of the discs $D(p_5, r_3)$ and $D(p_7, r_3)$ (Figure 16).
By Lemma 1, $1 - 2r_3 \leq c'_{2x} \leq 2r_3$. Let p_{221} be the intersection point of the circle $C(s_1, r_3)$ and the segment $(3r_3 - 1, 0)(3r_3 - 1, 1)$. Let p_{222} be the intersection point of the circles $C(p_5, r_3)$ and $C(p_7, r_3)$ as in Figure 16. Observe $p_{222}(2/3, 2/3)$. Thus p_9, p_{10} and p_{221} are covered neither by D'_1 nor by D'_2 on $\mathcal{T}$. After lifting it may be assumed that p_{221} lies in the fundamental domain. The two possible lifts for p_9 are p_9 or $p_9 - (0, 1)$. The two possible lifts for p_{10} are p_{10} or $p_{10} - (0, 1)$. This leads to two distinguished subcases: Subcase 2.2.1 where the point c'_3 lies in $D(p_9, r_3) \cap D(p_{221}, r_3)$ and Subcase 2.2.2 where the point c'_3 lies in $D(p_3, r_3) \cap D(p_{221}, r_3)$.

Subcase 2.2.1. The point c'_3 lies in the intersection of the discs $D(p_9, r_3)$ and $D(p_{221}, r_3)$ (Figure 17).
By Lemma 1, $c'_{3x} \geq 1 - 2r_3$. Let p_{2211} be the intersection point of the circles $C(s_1, r_3)$ and $C(p_9, 2r_3)$. Since $p_9p_{2211} > 2r_3$ and $p_7p_{2211} > 2r_3$, the point p_{2211} is covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 2.2.2. The point c'_3 lies in the intersection of the discs $D(p_3, r_3)$ and $D(p_{221}, r_3)$ (Figure 18).
By Lemma 1, $c'_{3x} \geq 1 - 2r_3$ and $c'_{3y} \geq 1 - 2r_3$. Since $p_3p_{10} = 2r_3$, the point p_{10} is

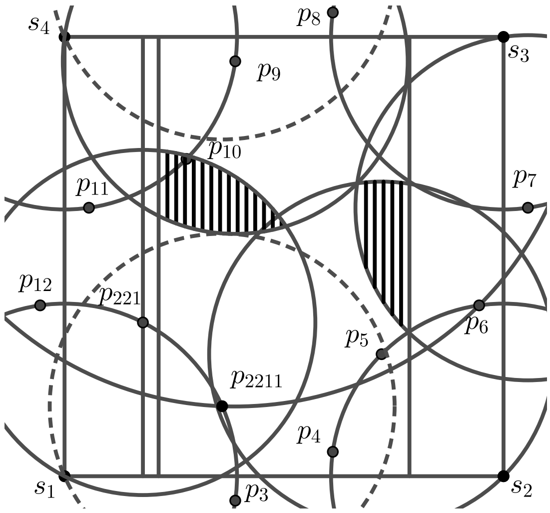

Figure 17: Case 2.2.1

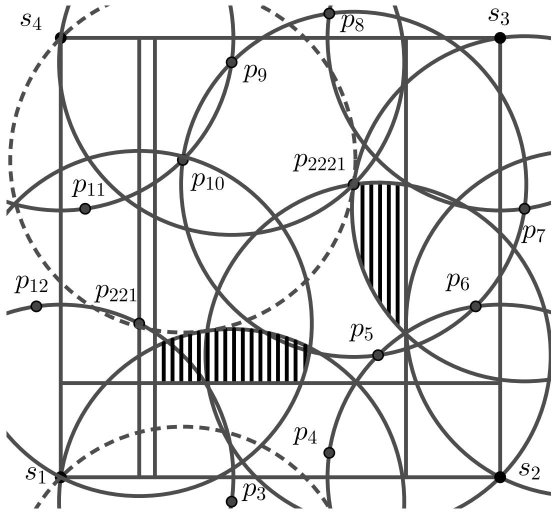

Figure 18: Subcase 2.2.2

covered neither by D'_1 nor by D'_2 nor by D'_3 on $\mathcal{T}$, a contradiction.

Subcase 2.3. The point c'_2 lies in the intersection of the discs $D(p_5+(0,1),r_3)$ and $D(p_7,r_3)$ (Figure 19).
Let A_1 be the segment $(0,2r_3)(1,2r_3)$. By Lemma 1, $c'_{2y} \leq 2r_3$, a contradiction.

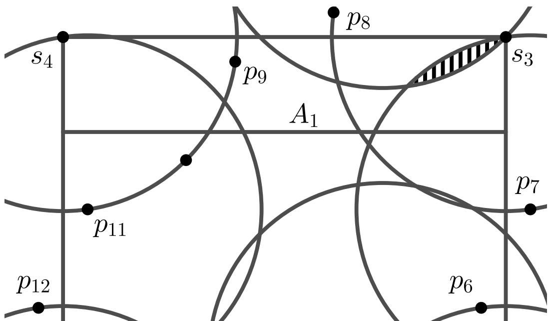

Figure 19: Case 2.3.

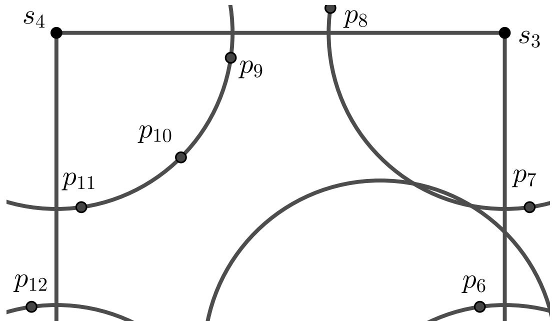

Figure 20: Case 3.1.

Case 3. The points p_3 and p_5 are not covered by the disc D'_2.
The points p_3 and p_5 are covered by D'_3. After lifting it may be assumed that p_5 lies in the fundamental domain. The three possible lifts for p_3 are p_3 or p_9 or $p_3+(1,0)$. This leads to three distinguished subcases: Subcase 3.1 where the center c'_3 lies in $D(p_3,r_3) \cap D(p_5,r_3)$, Subcase 3.2 where the center c'_3 lies in $D(p_5,r_3) \cap D(p_9,r_3)$ and Subcase 3.3 where the center c'_3 lies in $D(p_3,r_3) \cap D(p_5-(1,0),r_3)$.

Subcase 3.1. The center c'_3 lies in the intersection of the discs $D(p_3,r_3)$ and $D(p_5,r_3)$ (Figure 20).

Observe the reflected image of p_7 (p_5 respectively) is p_3 (p_5 respectively) over the line s_2s_4. If s_2 is changed for s_3 in the proof of Subcase 2.2, then Subcase 3.1 is proved.

Subcase 3.2. The center c_3' lies in the intersection of the discs $D(p_5, r_3)$ and $D(p_9, r_3)$ (Figure 21).
Since the reflected image of p_{11} (p_5 respectively) is p_9 (p_5 respectively) over the line s_2s_4, the proof comes from the proof of Subcase 2.1.

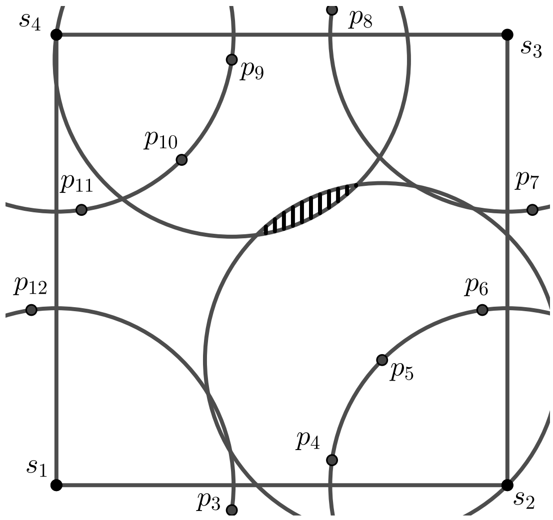

Figure 21: Case 3.2.

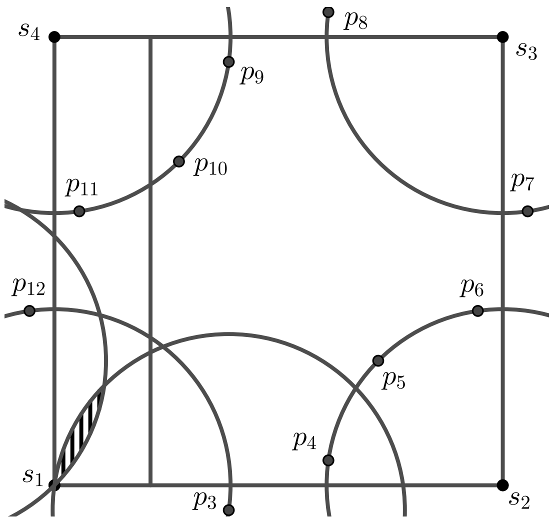

Figure 22: Case 3.3.

Subcase 3.3. The center c_3' lies in the intersection of the discs $D(p_3, r_3)$ and $D(p_5 - (1, 0), r_3)$ (Figure 22).
Since the reflected image of $p_5 - (1, 0)$ (p_3 respectively) is $p_5 + (0, 1)$ (p_7 respectively) over the line s_2s_4, the proof comes from the proof of Subcase 2.3.

Thus $r(3) = 5\sqrt{2}/18$. □

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